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OPTION PRICING AND STOCHASTIC PROCESSES

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by

Ton Vorst

Abstract

In this paper I will try to describe how the theory of stochastic processes and especially of stochastic differential equations has influenced option pricing theory. In my view, this is one of the best examples of the application of sophisticated mathematics to a purely economic, or financial, problem. This is not only because of the fact that the theory describes the economic phenomena very well, but merely since the main results are used in every day practice by market makers. I will discuss the pricing of options on stocks and bonds and mention some other examples.

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1. Introduction

Stochastic processes and especially the theory of stochastic differential equations have played a fundamental rôle in the theory of option pricing. In general an option gives the holder or owner the right to buy or sell something at his discretion before a certain prescribed date at a price, which has been specified in advance. Black and Scholes (1973) were the first who derived the equilibrium price of an option which gives the owner the right to buy a stock before a certain date at a fixed price. They used a stochastic process to model the price of the stock. It was Merton (1973), who, in his fundamental paper on the theory of rational option pricing, made their arguments mathematically more rigorous and made a more extensive use of the stochastic theory. Although they all used the theory of stochastic processes, they didn't use the theory in its full strength at that time. The theory was used to derive a partial differential equation to describe the option price as a function of the stock price and the time that remained till the date on which the option would expire.

However to solve the partial differential equation it was transformed into the heat equation of which the solution has been known in physics for a long time. At that time, it seemed that the option price formula could only be derived using physics. But this wasn't true. The theory of stochastic differential equations was developed far enough to give a direct solution of the partial differential equation in terms of the expected vlaue of a certain stochastic variable, without any reference to physics or what so ever. The so called Kolmogorov-backward equation gives this direct solution. Nowadays the result of Kolmogorov is used as a standard method to describe solutions of partial differential equations that appear in financial theory (see e.g. Cox, Ingersoll and Ross (1985)).

The formula of Black and Scholes is not only widely accepted by academic theorists, but also used as a black-box by market participants to calculate prices for options they want to sell or buy. By black-box we mean that the derivation of the Black-Scholes formula is far to complicated for most market participants to understand in all its details. Hence they only use the formula or even a software program which requires some input variables and immediately gives the Black-Scholes price.

Sometimes the Black-Scholes black-box has been used in cases where this is not appropriate. Fortunately, the Black-Scholes formula cannot be used for all kinds of options. Otherwise the subject would have lost academic interest. However, in all cases one has to rely heavily on the theory of stochastic processes.

Stochastic differential equations are not only used to price options but are also applied in a more general way to describe optimal consumption and investment decisions in a continuous time setting (see e.g. Merton (1989)). However, it is remarkable that a theory which has been so succesfully applied in parts of economics and finance has not been used on a larger scale in other disciplines in the economic sciences. This is particularly striking since other mathematical disciplines as catastrophe theory and chaos have attracted a lot more attention, while they have never proved to be of any use in describing any economic phenomena. For chaos there still might be some hope, because it has only recently been introduced in economic theory. Catastrophe theory, which was very much promoted in the seventies, didn't bring any new insights. May be it are the names which attract the theorists and in that respect the theory of stochastic differential equations doesn't seem to have a big appeal.

The aim of this paper is not to give a thorough introduction into the theory of stochastic processes and stochastic differential equations, because this would require a book of its own. Furthermore the books of Arnold (1971) and Øksendal (1985) are excellent introductory texts. We will also not give an extensive survey of the general theory of option pricing since the subject is to detailed to be described in a text for a general audience. The main goal of this paper will be to give a flavor of the results and theorems from the theory of stochastic differential equations that are applied in option pricing theory and how they influenced this theory. We will not prove any of these theorems but only give references for their proofs. We hope to trigger the interest of the reader to this mathematical theory, that is so beautifully applied to a purely economoic problem.

This paper is organized as follows. In the next section we will derive the

Black–Scholes formula which can be seen as the most pure application of stochastic differential equations in option pricing theory. Further we will stick our attention in this section to the pricing of options on stocks. In section 3 we will discuss the pricing of options on bonds. This part of the theory is far less developed than the preceeding part, due to the fact that bonds mature after some finite time. In section 4 some other generalizations of options will be described, while the last section concludes the paper.

