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MARKET AND LOCATIONAL EQUILIBRIUM FOR TWO COMPETITORS*

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Abstract
We consider the following two stage location and allocation game involving two competing firms. The firms first select the location of their facility on a network. Then the firms optimally select the quantities each wishes to supply to the markets which are located at the vertices of the network. The criterion for optimality for each firm is maximizing its profit, which is the total revenue minus the production and transportation costs. Under reasonable assumptions regarding the revenue, the production cost and the transportation cost functions, we show that there is a Nash equilibrium for the quantities offered at the markets by each firm. Furthermore, if the quantities supplied (at the equilibrium) by each firm at each market are positive, then there is also a Nash locational equilibrium, i.e. no firm finds it advantageous to change its location.

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1. Introduction

In the competitive location problem, firms select their strategic variables in order to maximize their own profit. Various pricing strategies have been considered in spatial economics for the unit line segment. The problem becomes much more complicated on a network; this explains why most of the models proposed in Operations Research assume that the firms sell one uniform product at a constant price. In such a case, consumers pay the transportation costs and firms' strategic variables reduce to their location. This problem has been studied on a network by, among others, Dobson and Karmarkar (1988), Hakimi (1983, 1989), Hansen and Labbé (1988) and Wendell and McKelvey (1981). When firms are paying the transportation costs, then the firms are allowed to select a delivered price which depends on the location of the consumers. Lederer and Thisse (1988) study such a situation where firms are allowed to select their location and the price at which they are willing to sell the product at each market. These authors consider a two-stage game in which firms select their location at the first stage and the price at the second stage, and show the existence of a sub-game perfect Nash equilibrium. In their study, the demand at each point is assumed to be constant.

Here we consider an alternative specification where the demand at each point depends on the price level. Specifically, we study a two stage game in which two firms select first their location and then the quantities they will offer to each market. Because firms compete in quantities rather than in price, the second stage is a Cournot game (see e.g. Friedman (1977)). For simplicity, we assume that the unit price on each market is a linear decreasing function of the total quantity offered at that market. Moreover, we assume that firms produce at constant marginal production cost. This implies that markets can be treated independently when the locations of the firms are fixed. The markets are located at the nodes of a network $N=(V,E)$. For any pair of locations x_1 and x_2 on the network, the second stage of the game, in which firms choose the quantities, is a non-zero-sum non cooperative two-person game for which there exists a Nash equilibrium (see Harker (1986) and Kuo and Hakimi (1988)). Furthermore, the linearity assumption for the price implies that these equilibrium quantities are unique and can be explicitly determined. Then, replacing these quantities in the profit functions of the two firms leads to functions which depend only upon the locations x_1 and x_2 that firms choose in the first stage. This yields a new non-zero-sum non cooperative two-person game. If there exists a Nash equilibrium pair of locations, this pair, together with the corresponding equilibrium quantities is a *subgame perfect Nash equilibrium*.

Assuming that the transportation costs are concave in the distance, that the marginal production costs are concave along any edge of the network, and provided that for every pair of possible locations x_1 and x_2, all the equilibrium quantities are positive, we show that there exists a subgame perfect Nash equilibrium. Furthermore, the corresponding pair of equilibrium locations always occurs at vertices of the network. For the case where all markets are not always served by both firms, we provide examples with no Nash equilibrium in location or with Nash equilibrium locations which consist of the interior points of edges.

This paper is organized as follows. The model is presented at the next section. The second stage of the game is studied at Section 3. The existence of a subgame perfect Nash equilibrium is discussed at Section 4. Some concluding remarks are presented in Section 5.

2. Model

Let $N=(V,E)$ be a network with vertex set V and edge set E. At each vertex $v_k \in V$ is located a *market* where a given product is sold at unit price p_k. This price is a linear decreasing function of the total quantity q_k offered at v_k and is given by:

$$p_k(q_k) = \begin{cases} \alpha_k - \beta_k q_k, & \text{if } 0 \le q_k \le \alpha_k/\beta_k \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

with $\alpha_k \ge 0$ and $\beta_k \ge 0$.

Each edge $(v_i,v_j)\in E$ has a positive length and is assumed to be rectifiable. The distance between two points (vertices or points along edges) x and y is noted $d(x,y)$ and is the length of a shortest path joining them.

