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AN ALGORITHM FOR MAXIMUM LIKELIHOOD ESTIMATION OF A NEW COVARIANCE MATRIX SPECIFICATION FOR SUM-CONSTRAINED MODELS

by

P.M.C. de Boer and R.Harkema*

Abstract

Maximum likelihood procedures for estimating sum-constrained models like demand systems, brand choice models and so on, break down or produce very unstable estimates when the number of categories n is large as compared with the number of observations available T. In empirical studies this difficulty is mostly resolved by postulating the contemporaneous covariance matrix of the dependent variables to be equal to $\sigma^2(I_n - n^{-1}\iota_n\iota_n')$. In this paper we develop a maximum likelihood procedure based on a contemporaneous covariance matrix which allows that the variances per category may be different, while the number of observations required is substantially less than the number that would be required in the case of a completely unrestricted contemporaneous covariance matrix.

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Contents

1. INTRODUCTION AND SUMMARY

Sum-constrained models, i.e., models in which subsets of the dependent variables sum to a fixed number, occur in almost every field of applied economic research. In demand analysis the amounts spent on the categories of consumer goods and services that are distinguished add up to total expenditure, in production theory the cost shares of the various factors of production add up to unity, in marketing analysis the market shares of all brands add up to unity, in international trade the flows of imports of a specific country from different destinations add up to total imports, and so on. Sum-constrained models may generally be represented by:

(1.1) $$y_{ti} = f_i(x_{ti1},\ldots,x_{tik_i};\ \beta_i) + u_{ti} \quad (t = 1,\ldots,T;\ i = 1,\ldots n)$$

where y_{ti} denotes the t^{th} observation on the i^{th} dependent variable, f_i represents a non-stochastic function, $(x_{ti1},\ldots,x_{tik_i})$ denotes the t^{th} observation on a set of k_i explanatory variables which are supposed to be specific for the i^{th} dependent variable, β_i is a vector of unknown parameters to be estimated, u_{ti} represents a zero-mean disturbance and n, the number of categories that are distinguished, is supposed to be larger than 3*. The adding-up restrictions imply that the dependent variables y_{ti} add up to a fixed number m_t. Hence

(1.2) $$\sum_{i=1}^{n} y_{ti} = m_t \quad (t = 1,\ldots,T)$$

Summing (1.1) over i and taking expectations it follows that

(1.3) $$\sum_{i=1}^{n} u_{ti} = 0 \quad (t = 1,\ldots,T)$$

and

(1.4) $$\sum_{i=1}^{n} f_i(x_{ti1},\ldots,x_{tik_i};\beta_i) = m_t \quad (t = 1,\ldots,T)$$

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* In case n=2 the parameters of the newly proposed specification of the covariance matrix are not identified; when n=3 the newly proposed specification coincides with the unrestricted contemporaneous covariance matrix.

Evidently, (1.3) reflects the well known fact that the vectors of disturbances in sum-constrained models are linearly dependent. The restrictions (1.4) are usually accomodated by imposing constraints on the functional form f_i and/or on the observations $(x_{ti1},\ldots,x_{tik_i})$ and/or on the vectors of parameters β_i. The vector of disturbances $u' = [u_1' \ldots u_n']$ with $u_i' = [u_{1i} \ldots u_{Ti}]$ is generally assumed to be normally distributed and to exhibit contemporaneous correlation only, i.e.:

(1.5) $$u \sim n(0, \Omega_n \otimes I_T),$$

where Ω_n is a positive semi-definite matrix of rank (n-1) in view of (1.3). A major difficulty in estimating sum-constrained models is caused by the fact that the widely used method of maximum likelihood is very demanding with respect to the number of observations that is required. Maximum likelihood procedures frequently break down or produce very unstable estimates because of lack of data even when only a moderate number of categories is considered. Laitinen (1978), for example, has shown that the minimum number of observations required for maximum likelihood estimation of the unrestricted Rotterdam model (see e.g. Theil (1975)) equals 2n. In applied research this problem is usually resolved by imposing far-reaching restrictions on the contemporaneous covariance matrix of the disturbances. McGuire et al. (1968), Solari (1971), Deaton (1985), and Deaton and Muellbauer (1980), for example, impose

(1.6) $$\Omega_n = \sigma^2(I_n - \frac{1}{n}\iota_n\iota_n').$$

The disadvantage of (1.6) is that all variances, as well as all covariances, are assumed to be equal.