2. Basic option pricing

In this section we derive the Black–Scholes formula for the price of a call option on a stock. A call option is characterized by the stock on which it is written, its maturity date T and its exercise price K. The holder of the call option has the right to buy one share of stock before time T at the fixed price K, the so called exercise price, from the issuer (or in option terms writer) of the option. Hence at the maturity date T, the value of the option is equal to

$$\text{(2.1)} \qquad \max\,(S(T)-K,0),$$

where S(t) is the price of the stock at time t. It is clear that at time t=0, the prices S(t), t>0, are unknown. It is common practice to assume that the uncertain future prices S(t) can be described by a stochastic process. Particulary it is assumed that the process S(t) is the solution of the following stochastic differential equation.

$$\text{(2.2)} \qquad dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),$$

where α and σ are positive constants and W(t) is a Wiener process or Brownian motion. As said before we will not go into the details of stochastic differential equations, but only give some idea of the general theory. For a good introduction one may study Arnold (1974), Øksendal (1985) or Malliaris and Brock (1982). The last one has been written with applications from economics and finance in mind.

Here we only try to give a flavor of the most important ideas of the theory.

A Wiener process is a stochastic process which has the following properties.

a. W(t) is a stochastic variable with values in $\mathbb{R}$ for every $t \geq 0$.

b. For all $0<t_1<t_2<...<t_n$, the variable $(W(t_1),...,W(t_n))^T \in \mathbb{R}^n$ is multivariate normally distributed with mean $0 \in \mathbb{R}^n$ and covariance matrix.

$$\begin{bmatrix} t_1 & t_1 & \dots & t_1 & t_1 \\ t_1 & t_2 & \dots & t_2 & t_2 \\ & & & & \\ t_1 & t_2 & \dots & t_{n-1} & t_{n-1} \\ t_1 & t_2 & \dots & t_{n-1} & t_n \end{bmatrix} \tag{2.3}$$

c. Almost every realisation, i.e. time path, $W:[0,\infty] \rightarrow \mathbb{R}$ of the stochastic process is continuous and nowhere differentiable.

It follows from (2.3) that Var(W(t))=t.

Now consider a more general stochastic process X(t), i.e. a process with properties a and c and a continuous function $s:\mathbb{R}^2 \rightarrow \mathbb{R}$. Itô (1944) has defined a stochastic integral

$$\int_0^t s(\tau,X(\tau)dW(\tau) \tag{2.4}$$

which is a stochastic generalization of the standard Riemann integral

$$\int_0^t s(\tau,X(\tau))d\tau \tag{2.5}$$

Having defined a stochastic integral we can talk about stochastic differential equations as follows. For given continuous functions $f,\sigma:\mathbb{R}^2 \rightarrow \mathbb{R}$ we say that the stochastic process X(t) is a solution of the stochastic differential equation.

$$dX(t) = f(t,X(t))dt + \sigma(t,X(t)dW(t) \tag{2.6}$$

if and only if

(2.7) $$X(t) = X(0) + \int_0^t f(\tau,X(\tau))d\tau + \int_0^t \sigma(\tau,X(\tau))dW(\tau)$$

Under some general conditions on f and σ one can show that a solution X(t) of the stochastic differential equation (2.6) always exists and that the solution X(t) has the following properties

(i) X(t) has the Markov-property, i.e. for $t_1<t_2< \ldots <t_n<s$ and $A \subseteq \mathbb{R}$,

(2.8) $$Pr(X(s) \in A|X(t_1),...,X(t_n) = Pr(X(s) \in A|X(t_n))$$

This means that the future of X only depends on the current state and not on the past. Hence it makes sence to define the transition probability.

(2.9) $$P(t,x;s,A) = Pr[X(s) \in A|X(t) = x] \text{ for } t<s.$$

i.e. P(t,x;s,A) gives the conditional probability that $X(s) \in A$ given that $X(t)=x$.

(ii) For every $x \in \mathbb{R}$ and $\varepsilon > 0$

(2.10) $$\lim_{s\downarrow t} \frac{1}{s-t} \int_{|y-x|>\varepsilon} P(t,x;s,dy) = 0;$$

(2.11) $$\lim_{s\downarrow t} \frac{1}{s-t} \int_{|y-x|\leq\varepsilon} (y-x)P(t,x;s,dy) = f(t,x);$$

(2.12) $$\lim_{s\downarrow t} \frac{1}{s-t} \int_{|y-x|\leq\varepsilon} (y-x)^2P(t,x;s,dy) = \sigma^2(t,x).$$

Equation (2.10) states that the probability of large changes in an infinitesimal time interval is zero, while (2.11) implies that the expected change in X over an infinitesimall time interval is given by f(t,x) times the length of the interval and (2.12) gives the second moment of the change over an infinitesimal interval. A process with properties (i), (ii) and (iii) is called a diffusion proces. f(t,x) is called the drift of the process and $\sigma^2(t,x)$ the diffusion coefficient.

One of the main results in the theory of stochastic differential equations is Itô's lemma. Let X(t) be a diffusion process which is a solution of the stochastic differential equation (26).