Two firms locate at the network at points x_1 and x_2 respectively, produce the product and ship it to the markets $v_k\in V$. The unit transportation cost between firm i's location x_i $(i=1,2)$ and market $v_k\in V$ is denoted $T(d(x_i,v_k))$ and is concave and increasing in the distance.

The marginal production cost, which is assumed to be independent from the quantity produced, is denoted by $C(x_i)$. Furthermore, it is a concave function along each edge of the network. This last assumption corresponds to the following situation.

Firm i, $i=1,2$, produces the product by using J inputs $j=1,\ldots,J$. There are H_j possible sources of input j in N, they are denoted by y_{jh}, $h=1,\ldots,H_j$. The price of input j at y_{jh} is given and denoted by p_{jh}. The cost of transporting one unit of input j from y_{jh} to x_i is $t_j(d,y_{jh},x_i))$, which is assumed to be increasing and concave in the distance $d(y_{jh},x_i)$. Let $a_j(i)$ be the amount of input j used by firm i to produce one unit of the product. Then, the marginal production cost at x_i is given by:

$$C(x_i)=\sum_{j=1}^{J}\left[\min_{h=1,\ldots,H_j}\{p_{jh}+t_j(d(y_{jh},x_i))\}\right]a_j(i),$$

which, as a weighted sum of minima of concave functions, is a concave function of the distances $d(y_{jh},x_i)$. Next, since each distance $d(y_{jh},x_i)$ is a concave function of x_i as x_i moves along an edge, $C(x_i)$ is also concave.

The quantity offered by firm i, $i=1,2$, at market $v_k\in V$ is denoted by q_{ik}, let $\underline{q}_i$ represent the vector of all quantities q_{ik}, $v_k\in V$. Moreover, $q_k=q_{1k}+q_{2k}$.

We can now state the profits as a function of the locations and the quantities.

$$\Pi_1(x_1,x_2;\underline{q}_1,\underline{q}_2)=\sum_{v_k\in V}[p_k(q_k)-T(d(x_1,v_k))]q_{1k}-C(x_1)\sum_{v_k\in V}q_{1k}, \tag{2}$$

and

$$\Pi_2(x_1,x_2;\underline{q}_1,\underline{q}_2)=\sum_{v_k\in V}[p_k(q_k)-T(d(x_2,v_k))]q_{2k}-C(x_2)\sum_{v_k\in V}q_{2k}. \tag{3}$$

3. The second stage

Let x_1 and x_2 be a pair of fixed locations for the firms. The second stage problem is a non-zero-sum non cooperative two-person game in which firms determine the quantity vectors that maximize their profit given by (2) and (3). The solution is a Nash equilibrium, i.e. a pair of nonnegative quantity vectors $\overset{*}{\underline{q}}_i=(\overset{*}{q}_{ik};v_k\in V)$, $i=1,2$, such that

$$\Pi_1(x_1,x_2;\overset{*}{\underline{q}}_1,\overset{*}{\underline{q}}_2)\geq\Pi_1(x_1,x_2;\underline{q}_1,\overset{*}{\underline{q}}_2),$$

and

$$\Pi_2(x_1,x_2;\overset{*}{\underline{q}}_1,\overset{*}{\underline{q}}_2)\geq\Pi_2(x_1,x_2;\overset{*}{\underline{q}}_1,\underline{q}_2)$$

for any vector $\underline{q}_i$, $i=1,2$, of nonnegative quantities. In words, this means that at equilibrium no firm has an incentive to change its market allocation strategy.

***Proposition* 1.** Let $c_k(x_i)=c(x_i)+T(d(x_i,v_k))$, for $i=1,2$ and $v_k\in V$. Then there exists a unique pair of equilibrium quantities given by:

$$\overset{*}{q}_{1k}(x_1,x_2)=\begin{cases}\dfrac{\alpha_k-c_k(x_1)}{2\beta_k} & \text{if } c_k(x_1)\leq\min\{\alpha_k,2c_k(x_2)-\alpha_k\},\\[2ex] \dfrac{\alpha_k-2c_k(x_1)+c_k(x_2)}{3\beta_k} & \text{if } \min\{\alpha_k,2c_k(x_2)-\alpha_k\}\leq c_k(x_1)\\ & \quad\leq\min\{\alpha_k,\dfrac{\alpha_k+c_k(x_2)}{2}\}\\[2ex] 0 & \text{otherwise;}\end{cases} \tag{4}$$