In this paper we present a more flexible specification of the covariance matrix than (1.6) which allows for n parameters to be estimated freely and that possesses the attractive property that the minimum number of observations that is required in order to prevent the estimated covariance matrix from becoming singular is substantially less than in case of a completely unrestricted covariance matrix. In the subclass of linear models, for instance, the minimum number of observations required equals $\max_i (k_i+1)$. For the case considered by Laitined this means that only n+2 observations are required instead of 2n.

This specification reads:

(1.7) $$\Omega_n = D_n - d^{-1}\delta_n\delta_n'$$

with

$$D_n = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix} \qquad \delta_n' = [d_1 \ldots d_n] \qquad d = \sum_{i=1}^{n} d_i$$

Obviously, (1.6) is a special case of (1.7), i.e. when

(1.8) $$d_i = \sigma^2 \qquad i = 1,\ldots,n.$$

Recently, Don (1986) has shown that specification (1.7) may be interpreted as corresponding to the least informative error distribution in the sense of having maximum entropy within the class of all error distributions with finite variances.

We have applied the specification proposed, (1.7), as well as (1.6) to the following three sum-constrained linear models:

1. The Rotterdam model of consumer demand using data taken from Theil (1975, table 6.4, p. 265) consisting of 31 yearly observations on 14 demand categories in the Netherlands 1923-1939; 1948-1963; see De Boer and Harkema (1986).

2. Idem (including a constant term) using data obtained from De Heus and Van Zundert (1982) consisting of 19 yearly observations on 15 demand categories in the Netherlands 1960-1979; cf. De Boer and Harkema (1985).

3. The AIDS model, due to Deaton and Muellbauer (1980), applied to import demand functions for the UK using a data set obtained from Winters (1984)

consisting of 28 yearly observations on imports from 10 countries, 1952-1979; cf. De Boer, Harkema and Van Heeswijk (1986).

We have estimated three versions of these models, viz. under the economic theoretical constraints of additivity, of homogeneity and of symmetry. In table 1 we summarize our findings.

In view of the considerable gain in loglikelihood and the value of twice this gain as compared to the critical value of the test statistic of (1.6) against (1.7), at a level of significance of 0,1%, we conclude that whenever one disposes of a relatively small number of observations and wishes to distinguish quite some categories and one decides to overcome this problem by imposing restrictions on the covariance matrix, it is worthwhile to use the specification that we propose.
In a previous paper (De Boer and Harkema (1986)) we presented for the applied research worker, whose main interest is how to estimate the covariance matrix, the algorithm to solve the first-order conditions for a maximum of the loglikelihood function. The purpose of the present paper is to present to the more theoretically inclined research worker the proofs underlying the algorithm. The organization of the papers is as follows. In section 2 we derive the first-order conditions for obtaining the maximum likelihood estimates of the covariance parameters. In section 3 we present the solution and an algorithm that is easy to implement on a computer and that works very fast. Section 4 contains the proofs of the propositions underlying the algorithm in section 3. As a final remark, in this paper we do not deal with the estimation of the parameters β_i, nor with matters of statistical inference.

Table 1. Summary of the results

Model	Value of loglikelihood		Gain in loglikelihood		Critical value of χ^2-statistic	
	Specification (1.6)	Specification (1.7)	absolute	percentage	d.o.f.	$\alpha = 0.001$
1. additivity	2037.04	2190.52	153.48	7.53	13	34.53
homogeneity	1982.27	2154.28	172.01	8.68		
symmetry	1818.34	2079.46	261.12	14.36		
2. additivity	1506.61	1815.09	308.48	20.48	14	36.12
homogeneity	1485.87	1735.98	250.11	16.83		
symmetry	1336.55	1529.17	192.62	14.41		
3. additivity	897.96	971.32	73.36	8.16	9	27.88
homogeneity	874.35	939.58	65.23	7.46		
symmetry	821.15	888.32	67.17	8.18		

2. MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS OF THE COVARIANCE MATRIX

From (1.7) one easily verifies that $\Omega_n \iota = 0$. As a consequence the density function of the vector u will be degenerate as it should be. Barten (1969), however, has shown that this problem may be handled by simply deleting one category. Choosing without any loss of generality the last one, we delete the last row and column of Ω_n. Denoting the resulting matrix by Ω_{n-1}, straightforward matrix calculation shows that

$$(2.1) \qquad \Omega_{n-1}^{-1} = \begin{bmatrix} d_1^{-1} + d_n^{-1} & d_n^{-1} & \cdots & d_n^{-1} \\ d_n^{-1} & d_2^{-1} + d_n^{-1} & & d_n^{-1} \\ \vdots & \vdots & & \vdots \\ d_n^{-1} & d_n^{-1} & \cdots & d_{n-1}^{-1} + d_n^{-1} \end{bmatrix}$$

In section 4 we prove:

Proposition 1. The determinant of the reduced covariance matrix is:

$$(2.2) \qquad |\Omega_{n-1}| = d^{-1} \prod_{i=1}^{n} d_i$$

From our assumptions about the distribution of the vector u, it follows that the density function of the vectors $u_1 \ldots u_{n-1}$ may be written as

$$(2.3) \qquad \begin{aligned} f(u_1 \ldots u_{n-1}) &= (2\pi)^{-T(n-1)/2} |\Omega_{n-1} \otimes I_T|^{-\frac{1}{2}} \cdot \\ &\exp\{-\tfrac{1}{2}[u_1' \ldots u_{n-1}'][\Omega_{n-1} \otimes I_T]^{-1}[u_1' \ldots u_{n-1}']'\} \end{aligned}$$

On substituting (1.1), (2.1), and (2.2) and applying some rearrangements we obtain the following likelihood function

$$(2.4) \qquad \begin{aligned} \ell(y_1 \ldots y_{n-1}) &= (2\pi)^{-T(n-1)/2} [d^{-1} \prod_{i=1}^{n} d_i]^{-T/2} \cdot \\ &\exp\{-\tfrac{1}{2}[\sum_{i=1}^{n-1} d_i^{-1}[y_i - f_i(\beta_i)]'[y_i - f_i(\beta_i)] + d_n^{-1} \sum_{i=1}^{n-1} [y_i - f_i(\beta_i)]' \sum_{j=1}^{n-1} [y_j - f_j(\beta_j)]]\} \end{aligned}$$

where y_i and $f_i(\beta_i)$ denote T-dimensional vectors with elements y_{ti} and $f_i(x_{ti1},\ldots,x_{tik_i};\ \beta_i)$, respectively. From the restrictions (1.2) through (1.4) one easily verifies that the loglikelihood function may be written as

$$\log \ell(y_1\ldots y_{n-1}) = -\frac{T(n-1)}{2}\log 2\pi - \frac{T}{2}\log[d^{-1}\prod_{i=1}^{n} d_i] + \tag{2.5}$$
$$-\tfrac{1}{2}\sum_{i=1}^{n}[d_i^{-1}(y_i - f_i(\beta_i))'(y_i - f_i(\beta_i))]$$

From (2.5) it is clear that the loglikelihood function and hence the resulting maximum-likelihood estimators are invariant with respect to the category that is deleted as it should be.

Conditional on some value, say $\hat{\beta}_i$, for β_i $(i = 1,\ldots,n)$, the maximum likelihood estimates of the covariance parameters d_i $(i = 1,\ldots,n)$ can be obtained from the following system of first-order derivatives of (2.5) with respect to d_i $(i = 1,\ldots,n)$:

$$\hat{d}_i - \frac{\hat{d}_i^2}{\hat{d}} = \frac{\hat{u}_i'\hat{u}_i}{T} \overset{\text{def}}{=} \hat{\alpha}_i \qquad i = 1,\ldots,n \tag{2.6}$$

where $\hat{u}_i = y_i - f_i(\hat{\beta}_i)$ with $\sum_{i=1}^{n} \hat{u}_i = 0$.

Maximum likelihood estimates for β_i as well as d_i $(i = 1,\ldots,n)^*$, may be obtained by iterating between the maximum likelihood estimates of d_i $(i = 1,\ldots,n)$, given β_i, and the (constrained) maximum likelihood estimate of β_i, given d_i $(i = 1,\ldots,n)$. In order to start up the procedure, one may take $d_i = 1$ $(i = 1,\ldots,n)$, which implies that the first-round estimate of β_i $(i = 1,\ldots,n)$ is the (constrained) ordinary (non-linear) least-squares estimate.