Itô's lemma. Let $u(t,x)\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function with continuous first and second order partial derivatives u_t, u_x, u_{xx}.
Define the stochastic proces $Y(t) = u(t,X(t))$. Then

$$dY(t) = u_t(t,X(t))dt + u_x(t,X(t))dX(t) + \tfrac{1}{2}u_{xx}(t,X(t))\sigma^2(t,X(t))dt \tag{2.13}$$

We will illustrate Itô's lemma by an example whith $X(t)=W(t)$.
W(t) evidentily is a solution of (2.6) if we take f=0 and σ=1.
Define

$$Y(t) = u(t,W(t)) = Y(0)\exp\{(\alpha-\tfrac{1}{2}\sigma^2)t + \sigma W(t)\}, \tag{2.14}$$

where α,σ and Y(0) are constants.
Then it follows from Itô's lemma that

$$\begin{aligned} dY(t) &= u_t(t,W(t))dt + u_w(t,W(t))dW(t) + \tfrac{1}{2}u_{ww}(t,W(t))dt = \\ &= (\alpha-\tfrac{1}{2}\sigma^2)Y(t)dt + \sigma Y(t)dW(t) + \tfrac{1}{2}\sigma^2 Y(t)dt = \\ &= \alpha Y(t)dt + \sigma Y(t)dW(t) \end{aligned} \tag{2.15}$$

Hence Y(t) is the solution of the stochastic differential equation (2.2) and we can take

$$S(t) = S(0)\exp((\alpha-\tfrac{1}{2}\sigma^2)t + \sigma W(t)) \tag{2.16}$$

Combining this with the properties of the Brownian motion, we see that S(t) is lognormally distributed for each t and

$$E(\log S(t)/S(0)) = (\alpha-\tfrac{1}{2}\sigma^2)t \tag{2.17}$$

$$\mathrm{Var}(\log S(t)/S(0)) = \sigma^2\mathrm{Var}\ W(t)) = \sigma^2 t. \tag{2.18}$$

σ is called the volatility of the stock.
Besides stocks we will also use riskless bonds in the derivation of the option

pricing formula. We will assume that the riskless interest rate ν is constant over time, at least till the maturity date T of the option. Hence the increase in value of a riskless investment can be described by the differential equation.

(2.19) $$dB = rBdt,$$

where $r = 1 + \nu$.

In fact (2.19) can also be seen as a stochastic differential equation with σ=0.

In what follows we will assume that the stock doesn't pay dividend before the maturity date of the option and ignore transaction cost and taxes. Furthermore we will assume that all assets are perfectly divisible and that the markets are frictionless. Since all other parameters, T, K, r, α, and σ are constant the only variables on which the option price depends are the stock price S and the actual time t. Let C(t,S) denote the option price. From Itô's lemma it follows that:

(2.20) $$dC = C_t dt + C_s dS + \frac{1}{2}C_{ss}\sigma^2S^2dt = (C_t + C_s\alpha S + \frac{1}{2}C_{ss}\sigma^2S^2)dt + \sigma SC_s dW,$$

where we have suppressed the variables on which the functions C, S and W depend and C_t, C_s and C_{ss} are again first and second order partial derivatives.

Consider at time t the portfolio P consisting of C_s shares of stock and borrowing the amount (SC_s–C) against the riskless rate r. The initial value of this portfolio is $C_s.S-(SC_s-C)=C$ exactly the same as the option price, while the portfolio value follows the stochastic differential equation

(2.21) $$dP = C_s dS - (SC_s-C)rdt = [\alpha SC_s + r(C-SC_s)]dt + \sigma SdWC_s$$

We see that the portfolio bears the same instantaneous risk as the option i.e.

(2.22) $$dP - dC = [r(C-SC_s) - C_t - \frac{1}{2}\sigma^2S^2C_{ss}]dt$$

doesn't have a stochastic term in its specification.

Hence the combination of P and C is riskless and requires no initial investment. But then it should give a zero expected profit, because otherwise there would be riskless arbitrage opportunities. This means that the righthandside of (2.22) must be zero:

(2.23) $C_t + \frac{1}{2}\sigma^2 S^2 C_{ss} + r(SC_s - C) = 0$

This is the fundamental partial differential equation that every option obeys.
Remark that the portfolio P has to be adjusted continuously in order that it bears the same risk as the option. This is the reason that it is often called a self-financing dynamic duplicating portfolio.
We will now assume that the option can only be exercised, i.e. changed against the stock for the exercise price K, at the maturity date T. This kind of option is called a European option. In practice options can be exercised also before the maturity date. These options are called American. The value of the European call option is determined by the solution of (2.23) which satisfies the boundary condition.