and

$$\overset{*}{q}_{2k}(x_1,x_2) = \begin{cases} \dfrac{\alpha_k - c_k(x_2)}{2\beta_k} & \text{if } c_k(x_2) \le \min\{\alpha_k, 2c_k(x_1) - \alpha_k\}, \\[2ex] \dfrac{\alpha_k - 2c_k(x_2) + c_k(x_1)}{3\beta_k} & \text{if } \min\{\alpha_k, 2c_k(x_1) - \alpha_k\} \le c_k(x_2) \\ & \le \min\{\alpha_k, \dfrac{\alpha_k + c_k(x_1)}{2}\} \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{5}$$

Proof: First, from equations (1),(2) and (3), we observe that firms will always produce quantities such that the resulting price $p_k(q_k)$ is strictly positive. Hence, we can restrict the set of feasible solutions to

$$Q = \{(\underline{q}_1, \underline{q}_2) : q_{1k} \ge 0 \text{ and } q_{1k} + q_{2k} \le \alpha_k/\beta_k, \ i = 1,2, \text{ and } v_k \in V\}.$$

Then, the profit functions can be rewritten as follows:

$$\Pi_1(x_1, x_2; \underline{q}_1, \underline{q}_2) = \sum_{v_k \in V} (\alpha_k - \beta_k(q_{1k} + q_{2k}) - c_k(x_1))q_{1k}$$

and

$$\Pi_2(x_1, x_2; \underline{q}_1, \underline{q}_2) = \sum_{v_k \in V} (\alpha_k - \beta_k(q_{1k} + q_{2k}) - c_k(x_2))q_{2k}.$$

Next, from these expressions, it is easy to see that each market v_k can be treated separately from the others. Furthermore, the contribution of each market to these profits is a strictly concave function in the quantities. This property together with the fact that Q is compact and convex implies the existence and the uniqueness of the equilibrium in quantities.

Now consider the following two quadratic programs:

$$\overset{*}{q}_{1k} = \arg\max_{q_{1k} \ge 0} \{(\alpha_k - \beta_k(q_{1k} + \overset{*}{q}_{2k}) - c_k(x_1))q_{1k}\},$$

and

$$\overset{*}{q}_{2k} = \arg\max_{q_{2k} \ge 0} \{(\alpha_k - \beta_k(\overset{*}{q}_{1k} + q_{2k}) - c_k(x_2))q_{2k}\}.$$

The solutions given by (4) and (5) satisfy simultaneously the Kuhn–Tucker conditions of these two programs and the condition $q^*_{1k}+q^*_{2k}\leq\alpha_k/\beta_k$, which is relaxed in these two programs, is always satisfied.

□

4. The First Stage

For any pair $x_1,x_2\in N$ of firms' locations, we can now replace the quantities by expressions (4) and (5) in the profit functions and obtain functions depending only on the locations.

In this way we obtain the game of the first stage, in which the profit functions are:

$$\Pi_i(x_1,x_2)=\Pi_i(x_1,x_2;\underline{q}^*_1(x_1,x_2),\underline{q}^*_2(x_1,x_2)) \tag{6}$$

for $i=1,2$.

The solution of the total competitive process, which is called a *subgame perfect Nash equilibrium*, is then given by a pair of Nash equilibrium locations for the first stage game together with the equilibrium quantities for the second stage game (see Selten (1975) for a discussion of this solution concept). Formally, a pair $x^*_1,x^*_2\in N$ of locations and the corresponding quantities $q^*_{1k}(x^*_1,x^*_2)$ and $q^*_{1k}(x^*_1,x^*_2)$ given by (4) and (5) for each $v_k\in V$ constitute a subgame perfect Nash equilibriium if and only if

$\Pi_1(x^*_1,x^*_2)\geq\Pi_1(x_1,x^*_2)$,

and

$\Pi_2(x^*_1,x^*_2)\geq\Pi_2(x^*_1,x_2)$,

for any $x_1\in N$ and $x_2\in N$.