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* Note that in case of cross-equation constraints, like the symmetry conditions in demand systems, the maximum likelihood estimate of β_i depends on d_j $(j = 1,\ldots,n)$, which is the most important reason why maximum likelihood estimates of d_i $(i = 1,\ldots,n)$ have to be extracted from (2.6). In addition, maximum likelihood estimates of d_i $(i = 1,\ldots,n)$ are necessary in order to evaluate the loglikelihood function.

3. THE ALGORITHM FOR SOLVING THE FIRST-ORDER CONDITIONS

In this section we elaborate on how to obtain the estimates $\hat{d}_i$ for the covariance parameters from the first-order conditions (2.6). Without any loss of generality we assume that*:

$$(3.1) \qquad \hat{\alpha}_n \geq \hat{\alpha}_i > 0 \qquad i=1,\ldots,n,$$

with $\hat{\alpha}_i$ being defined in (2.6) as $\hat{\alpha}_i = T^{-1}\hat{u}_i'\hat{u}_i$.

By virtue of the inequality of Gauchy-Schwarz it holds true that

$$(3.2) \qquad T^{-1}\hat{u}'_i\hat{u}_n \geq -\{(\frac{\hat{u}_n'\hat{u}_n}{T})(\frac{\hat{u}_i'u_n}{})\}^{\frac{1}{2}} = -(\hat{\alpha}_n\hat{\alpha}_i)^{\frac{1}{2}}$$

Summing (3.2) over i=1,...,n and substituting $\sum_{i=1}^{n-1} \mu_i' = \hat{\mu}_n'$, we obtain

$$(3.3) \qquad \hat{\alpha}_n = \leq (\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}})^2$$

From (2.5) it follows that, conditional on $\hat{\alpha}_i$, the estimates d_i are obtained by minimizing

$$(3.4) \qquad f(d_1 \ldots d_n) = \log\Big(d^{-1}\prod_{i=1}^{n} d_i\Big) + \sum_{i=1}^{n} d_i^{-1}\hat{\alpha}_i$$

Since we deal with covariance matrices, we are only interested in those vectors of covariance parameters that yield a positive-definite covariance matrix. In section 4 we prove:

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* If there would be multiple maxima, we may choose arbitrarily one of them to be α_n.

Proposition 2. The reduced covariance matrix Ω_{n-1} is positive-definite if and only if either all d_i's are positive or at most one d_i is negative with d being negative as well.

Let us now consider any admissible vector $(d_1,\dots,d_n)$ with $d_k (k \neq n)$ being negative
As $\hat{\alpha}_n \geq \hat{\alpha}_k$, the value of f will centainly not increase when the values of d_k and d_n are interchanged. Therefore, we can restrict the analysis to the set of vectors $S = S_1 \cup S_2$ with S_1 and S_2 being defined by

(3.5)
$$S_1 = \{(d_1 \dots d_n) \in R^n | d_i > 0 \ \forall \ 1 < i < n\}$$
$$S_2 = \{(d_1 \dots d_n) \in R^n | d_i > 0 \ \forall \ 1 < i < n-1; \ d = \sum_{i=1}^{n} d_i < 0\}$$

On the set S we can prove:

Proposition 3. Stationary points of f in S_1 can only exist when $\hat{\alpha}_n < \sum_{i=1}^{n-1} \hat{\alpha}_i$ and stationary points of f in S_2 can only exist when $\hat{\alpha}_n > \sum_{i=1}^{n-1} \hat{\alpha}_i$.