(2.24) $C(T,S) = \max(S(T)-K,0))$

To solve the partial differential equation we first transform it.
Define

(2.25) $F(t,S) = e^{r(T-t)}C(t,S)$

Then (2.23) and (2.24) can be rewritten for F as follows:

(2.26) $F_t + \frac{1}{2}C_{ss}\sigma^2 S^2 + rSC_s = 0$

(2.27) $F(T,S) = \max\ (S(T)-K,0)$

Now we can use an important result from the theory of stochastic processes (Arnold [1974], pp.41–43). Let P(t,x;s,A) be the transition probability of a

diffusion process X(t) with continuous drift f(t,x) and continuous diffusion coëfficient $\sigma^2(t,x)$. Furthermore if g(x) denotes a continuous function such that

(2.28) $u(t,x) = E_{t,x}g(X_s) = \int g(y)P(t,x;s,dy)$

for t<s, and s fixed, is continuous, as are its partial derivatives u_x and u_{xx} then u satisfies Kolmogorov's backward equation:

$$u_t + fu_x + \frac{1}{2}\sigma^2 u_{xx} = 0 \tag{2.29}$$

with boundary condition

$$\lim_{t \uparrow s} u(t,x) = g(x) \tag{2.30}$$

$E_{t,x}$ is the conditional expectation given that X(t)=x. In our case we consider the diffusion process given through the stochastic differential equation

$$dS = rSdt + \sigma SdW, \tag{2.31}$$

where we again suppressed the variable t as the argument of the functions. We take s=T and $g(S_s) = \max(S(T)-K,0)$.

Now Kolmogorov's backward equation states that if

$$u(t,S) = E_{t,s}g(S(T)) \tag{2.32}$$

is continuous, as are its partial derivatives u_s and u_{ss}, with respect to S, then u satisfies

$$u_t + rSu_s + \frac{1}{2}\sigma^2 S^2 u_{ss} = 0 \tag{2.33}$$

which is exactly the partial differential equation (2.26). To determine (2.32) we remark that it follows from the example by which we illustrated Itô's lemma that log(S(T)/S(t)) is normally distributed with mean $(r-\frac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t)$. Using this, it is an elementary exercise in probability theory to show that

$$u(t,S) = E_{t,S}g(S(T)) = \int_{S(T)\geq K} (S(T)-K)f(S(T))dt = \tag{2.34}$$

$$Se^{r(T-t)}N\left[\frac{\ln(S/K) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right] - KN\left[\frac{\ln(S/K) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right]$$

where f is the density function of the lognormal distribution described in the previous sentence and N is the standard cumulative normal distribution function. It requires some tedious calculations to show that u_s and u_{ss} exist and are continuous. Hence we know that u is the solution of (2.26) and (2.27). Using (2.25) we deduce that

$$(2.35) \qquad C(S,t) = SN[d_1] - Ke^{-r(T-t)}N[d_1 - \sigma\sqrt{T-t}]$$

with

$$(2.36) \qquad d_1 = \{\ln(S/K) + (r+\tfrac{1}{2}\sigma^2)(T-t)\}/\sigma\sqrt{T-t}$$

This formula is the Black-Scholes formula for the price of a European call option. We like to make a few remarks about this formula. Combining (2.25) and (2.34) it follows that the option price is simply the discounted expected value of max(S(T)-K,0). However S(T) no longer represents the price of the stock, but it is the solution of the stochastic process (2.31) which can be seen as describing the price of a stock which has the same variance as the underlying stock but earns only the riskless rate r instead of the rate α of the underlying stock.

Besides call options there also exist put options. The owner of a put option has the right to sell a stock before time T at a fixed exercise price K. A put option is not simply the negative of a call option because a call option will only be exercised if the stock price exceeds K and a put option only if S is below K. To price a European put option P, i.e. one that can only be exercised at the maturity date T, we derive the partial differential equation for the put price P.

$$(2.37) \qquad P_t + \tfrac{1}{2}\sigma^2 S^2 P_{ss} + r(SP_s - P) = 0 \qquad\qquad 0 \le t \le T,\ S \ge 0$$

This is the same equation as (2.23), because up to (2.23) we never used the fact that we were pricing a call option. We only have another boundary condition

$$(2.38) \qquad P(S,T) = \max(K - S(T), 0).$$

One can use again Kolmogorov's backward equation to find the value of the put option. However, there is an easier way around, which uses the fact that we already know the value of the call option. Consider two investors A and B. A buys a stock and a put option on it, while B buys a call option on that stock with the same exercise price K and maturity date T as the put option and invests $Ke^{-r(T-t)}$ in riskless bonds. It is an easy exercise to show that the portfolios of both investors have, independent of the final value of S, the same value at the maturity date T. Hence also the initial investments for both portfolios must be equal i.e.

$$P + S = C + Ke^{-r(T-t)} \tag{2.39}$$

Hence the price P of the put option can be derived from the call value C, the stock price S and the riskless interest rate r. Equation (2.39) is known as the put-call parity.