To establish the existence of an equilibrium at the first stage, we now concentrate on the case where at the second stage, for every pair x_1,x_2 of locations, the equilibrium quantities $q^*_{ik}(x_1,x_2)$ are strictly positive for $i=1,2$ and all $v_k\in V$ with $\alpha_k>0$.

Looking at expressions (4) and (5), it can be seen that $q^*_{ik}(x_1,x_2)>0$ for $i=1,2$ and $v_k \in V$ if and only if for every $v_k \in V$ with $\alpha_k>0$,

$$2 \max_{x\in N} c_k(x) - \min_{x\in N} c_k(x) < \alpha_k \ . \tag{7}$$

Intuitively, this means that for any location $x\in N$, the total costs must be sufficiently small so that it is always profitable to offer some quantity at each market.

Under condition (7), we can rewrite the profit functions of the first stage. Specifically, let $V'=\{v_k\in V:\alpha_k>0\}$. We then have:

$$\Pi_1(x_1,x_2) = \sum_{v_k\in V'} (\alpha_k - 2c_k(x_1)+c_k(x_2))^2/9\beta_k \ , \tag{8}$$

and

$$\Pi_2(x_1,x_2) = \sum_{v_k\in V'} (\alpha_k - 2c_k(x_2)+c_k(x_1))^2/9\beta_k \ . \tag{9}$$

***Lemma* 2.** Under condition (7), $\Pi_1(x_1,x_2)$ is convex when x_1 moves along an edge $(v_i,v_j)\in E$ and $\Pi_2(x_1,x_2)$ is convex when x_2 moves along an edge $(v_i,v_j)\in E$.

Proof. We need only to provide the proof for $\Pi_1(x_1,x_2)$. First, observe that for each $v_k\in V'$, $c_k(x_1)=c(x_1)+T(d(x_1,v_k))$ is concave along (v_i,v_j). Hence $(\alpha_k-2c_k(x_1)+c_k(x_2))^2/9\beta_k$ is convex when x_2 is fixed and x_1 moves along (v_i,v_j). Finally, under (7), $\Pi_1(x_1,x_2)$ is given by (8), thus it is a sum of convex functions.

□

***Corollary* 3.** Under condition (7), each firm maximizes its profit at some vertex of N, regardless where its competitor locates, i.e.

$$\arg\max_{x_i\in N} \Pi_i(x_1,x_2)\in V.$$

Proof. This is a direct consequence of the convexity of $\Pi_i(x_1, x_2)$ along an edge.

□

We can now establish the existence of a subgame perfect Nash equilibrium when the total costs are not too large with respect to the prices on the markets.

***Theorem* 4.** Under condition (7), there always exists a pair of equilibrium locations for the first stage game. Furthermore, such an equilibrium consists of a pair of vertices.

Proof. From Corollary 3, we can restrict the set of candidate locations to the set V of vertices.

Given any starting location of a firm, we show that if each firm, in turn, responds by relocating at a vertex that maximizes its profit, then this process converges, in a finite number of iterations, to an equilibrium. By contradiction, assume that there exists two subsets $\{v_{1,1}, v_{1,2}, \cdots, v_{1,p}\}$ and $\{v_{2,1}, v_{2,2}, \cdots, v_{2,p}\}$ of vertices (which represents the sequences of choices of locations by firms 1 and 2 respectively with firm 1 being the first player) such that

$$\begin{aligned}
&\Pi_1(v_{1,2}, v_{2,1}) > \Pi_1(v_{1,1}, v_{2,1}),\\
&\Pi_2(v_{1,2}, v_{2,2}) > \Pi_2(v_{1,2}, v_{2,1}),\\
&\qquad\vdots\\
&\Pi_1(v_{1,i+1}, v_{2,i}) > \Pi_1(v_{1,i}, v_{2,i}),\\
&\Pi_2(v_{1,i+1}, v_{2,i+1}) > \Pi_2(v_{1,i+1}, v_{2,i}),\\
&\qquad\vdots\\
&\Pi_1(v_{1,1}, v_{2,p}) > \Pi_1(v_{1,p}, v_{2,p}),\\
&\Pi_2(v_{1,1}, v_{1,2}) > \Pi_2(v_{1,1}, v_{2,p});
\end{aligned}$$

i.e. the process cycles.