In order to trace the stationary points, we prove propositions 4 and 5; to that purpose we first define:

(3.6)
$$f_1(d) = \sum_{i=1}^{n} \left(1 - \frac{4\hat{\alpha}_i}{d}\right)^{\frac{1}{2}} - (n-2) \qquad \text{for } d > 4\hat{\alpha}_n$$

(3.7)
$$f_2(d) = \sum_{i=1}^{n-1} \left(1 - \frac{4\hat{\alpha}_i}{d}\right)^{\frac{1}{2}} - \left(1 - \frac{4\hat{\alpha}_n}{d}\right)^{\frac{1}{2}} - (n-2) \quad \text{for } d > 4\hat{\alpha}_n \text{ or } d < 0$$

(3.8)
$$\hat{\gamma} = f_1(4\hat{\alpha}_n) = f_2(4\hat{\alpha}_n) = \sum_{i=1}^{n-1} \left(1 - \frac{\hat{\alpha}_n}{\hat{\alpha}_n}\right)^{\frac{1}{2}} - (n-2).$$

The graph of the functions $f_1(d)$ and $f_2(d)$ is shown in figure 1.

Proposition 4. If $\hat{\alpha}_n < \sum_{i=1}^{n-1} \hat{\alpha}_i$, there is a unique stationary point in S_1, that can be found as follows:

a. if $\hat{\gamma} \leq 0$, solve on $[4\hat{\alpha}_n, \infty)$ $f_1(\hat{d}) = 0$ (see I in figure 1) and substitute the solution obtained into:

$$(3.9) \qquad \hat{d}_i = \frac{\hat{d}}{2}\left[1 - \left(\frac{4\hat{\alpha}_i}{\hat{d}}\right)^{\frac{1}{2}}\right] \qquad i=1,\dots,n$$

b. if $\hat{\gamma} > 0$, solve on $[4\hat{\alpha}_n, \infty)$ $f_2(\hat{d}) = 0$ (see II in figure 1) and subtitute the solution found into:

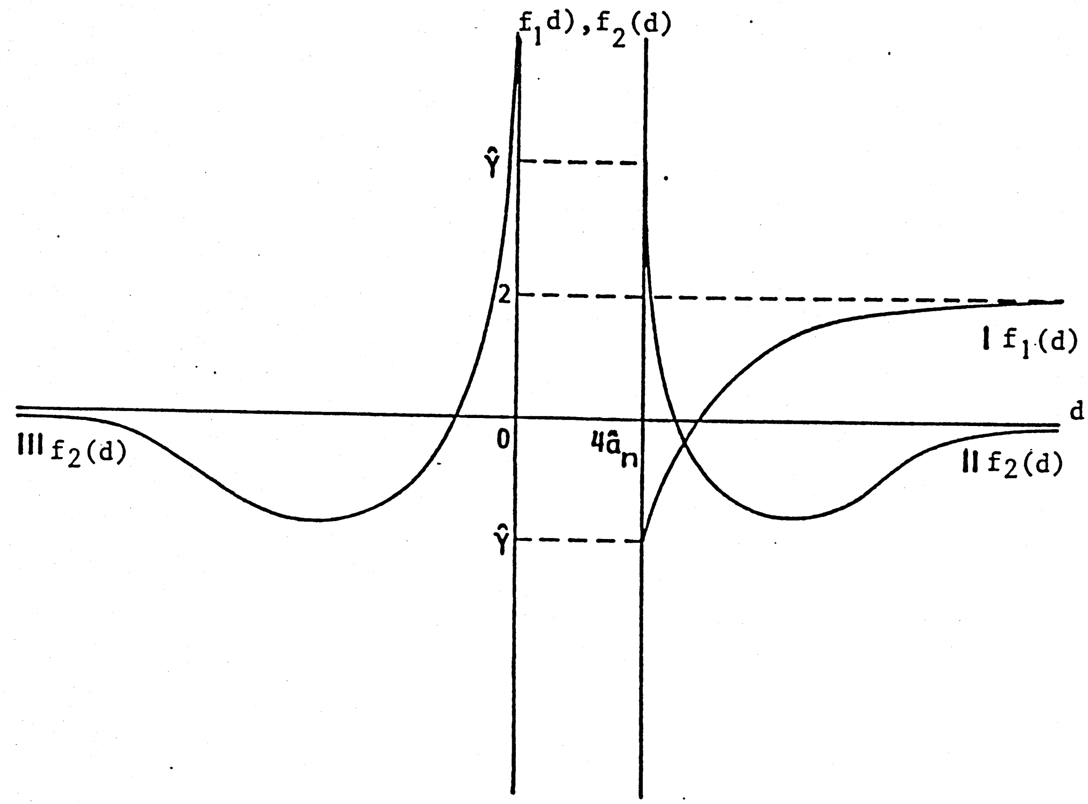
$$(3.10) \qquad \hat{d}_i = \frac{\hat{d}}{2}\left[1 - \left(1 - \frac{4\hat{\alpha}_i}{\hat{d}}\right)^{\frac{1}{2}} \qquad i = 1, \dots, n$$

$$\hat{d}_n = \frac{\hat{d}}{2}\left[1 + \left(1 - \frac{4\hat{\alpha}_n}{\hat{d}}\right)^{\frac{1}{2}}\right]$$