Up to now we have only considered European options, i.e. options that can only be exercised at the maturity date. In reality, however, options are never European but always American and can always be exercised before the maturity date. Hence the holder of the option has the extra right to decide when he wants to exercise his option. This will also mean that American options will always be more expensive than ceteris paribus, their European counterparts, since these extra rights will have their price. Only if it is certain that it is never optimal to exercise early the prices of both options will be the same. The holder is confronted with an optimal stopping problem, where by stopping in this case is meant that he exerises the option. General references for optimal stopping theory are Shiryayev (1978) and Øksendahl (1985), while Van Moerbeke (1973) considers problems which are very similar to the ones we are confronted with here. It follows from the results of Van Moerbeke that if the underlying stock pays no dividend before the maturity date of the option, it is never optimal to exercise a call option early. There is also an easier way to see this. Assume that the holder of the option at some moment thinks that it is optimal to exercise the option. Instead of excercising the option at that very moment he can decide to exercise the option at the maturity date, whatever might happen. In both cases he will have the stock at the maturity date. Only in the second case he has to pay the exercise price K later than in the first case. Since we assume that interest rates are positive, it is clear that the second strategy has to be prefered. Hence instead of exercising, it is better to decide to exercise at the maturity date and to hold the option. Furthermore by not revealing one's decision to the issuer of the option one still has the possibility not to exercise at the maturity date. To put some more perspective on this argument, let us look at the case of a put option. In case one exercises a put option early, one receives the exercise price K before the maturity date instead of at the maturity date. Depending on the interest rate this might by very profitable since on can reinvest the amount K until the maturity date T and end up with a sum which is more than K. However by early exercising one gives up ones right to postpone the decision and to see what happens with the stock price. Hence for a put option it is not immediately clear when it is optimal to exercise. The put price P still satisfies the partial differential equation (2.37) and the boundary condition (2.38), but furthermore it follows from optimal stopping theory that for every t there exists a level $S^*(t)$ below which it is optimal to exercise the put option and above which it is not optimal. This level $S^*(t)$ is given by the following conditions:

(2.40) $$P(S^*(t),\ t)) = K - S^*(t)$$

(2.41) $$P_s(S^*(t),\ t)) = -1$$

Condition (2.40) states that when it is just optimal to exercise, the value of the put option is exactly the cashflow which results from exercising. Hence for $S(t) \leq S^*(t)$ the value of the option is given by $K-S(t)$ and for $S(t) \geq S^*(t)$ the value is given by $P(S(t),t)$. Condition (2.41) states that the option value is differentiable with respect to S in the critical point $S^*(t)$.
Conditions (2.39) and (2.40) in fact replace the boundary condition $P(0,t)=K$ for the European put option which we didn't write down explicitly. The partial differential equation (2.36) no longer holds for all $S(t) \geq 0$, but only for $S \geq S^*(t)$, where $S^*(t)$ has to be determined by simultaneously solving (2.37), (2.38), (2.40) and (2.41). This is called a free boundary problem since the partial differential equation doesn't have to be solved on a fixed domain but on a domain of which the boundary has to be determined at the same time. There isn't an analytic solution for this problem. To solve the system of equations (2.37), (2.38), (2.40) and (2.41) one has to use numerical procedures.
If there are dividends before the maturity of the option it might also for the holder of a call option be optimal to exercise early, since the dividends accrue to the stockholder and not to the holder of the call option. However, using the same argument as before one can show that it is only optimal for a

call holder to exercise just before a dividend date. For the holder of a put option dividends might be an incentive not to exercise the option. In the literature there are several methods to value an option on a stock which pays dividends before the maturity date. The compound option method as proposed by Geske (1979) is one of the best-known methods.

3. Bond Options

In the preeeding section we described the pricing of options on stocks. For non-dividend paying stocks the Black-Scholes formula gives an analytic formula for the price of a call option. Not only scientists agree on this formula but also in practice market-participant use the Black-Scoles formula to calculate their prices. They only need to have an estimate of the volatility of the stock, since all other variables in the Black-Scholes formula are known from the specification of the option (K and (T-t)) or are determined in other markets (S and r).

