A Note on a Stochastic Location Problem

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Abstract

In this note we give a short and easy proof of the equivalence of Hakimi's one-median problem and the K-server-single-facility-loss median problem as discussed by Chiu and Larson (cf. [1]). The proof makes only use of a stochastic monotonicity result for birth and death processes and the insensitivity of the $M|G|K$ loss model.

1. Model formulation and results

Consider the K-server-single-facility-loss median model ($K-SFLM$) as discussed by Chiu and Larson (cf. [1],[2]) and define

$\underline{n}_t^{(K)}(x) :=$ number of customers being served at time t by one of the servers if the facility is located at $x \in F$

where the set $F \subseteq \mathbb{R}^2$ denotes some feasible region. As argued in [1] the queueing process underlying the $K-SFLM$ location problem is a $M|G|K$ loss-model. For this queueing process it is well-known (cf. [3]) that for fixed $x \in F$ $\underline{n}_t^{(K)}(x)$ converges in distribution to some random variable $\underline{n}_\infty^{(K)}(x)$ representing the number of customers being served in steady state if the location is $x \in F$. Moreover, the distribution of $\underline{n}_\infty^{(K)}(x)$ only depends on the first moment $m_1(x)$ of the service time (Erlang's Loss Formula). If the facility is located at $x \in F$ the costfunction $Z(x)$ takes the following form (cf. [1])

$$Z(x) = P_K(x)Q + (1 - P_K(x))m_1(x) \tag{1}$$

with

$P_K(x) :=$ $P\{$customer "arriving" in steady state at location x is lost$\}$

$m_1(x) :=$ expected service time of arbitrary customer whenever facility is located at x

and

$Q :=$ cost per lost customer, $Q \geq 0$.

The main result proved in [1] using lengthy calculations states that the costfunction $Z(x)$ is increasing in $m_1(x)$. This implies that the $K-SFLM$ location problem is solved by determining that location $x \in F$ which minimizes the expected service time $m_1(x)$. Hence, in the special case where F denotes some network N this boils down to finding the so-called Hakimi median (cf. [5]) at one of the nodes of N.

The above result can be verified easily without any calculations by using a well-known stochastic monotonicity result for birth and death processes. Before proving this we need the following observations. By Little's formula $(1-P_K(x)m_1(x)$ equals $\frac{1}{\lambda}\ \mathcal{E}(\underline{n}_{\infty}^{(K)}(x))$ with λ the arrival rate of the Poisson process. Moreover, by the PASTA property (Poisson Arrivals See Time Averages)

$$P_K(x)=P\{\underline{n}_{\infty}^{(K)}(x)=K\}.$$

Hence by (1)

$$Z(x)=QP\{\underline{n}_{\infty}^{(K)}(x)=K\}+\frac{1}{\lambda}\ \mathcal{E}(\underline{n}_{\infty}^{(K)}(x)) \tag{2}$$

Lemma 1.
Let $x,x^{*}\in F$. If $m_1(x)\leq m_1(x^{*})$ we obtain $Z(x)\leq Z(x^{*})$.

Proof.
In order to prove $Z(x)$ is increasing in $m_1(x)$ it is sufficient to verify that $\mathcal{E}(\underline{n}_{\infty}^{(K)}(x))\leq\mathcal{E}(\underline{n}_{\infty}^{(K)}(x^{*}))$ and $P\{\underline{n}_{\infty}^{(K)}(x)=K\}\leq P\{\underline{n}_{\infty}^{(K)}(x^{*})=K\}$ whenever $m_1(x)\leq m_1(x^{*})$.
By the sensitivity of the $M|G|K$ loss system with respect to only the first moment $m_1(x)$ this breaks down to prove the above inequalities for the Markovian loss models $\Sigma_1=M|M_1|K$ and $\Sigma_2=M|M_2|K$, i.e. loss models with the same Poisson arrival process and negative exponential service times with parameters $\mu_1(x)=(m_1(x))^{-1}$ for Σ_1 and $\mu_1(x^{*})=(m_1(x^{*}))^{-1}$ for Σ_2. Denote by $\underline{n}_t^{(K)}(\Sigma_i)$ the number of customers in system Σ_i, $i=1,2$, at time t, $0<t<\infty$ and by $\underline{n}_{\infty}^{(K)}(\Sigma_i)$, $i=1,2$ the corresponding limiting random variables.
Clearly these stochastic processes are birth and death processes on the finite state space $\{0,1,\dots,K\}$ with transition rates $q_{ii+1}=\lambda$, $i=0,\dots,K-1$, $q_{ii-1}=i\mu_1(x)$, $i=1,\dots,K$ for Σ_1 and $q_{ii+1}=\lambda$, $i=0,\dots,K-1$, $q_{ii-1}=i\mu_1(x^{*})$, $i=1,\dots,K$ for Σ_2.
Since $\mu_1(x)\geq\mu_1(x^{*})$ we obtain by a well-known stochastic monotonicity result for birth and death processes (cf. [4,prop.4.2.10]) that for every $t>0$

$$\underline{n}_t^{(K)}(\Sigma_1)\underset{d}{\leq}\underline{n}_t^{(K)}(\Sigma_2)$$

This implies by a limit procedure $\mathcal{E}\underline{n}_\infty^{(K)}(\Sigma_1) \leq \mathcal{E}\underline{n}_\infty^{(K)}(\Sigma_2)$ and $P\{\underline{n}_\infty^{(K)}(\Sigma_1) = K\} \leq P\{\underline{n}_\infty^{(K)}(\Sigma_2) = K\}$. Hence the desired result is proved.

□

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