Proposition 5. If $\sum_{i=1}^{n-1} \hat{\alpha}_i < \hat{\alpha}_n < \left(\sum_{n=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}}\right)^2$, there is a unique stationary point in S_2 that can be found as follows:

Solve on $(-\infty, 0)$ $f_2(\hat{d}) = 0$ (see III in figure 1) and substitute the solution obtained into (3.10).

Figure 1



In order to obtain the infimum of f with respect to S and hence the maximum likelihood estimates of the covariance parameters, we finally prove the following proposition.

Proposition 6. Let f be as defined in (3.4) and S_1 and S_2 as defined in (3.5). Then we have

A.

$$\inf_{x \in S_1} f(x) = \begin{cases} f(x^0) < \sum_{i=1}^{n-1} \log \hat{\alpha}_i + n-1 & \text{if } \hat{\alpha}_n \le \sum_{i=1}^{n-1} \hat{\alpha}_i \\ \sum_{i=1}^{n-1} \log \hat{\alpha}_i + n-1 = \lim_{d_n \to \infty} f(\hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1}, d_n) & \text{if } \sum_{i=1}^{n-1} \hat{\alpha}_i \le \hat{\alpha}_n \le \left(\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}}\right)^2 \end{cases}$$

where x^0 denotes the unique stationary point of f in S_1 (Proposition 4).

B

$$\inf_{x \in S_2} f(x) = \begin{cases} \sum_{i=1}^{n-1} \log \hat{\alpha}_i + n-1 = \lim_{d_n \to -\infty} f(\hat{\alpha}_1, \ldots, \hat{\alpha}_{n-1}, d_n) & \text{if } \hat{\alpha}_n \le \sum_{i=1}^{n-1} \hat{\alpha}_i \\ f(x^1) < \sum_{i=1}^{n-1} \log \hat{\alpha}_i + n-1 & \text{if } \sum_{i=1}^{n-1} \hat{\alpha}_i < \hat{\alpha}_n < \left(\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}}\right)^2 \\ -\infty = \lim_{\lambda \downarrow 0} f\left(\lambda \hat{\alpha}_1^{\frac{1}{2}}, \ldots, \lambda \hat{\alpha}_{n-1}^{\frac{1}{2}}, -\lambda \sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}} - \lambda^2\right) & \text{if } \hat{\alpha}_n = \left(\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}}\right)^2 \end{cases}$$

where x^1 denotes the unique stationary point of f in S_2 (proposition 5).

From proposition 6 it follows that the maximum likelihood estimates are given by x^0 when $\hat{\alpha}_n < \sum_{i=1}^{n-1} \hat{\alpha}_i$ and by X^1 when $\sum_{i=1}^{n-1} \hat{\alpha}_i < \hat{\alpha}_n < \left(\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}}\right)^2$.

When $\hat{\alpha}_n = \sum_{i=1}^{n-1} \hat{\alpha}_i$, they are given by

$$\hat{d}_i = \hat{\alpha}_i \qquad\qquad i = 1, \ldots, n-1$$

$$\hat{d}_n = \pm\infty$$

which leads to the following estimates for the elements of the covariance matrix Ω_n

$$\lim_{\hat{d}_n \to \pm\infty} \left(\hat{d}_i - \frac{\hat{d}_i^2}{\hat{d}}\right) = \hat{\alpha}_i \qquad\qquad i = 1, \ldots, n-1$$

$$\lim_{\hat{d}_n \to \pm\infty} - \frac{\hat{d}_i \hat{d}_j}{\hat{d}} = 0 \qquad\qquad i,j = 1, \ldots, n-1;\ i \neq j$$

$$\lim_{\hat{d}_n \to \pm\infty} - \frac{\hat{d}_i \hat{d}_n}{\hat{d}} = -\hat{\alpha}_i \qquad\qquad i = 1, \ldots, n-1$$

$$\lim_{\hat{d}_n \to \pm\infty} \left(\hat{d}_n - \frac{\hat{d}_n^2}{\hat{d}}\right) = \lim_{\hat{d}_n \to \pm\infty} \frac{\hat{d}_n}{\hat{d}} \sum_{i=1}^{n-1} \hat{d}_i = \sum_{i=1}^{n-1} \hat{\alpha}_i = \hat{\alpha}_n$$