However for options on bonds, there is less agreement about a correct formula which can also be easily used by market-participants. In fact one might say that the pricing of options on bonds is still an open problem. The main problem with bonds is that they have a finite lifetime. Bonds are redeemed after some time, the maturity date T_B of the bond. For ease of exposition we will consider a discount, or zero coupon, bond i.e. a bond which doesn't pay coupons. The holder only receives the nominal value at the maturity date. These bonds are called discount bonds because there are issued at a price far below the nominal value to compensate the investor for not receiving interest payments during the lifetime of the bond. At any time t before T_B, the price of the bond B(t) depends on the prevailing interest rates, while at T_B the price is the nominal, or face, value of the bond. We will scale the bond prices such that its face value is 1, i.e. $B(T_B)=1$. Now it will be clear that we cannot use an equation like (2.2) to describe the bondprice process since this equations leads to a stochastic bond price at every moment, and especially at T_B. This is in contradiction with the fact that the price is known. Furthermore it follows from the solution of (2.2) as given in (2.17) and (2.18) that the variance of the future prices increases over time, while for a bond the variance decreases if one approaches maturity. In conclusion we cannot rely on the Black-Scholes formula to price options on bonds, since the stochastic process describing stock prices cannot be used for bonds. A second problem is that we assumed that there is a fixed interest rate r which is independent of the stock price. However, bondprices are determined by interest rates and hence are certainly not independent of the prevailing interest rates.

The proposed methods to describe the pricing of bond options can be divided into two groups, direct and indirect methods. Direct methods specify a stochastic process for the bondprice and derive from this an option price formula. Indirect methods first specify a stochastic process that describes the interest rates over time, then deduce from these interest rates bond prices and in the sequel determine option prices. We will not describe all proposed methods, but only some which illustrate the use of the theory of stochastic processes in option pricing.

A direct method for bond option pricing, which uses the theory of stochastic processes in a very elegant way, has been proposed by Ball and Torous (1983). They use the following stochastic differential equation to describe the price of a bond that matures at T_B and has face value 1:

$$\text{(3.1)} \qquad dB(t) = \alpha(B,t)B(t)dt + \sigma B(t)dW(t)$$

where

$$\text{(3.2)} \qquad \alpha(B,t) = \tfrac{1}{2}\sigma^2 - \frac{1}{(T_B-t)}\ln(B(t))$$

This stochastic differential equation differs by the one, which we used to describe a stockprice, only in the drift term which is no longer constant. The advantage of this specification can be seen most easily by using Itô's lemma to find following the stochastic differential equation for $Y(t) = \ln(B(t))$

$$\text{(3.3)} \qquad dY(t) = -\frac{1}{(T_B-t)}Y(t)dt + \sigma dW(t)$$

From (3.3) we see that if Y(t) is positive the drift term is negative and vice versa, if Y(t) is negative then the drift term is positive. Hence the process has a tendency to zero, which becomes stronger if t gets closer to T_B due to the $1/(T_B-t)$ factor. It can be proved (see Karlin and Taylor (1981)) that this tendency is that strong that $Y(T_B)=0$ with probability one. Hence

$B(T_B) = e^{Y(T_B)} = 1$, as is required for the bondprice. The process described in (3.3) is known as a Brownian bridge process, and for every $t \in (0,T_B)$, Y(t) is normally distributed with mean $Y(0)((T_B-t)/T_B$ and variance $t\sigma^2(1-t/T_B)$ (see e.g. Arnold pp. 131–132).

Having described the bond price dynamics we can try to proceed in the same way as we did for stocks. However, since interest rates are related to bond prices we can no longer assume that the interest rate r is fixed. Fortunately Merton (1973) has proved that the Black–Scholes formula also holds for stochastic interest rates. For bonds we proceed as follows. Let D(t) be the price of a discount bond with face value 1, which matures at T, the maturity date of the option. Hence $T < T_B$. We assume that the D(t) can be described by

$$(3.4)\quad dD(t) = \alpha_1(D,t)\, D(t)dt + \sigma_1 D(t)\, dW_1(t)$$

where

$$(3.5)\qquad \alpha_1(D,t) = \tfrac{1}{2}\sigma_1^2 - \frac{1}{(T-t)}\ln(D(t))$$

and $W_1(t)$ is another standard Brownian motion such that the correlation between $W_1(t)$ and W(t) is ρt.
Now, nothing stands in the way to apply the same arguments as we did for options on stocks. This results in the following formula (see Ball and Torous (1983)) for the price of an option on the bond with exercise price K:

$$(3.6)\qquad C(B,t) = BN(d_1) - KD\, N(d_1-\nu\sqrt{T-t})$$

where

$$(3.7)\qquad d_1 = \left\{\ln\,(B/K) - \ln D + \frac{\nu^2}{2}(T-t)\right\}/\nu\sqrt{T-t}$$

$$(3.8)\qquad \nu = \sigma^2 + \sigma_1^2 + 2\rho\sigma\,\sigma_1$$

We see that (3.6) bears close similarity with the price for options on stocks (2.35), with D replacing $e^{-r(T-t)}$ and ν replacing σ.
To determine the price of a particular bond one needs an estimate of ν. Bergstrom (1976) describes methods for getting unbiased estimates for the

parameters of the joint process (3.1) and (3.4). As with stocks (3.6) not only gives the value of a European call option but also of an American since we assumed that we had to do with a discount bound which doesn't pay any coupons. For bonds with coupons one again has an optimal stopping problem.