Adding up all these inequalities, we obtain:

$$\sum_{i=1}^{p} \Pi_1(v_{1,i+1}, v_{2,i}) + \sum_{i=1}^{p} \Pi_2(v_{1,i}, v_{2,i}) > \sum_{i=1}^{p} \Pi_1(v_{1,i}, v_{2,i}) + \sum_{i=1}^{p} \Pi_2(v_{1,i+1}, v_{2,i}),$$

where $v_{1,p+1} = v_{1,1}$.

Replacing the profits by their expressions (8) and (9), we have:

$$\sum_{i=1}^{p} \sum_{v_k \in V'} \frac{(\alpha_k - 2c_k(v_{1,i+1}) + c_k(v_{2,i}))^2}{9\beta_k} + \sum_{i=1}^{p} \sum_{v_k \in V'} \frac{(\alpha_k - 2c_k(v_{2,i}) + c_k(v_{1,i}))^2}{9\beta_k} >$$

$$\sum_{i=1}^{p} \sum_{v_k \in V'} \frac{(\alpha_k - 2c_k(v_{1,i}) + c_k(v_{2,i}))^2}{9\beta_k} + \sum_{i=1}^{p} \sum_{v_k \in V'} \frac{(\alpha_k - 2c_k(v_{2,i}) + c_k(v_{1,i+1}))}{9\beta_k}.$$

By permuting the summations over i and $v_k \in V'$ and putting all terms in the RHS, we get:

$$\sum_{v_k \in V'} \frac{1}{9\beta_k} \sum_{i=1}^{p} \{(\alpha_k - 2c_k(v_{1,i+1}) + c_k(v_{2,i}))^2 + (\alpha_k - 2c_k(v_{2,i}) + c_k(v_{1,i}))^2$$

$$- (\alpha_k - 2c_k(v_{1,i}) + c_k(v_{2,i}))^2 - (\alpha_k - 2c_k(v_{2,i}) + c_k(v_{1,i+1}))^2\} > 0 ,$$

or by developing:

$$\sum_{v_k \in V'} \frac{1}{3\beta_k} \left\{ \sum_{i=1}^{p} (c_k(v_{1,i+1}))^2 - \sum_{i=1}^{p} (c_k(v_{1,i}))^2 + 2\alpha_k \left[\sum_{i=1}^{p} c_k(v_{1,i}) - \sum_{i=1}^{p} c_k(v_{1,i+1}) \right] \right\} > 0 .$$

Finally, the fact that $v_{1,p+1} = v_{11}$ implies that all the terms of the RHS cancel and a contradiction occurs.

□

Theorem 4 provides an easy method to find a subgame perfect Nash equilibrium for networks satisfying condition (7). Indeed, it suffices first to compute the profit functions $\Pi_1(v_i, v_j)$ and $\Pi_2(v_i, v_j)$ given by (8) and (9) for each pair v_i, v_j of vertices. Then starting with the location of firm 1 at some vertex say v_{11}, find the vertex, say v_{21}, that maximizes the profit of firm 2. Then, given firm 2's optimal vertex location v_{21}, find firm 1's optimal vertex location v_{12} by maximizing its profit, and so on. From Theorem 4 this process converges in $O(|V|^2)$. Furthermore the computation of $\Pi_i(v_k, v_\ell)$ can be performed in $O(|V|)$ for each pair v_k, v_ℓ of vertices. Hence this simple algorithm is completed $O(|V|^3)$ time.

When condition (7) is not satisfied, we cannot guarantee the existence of a pair of Nash equilibrium in location, and furthermore in case of existence, all such pairs may consist of interior points of edges. The following two examples illustrate such situations.

Example **1.** Consider the network of Figure 1, where $\alpha_k = 12$ and $\beta_k = 1$ for $k = 2, 4$ and 6 and $\alpha_k = \beta_k = 0$ for $k = 1, 3$ and 5. Furthermore assume that $T(d(x, v_k)) = d(x, v_k)$ for all $x \in N$ and $v_k \in V$ and $C(x) = 12\ell(v_i, x)/\ell(v_i, v_j)$ for $x \in (v_i, v_j)$, $i = 1, 3, 5$ and $j = 2, 4, 6$.