The model of Ball and Torous is not unobjected for reasons to be explained below. Since interest rates are always positive it follows that $B(t) \leq 1$ for all $t \in (0,T_B)$, otherwise one would earn a negative interest rate on the discount bond. Hence $Y(t)$ has to be negative for all $t \in (0,T_B)$, with probability one. However, as remarked earlier, $Y(t)$ as given by (3.3) is normally distributed with mean $Y(0)(T_B-t)/T_B$ and variance $t\sigma^2(1-t/T_B)$ and so there is a positive probability that $Y(t) > 0$.

This is the main objection against the Ball–Torous model. It allows for negative interest rates with positive probability. As far as we know there have never been introduced in the literature a stochastic process which describes the bondprice dynamics and overcomes the objection against the previous model. A suitable process also needs to fulfil the condition that $B(t)$ is less than $D(t)$, where $D(t)$ still is the price of the discount bond which matures at $T < T_B$. This also follows from the requirement of nonegative interest rates, since if $D(t) < B(t)$, it would be cheaper to buy the bond which matures at T and keep its face value on a deposit without interest payments (zero interest) from T till T_B, than to buy the bond which matures at T_B. Hence the price of the last bond is to high and will decrease. If one wants a model which describes different options on bonds with different maturities one gets more of this kind of conditions. This is the reason why e.g. Cox, Ingersoll and Ross (1985) and Courtadon (1982) introduced models for pricing bond options which specify a stochastic process for the interest rates and deduce from these simultaneously the prices of bonds and options. In this way they can ensure that negative interest are not possible. These models are examples of indirect methods.

Cox, Ingersoll and Ross (1985) use the following process to describe the short–term interest rates.

$$(3.9) \qquad dr = k(\theta-r)dt + \sigma\sqrt{r}dW,$$

where $\sigma, k, \theta > 0$ are constants and we have again suppresed the variable t in the functions r and W. Given the sign of k this process has a tendency to go to θ and is for this reason called a mean–reverting process. θ can be seen as

the long term equilibrium interest rate. In order to ensure that the interest rates are positive, it is enough to require that $2k\theta \geq \sigma^2$, because in that case the stochastic process r(t) can never reach 0. (See e.g. Schuss (1982) for conditions on the specification of a stochastic differential equation that ensure that the process never crosses a certain boundary.) Using (3.9), Cox, Ingersoll and Ross derive partial differential equations like (2.23) and expressions involving expectations like (2.32) for bond and option prices. However, the problem is that from these equations and expressions one cannot derive analytic formulas like the Black-Scholes formula, and hence one doesn't get an explicit formula for the price of an option on a bond. The only way to derive this price is by using numerical procedures to solve the partial differential equation. This is a disadvantage of most indirect methods.
We can say that there is not a single method that is acknowledged by both practitioners and academists as the best one for the pricing of options on bonds. This becomes even more disturbing if one realizes that options are not only traded separately from bonds, but that a lot of bonds have an option attached to them. Examples are callable bonds, where the issuer has the right to redeem the bond before maturity by paying the face value to the holder of the bond. Hence the holder of the bond has a straightforward bond, but at the same time the issuer holds a call option on the bond with exercise price equal to the face value.

4. Options on other instruments

There are not only traded options on stocks and bonds but also on other financial instruments. Very popular are for example currency options. The holder of such an option has the right to exchange a certain amount of one currency against another currency at a fixed rate before the maturity date of the option contract. Garman and Kolbhagen (1983) derive an analytic formula for the price of a European currency option. The following duality for currency options can be used to understand their result, which will be given below, and at the same time it illustrates why the American feature of currency options is important. A call option to exchange Yens against dollars is the same as a put option to exchange dollars against Yens as long as both options have the same maturity date and fixed exchange rate. Both options will be exercised if and only if the dollar is cheaper measured in Yens than it is for the fixed exchange rate. If one views the Yen as a stock in dollar terms and tries to use the Black-Scholes formula, the interest rate in the United States will be a relevant variable. On the other hand for the put option the interest rate in Japan will be relevant. Hence it might come as no surprise that the price of a European currency call option is influenced by both interest rates. The formula of Garman and Kolbhagen (1983) for a call option to exchange Yens against dollars reads as follows