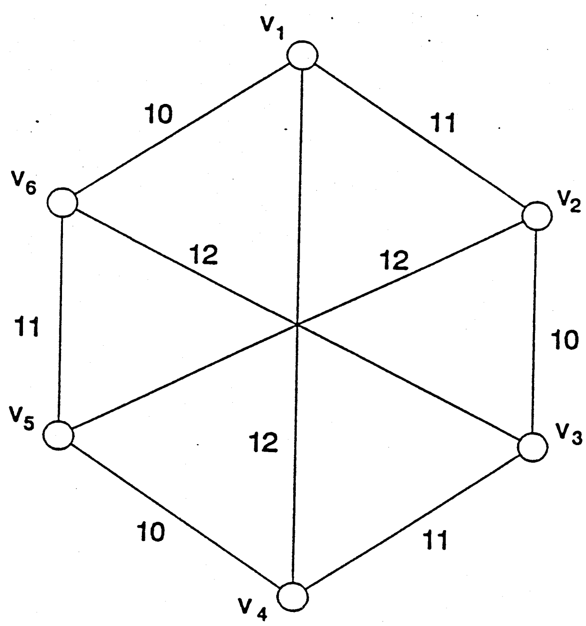

Figure 1. *A network with no equilibrium locations.*

First note that a firm will never locate at a point other than v_1, v_3 or v_5 because the total cost $c_k(x) = d(x, v_k) + C(x)$ of all markets is always smaller at one of these vertices than at any of the remaining ones. Indeed, let $x \in (v_i, v_j)$ and $x \neq v_i$ with $i = 1, 3, 5$ and $j = 2, 4, 6$.

Then for $v_k \neq v_j$, we have:

$$\begin{aligned} c_k(x) &= d(x,v_k)+12\ell(v_i,x)/\ell(v_i,v_j) \\ &> d(v_i,v_k) \\ &= c_k(v_i); \end{aligned}$$

and for $v_k = v_j$,

$$\begin{aligned} c_j(x) &= \ell(x,v_j)+12\ell(v_i,x)/\ell(v_i,v_j) \\ &\geq \ell(v_i,v_j), \text{ because } \ell(v_i,v_j)\leq 12 \\ &= c_j(v_i). \end{aligned}$$

We can thus restrict the set of candidate locations to $\{v_1,v_3,v_5\}$. Now, using formulas (4) and (5) we have:

$$\Pi_1(v_1,v_5)=\Pi_2(v_1,v_3)=\Pi_1(v_5,v_3)=1.25$$
$$\Pi_2(v_1,v_5)=\Pi_1(v_1,v_3)=\Pi_2(v_5,v_3)=1,$$
$$\Pi_1(v_k,v_\ell)=\Pi_2(v_\ell,v_k),\ k=1,3,5,\ \ell=1,3,5 \text{ and}$$
$$\Pi_1(v_k,v_k)=\Pi_2(v_k,v_k)=5/9,\ k=1,3,5.$$

Hence, there is no equilibrium in location.

***Example* 2.** Consider the network of Figure 2 where $\alpha_1=\alpha_2=21$, $\alpha_3=\alpha_4=23$ and $\beta_k=1$, $k=1,2,3,4$. Moreover, let $T(d(x,v_k))=d(x,v_k)$ and $C(x)=0$ for all $x\in N$ and $v_k\in V$. Hence $c_k(x)=d(x,v_k)$.

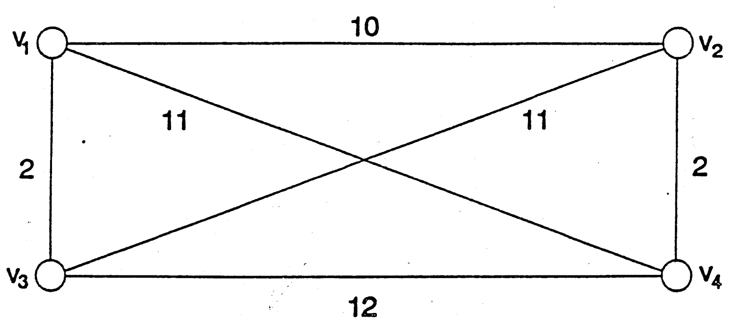

Figure 2. *A network with a pair of equilibrium locations which consists of interior points*

Throughout this example, we denote by (v_i, v_j, y) the point $x \in (v_i, v_j)$ such that $\ell((v_i, x) = y$.

First looking at Table 1, which contains the values of the profits when firms locate at v_1, v_2, v_3, v_4, $(v_1, v_3, 1)$ or $(v_2, v_4, 1)$, one remarks that no pair of points at most one of which is a non-vertex point can be a Nash equilibrium. Now, we show that the pair $(v_1, v_3, 1)$, $(v_2, v_4, 1)$ constitute such an equilibrium. Assume that firm 1 locates at $(v_1, v_3, 1)$. If firm 2's location x_2 is restricted to some edge (v_i, v_j), then from (5) and (6):

$$\Pi_2((v_1, v_3, 1), x_2) = \sum_{k=1}^{4} \Pi_{2k}((v_1, v_3, 1), x_2) \text{ with}$$

$$\Pi_{2k}((v_1, v_3, 1), x_2) = \begin{cases} \dfrac{(\alpha_k - d(x_2, v_k))^2}{4}, \\ \text{if } d(x_2, v_k) \leq \min\{\alpha_k, 2d((v_1, v_3, 1), v_k) - \alpha_k\} \\ \\ \dfrac{(\alpha_k - 2d(x_2, v_k) + d((v_1, v_3, 1), v_k))^2}{9}, \\ \text{if } \min\{\alpha_k, 2d((v_1, v_3, 1), v_k) - \alpha_k\} \leq d(x_2, v_k) \leq \\ \min\{\alpha_k, (\alpha_k + d((v_1, v_3, 1), v_k))/2\} \\ \\ 0, \text{ otherwise.} \end{cases}$$

Each such $\Pi_{2k}((v_1, v_3, 1), x_2))$ is thus along (v_i, v_j) a piecewise quadratic function in the distance with *breakpoints* (i.e. points where the function shape changes) defined by the following conditions.

$$\begin{aligned} &d(x_2, v_k) = \alpha_k, \text{ or} \\ &d(x_2, v_k) = 2d((v_1, v_3, 1), v_k) - \alpha_k, \text{ or} \\ &d(x_2, v_k) = (\alpha_k + d((v_1, v_3, 1), v_k))/2. \end{aligned} \tag{10}$$

Furthermore, because $d(x_2, v_k)$ is concave along (v_i, v_j), each $T_{2k}((v_1, v_3, 1), x_2)$ is convex along any subedge limited by two consecutive breakpoints. Hence, in order to maximize $\Pi_2((v_1, v_3, 1), x_2)$ we can restrict x_2 to be in the set of vertices or breakpoints (i.e. points satisfying (10)).