$$(4.1) \qquad C = Se^{-r^*(T-t)}N(d_1^*) - Ke^{-r(T-t)}N(d_1^* - \sigma\sqrt{T-t})$$

with

$$(4.2) \qquad d_1^* = \{\ln(S/K) + (r-r^*+\tfrac{1}{2}\sigma^2)(T-t)\}/\sigma\sqrt{T-t}$$

where K is the fixed exchange rate, stated in the option contract and S the figuring exchange rate. σ^2 is the variance in the exchange rate, r the interest rate in the United States and r^* the interest rate in Japan. If we compare this formula with the Black-Scholes formula we see that S is replaced by $Se^{-r^*(T-t)}$. The duality described above immeditely yields the fact that it might be optimal to exercise a currency option earlier because it can also be seen as a put option, for which we showed earlier that it might be optimal to exercise early. Hence (4.1) holds only for European currency options.

Another kind of intensively traded options are index options. An index is a weighted average of the prices of different stocks such as the Dow-Jones index or the Standard & Poors 500 index. Of course an index isn't a tradeble security as a stock or a bond. Hence, the holder of a call option on an index cannot buy the index against the fixed exercise price but he is entitled to the difference between the index and the exercise price, whenever this difference is positive. This is an option with cash settlement, i.e. the owner receives an amount of cash at the exercise date instead of a security. These options can be used by holders of a portfolio of stocks to protect their portfolio against a crash in the stock market.

Recently there have been introduced options which do not payoff the difference between the price of a security and the exercise price, whenever this is positive, but their final payoff is the difference between the average price of the security over some time interval and the exercise price. For example, the AB Svensk Exportkredit issued Yen/dollar currency options which entitle the owner of the option to the difference of the average Yen/dollar spot exchange rate over the contract year and a fixed exchange rate, whenever this is positive. Other examples of these kind of contracts are based on the average oil price over a specific time interval, or the average gold price. Often one meets these options as part of a bond contract. The venture capital company Oranje Nassau issued bonds with a 6% coupon which entitle the holders at the maturity date to the maximum of the face value and the average price of 10,5 barrels North See Oil over the last year of the bond contract. Hence holders of the bond have a straightforward bond and an option on the average oil price over the last contract year with an exercise price equal to the face value of the bond. Reasons for using the average value after a year, instead of the oil price on the last contract day are that Oranje Nassau is protected against oilprice manipulation at this final day, and that the profits of Oranje Nassau are heavily influenced by the oil price and in this way they only have to pay a positive at the maturity date of the option if the oil price is high during the last year of the contract, in which case they will indeed have generated significant profits. Kemna and Vorst (1988) describe a pricing model for options based on average asset values over a fixed time interval. Also for this kind of option a partial differential equation like (2.23) and an expectation formula like (2.32) can be derived, but there cannot be found an analytic expression like the Black-Scholes formula. The reason is, that if the asset price is described by (2.2), the future prices are lognormally distributed and the average value of lognormally distributed variables is no longer lognormally distributed. If one would have taken the geometric average instead of the usual arithmetic average the average would also have been lognormally distributed and a Black-Scholes formula could be derived. Kemna and Vorst (1988) use a geometric average and Monte Carlo-simulation to approximate the price of an average value option. There also are firms that issued options based on geometric averages, but these options are so called over-the-counter-options, i.e. they are not traded on an exchange but directly sold by the issuer to some buyers.

Stulz (1982) describes the pricing of options on the minimum and maximum vlaue of two assets, while Margrabe gives a model to price options that give the right to exchange one asset for another. These kind of options are often part of a futures contract. For example in commodity forward contracts two parties agree that at the maturity date of the forward contract one of the parties

sells a certain amount of the commodity to the other party at a fixed price. Most of the time, the seller has the right to chose between several qualities of the commodity. Hence he has the option to chose the quality. A good example is the Treasury Bond Future contract. With this future contract the seller has to sell a Treasury Bond to the other party at some fixed price. However the seller can chose which bond he wants to sell. The only restriction is that the bond still has at least 15 years to maturity. Hence he has the option to sell the bond with the lowest value.
There are several other example of options. A good overview can be found in Cox and Rubinstein (1985).

5. Conclusion

In this paper we have shown how the theory of stochastic processes has influenced the theory of option pricing. Especially the problem of pricing call options on non-dividend paying stocks has been resolved using the theory of stochastic differential equations in such a way that both practitioners and theorists agree on the solution. However in case of options on bonds there still remains a lot of work to be done to get a pricing formula that is both theoretically sound and easy to use. In the last section we showed that there are not only options on stocks and bonds but there are a lot more options, although most of them are not traded as such, but are part of a financial contract. These options will be a fruitful area of future research.

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