x_1 \ x_2	v_1	v_2	v_3	v_4	$(v_1, v_3, 1)$	$(v_2, v_4, 1)$
v_1	$\begin{bmatrix}127.4\\127.4\end{bmatrix}$	$\begin{bmatrix}207.9\\207.9\end{bmatrix}$	$\begin{bmatrix}137.7\\121.7\end{bmatrix}$	$\begin{bmatrix}207.9\\209.6\end{bmatrix}$	$\begin{bmatrix}133.0\\119.0\end{bmatrix}$	$\begin{bmatrix}217.9\\207.3\end{bmatrix}$
v_2	$\begin{bmatrix}207.9\\207.9\end{bmatrix}$	$\begin{bmatrix}127.4\\127.4\end{bmatrix}$	$\begin{bmatrix}207.9\\209.6\end{bmatrix}$	$\begin{bmatrix}133.7\\121.7\end{bmatrix}$	$\begin{bmatrix}217.9\\207.3\end{bmatrix}$	$\begin{bmatrix}133.0\\119.0\end{bmatrix}$
v_3	$\begin{bmatrix}121.7\\133.7\end{bmatrix}$	$\begin{bmatrix}209.6\\207.9\end{bmatrix}$	$\begin{bmatrix}123.4\\123.4\end{bmatrix}$	$\begin{bmatrix}209.6\\209.6\end{bmatrix}$	$\begin{bmatrix}124.6\\122.6\end{bmatrix}$	$\begin{bmatrix}219.4\\221.1\end{bmatrix}$
v_4	$\begin{bmatrix}209.6\\207.9\end{bmatrix}$	$\begin{bmatrix}121.7\\133.7\end{bmatrix}$	$\begin{bmatrix}209.6\\209.6\end{bmatrix}$	$\begin{bmatrix}123.4\\123.4\end{bmatrix}$	$\begin{bmatrix}219.4\\221.1\end{bmatrix}$	$\begin{bmatrix}124.6\\122.6\end{bmatrix}$
$(v_1, v_3, 1)$	$\begin{bmatrix}119.0\\133.0\end{bmatrix}$	$\begin{bmatrix}207.3\\217.9\end{bmatrix}$	$\begin{bmatrix}122.6\\124.6\end{bmatrix}$	$\begin{bmatrix}221.1\\219.4\end{bmatrix}$	$\begin{bmatrix}122.8\\122.8\end{bmatrix}$	$\begin{bmatrix}221.0\\221.0\end{bmatrix}$
$(v_2, v_4, 1)$	$\begin{bmatrix}207.3\\217.9\end{bmatrix}$	$\begin{bmatrix}119.0\\133.0\end{bmatrix}$	$\begin{bmatrix}221.1\\219.4\end{bmatrix}$	$\begin{bmatrix}122.6\\124.6\end{bmatrix}$	$\begin{bmatrix}221.0\\221.0\end{bmatrix}$	$\begin{bmatrix}122.8\\122.8\end{bmatrix}$

Table 1: $\begin{bmatrix}\Pi_1(x_1, x_2)\\\Pi_2(x_1, x_2)\end{bmatrix}$ *for* x_1 *and* $x_2 \in \{v_1, v_2, v_3, v_4, (v_1, v_3, 1), (v_2, v_4, 1)\}$.

Moreover the row in Table 1 which corresponds to $x_1 = (v_1, v_3, 1)$ indicates that no vertex provides a better profit to firm 2 than $(v_2, v_4, 1)$ does. It remains to check that this is also the case for the breakpoints. The necessary information is presented in Tables 2 and 3.

	v_1	v_2	v_3	v_4
α_k	21	21	23	23
$2d((v_1,v_3,1),v_k)-\alpha_k$	−19	1	−21	1
$(\alpha_k+d((v_1,v_3,1),v_k))/2$	11	6	12	17.5

Table 2. *Values of* $d(x,v_k)$ for x to be a breakpoint (cf. (11)).

	$d(x_2,v_1)$	$d(x_2,v_2)$	$d(x_2,v_3)$	$d(x_2,v_4)$	$\Pi_2((v_1,v_3,1),x_2)$
$(v_1,v_2,9)$	9	1	11	3	195.7
$(v_1,v_4,10)$	10	3	12	1	196.6
$(v_2,v_3,1)$	11	1	10	3	195.2
$(v_2,v_3,2)$	11	2	9	4	172.1
$(v_2,v_4,1)$	11	1	13	1	221.0
$(v_3,v_4,9)$	11	5	9	3	151.2
$(v_3,v_4,11)$	12	3	11	1	196.6

Table 3. *Distances to the markets and profit when firm* 2 *locates at a breakpoint.*

5. Concluding Remarks

We have studied the competitive process of two firms which first locate and then decide the quantities they offer on the markets, where the unit price is a linear decreasing function of the total quantity offered. Under the assumption that the transportation costs plus the production costs are never "too large" (so that it is always profitable to offer a positive quantity at each market) we have proved the existence of a subgame perfect Nash equilibrium and shown that one can be find be looking at the vertices of the network. When the above assumption is relaxed, we provide examples showing that these properties do not hold anymore. Though these results concern the case where firms are allowed to locate anywhere on the network N, they also hold when firms have only a finite set of possible locations. This can be easily checked by looking at the proof of Theorem 4 and Example 1.

It appears interesting to expand to the cases where one might allow more than two firms to compete and/or firms to open several facilities. Finally, throughout this paper, no special assumption has been made about the network. Then, a natural question is: can we obtain stronger results when the network is a tree. We conjecture that the existence of a subgame perfect Nash equilibrium is then always guaranteed.

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