Hence, in this specific case, the estimate of the covariance matrix becomes as follows

$$(3.11) \qquad \hat{\Omega}_n = \begin{bmatrix} \hat{\alpha}_1 & \cdots & 0 & -\hat{\alpha}_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \hat{\alpha}_{n-1} & -\hat{\alpha}_{n-1} \\ -\hat{\alpha}_1 & \cdots & -\hat{\alpha}_{n-1} & \hat{\alpha}_n \end{bmatrix}$$

From a viewpoint of estimation, this corresponds to deleting one equation from the system and applying ordinary (non linear) least squares to the (n-1) remaining equations seperately.

Finally, when $\hat{\alpha}_n = \left(\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}}\right)^2$ the likelihood function is unbounded. In this case one has to resort to a less flexible specification of the contemporaneous covariance matrix like, for example, the specification (1.6).

In table 2 we summarize the algorithm presented in this section.

TABLE 2. ALGORITHM TO SOLVE (2.6)

Initial conditions*	Solution for $\hat{d} = \sum_{i=1}^{n} \hat{d}_i$	Solution for $\hat{d}_i$
$\hat{\alpha}_n < \sum_{i=1}^{n-1} \hat{\alpha}_i$ $\hat{\gamma} \leq 0$	Solve on $[4\hat{\alpha}_n, \infty)$ $f_1(d) = 0$ (see I in figure 1)	$\hat{d}_i = \frac{\hat{d}}{2}[1 - (1 - \frac{4\hat{\alpha}_i}{\hat{d}})^{\frac{1}{2}}]$ $i=1,\ldots,n$
$\hat{\alpha}_n < \sum_{i=1}^{n-1} \hat{\alpha}_i$ $\hat{\gamma} > 0$	Solve on $(4\hat{\alpha}_n, \infty)$ $f_2(d) = 0$ (see II in figure 1)	$\hat{d}_i = \frac{\hat{d}}{2}[1 - (1 - \frac{4\hat{\alpha}_i}{\hat{d}})^{\frac{1}{2}}]$ $i=1,\ldots,n-1$ $\hat{d}_n = \frac{\hat{d}}{2}[1 + (1 - \frac{4\alpha_n}{\hat{d}})^{\frac{1}{2}}]$
$\hat{\alpha}_n = \sum_{i=1}^{n-1} \hat{\alpha}_i$		$\hat{d}_i = \hat{\alpha}_i$ $i=1,\ldots,n-1$ $\hat{d}_n = \pm\infty$
$\sum_{i=1}^{n-1} \hat{\alpha}_i < \hat{\alpha}_n < (\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}})^2$	Solve on $(-\infty, 0)$ $f_2(d) = 0$ (see III in figure 1)	$\hat{d}_i = \frac{\hat{d}}{2}[1 - (1 - \frac{4\hat{\alpha}_i}{\hat{d}})^{\frac{1}{2}}]$ $i=1,\ldots,n-1$ $\hat{d}_n = \frac{\hat{d}}{2}[1 + (1 - \frac{4\alpha_n}{\hat{d}})^{\frac{1}{2}}]$
$\hat{\alpha}_n = (\sum_{i=1}^{n-1} \hat{\alpha}_i^{\frac{1}{2}})^2$	Likelihood function is unbounded	

* Without any loss of generality, it is assumed that $\hat{\alpha}_n \geq \hat{\alpha}_i (i = 1, \ldots, n-1)$.

4. PROOF OF THE PROPOSITIONS*

Proposition 1.

The first principal minor of $|\Omega_{n-1}|$ (c.f. (1.7)) is

$$|\Omega_1| = d_1 - \frac{d_1^2}{d} = \frac{\sum_{i=2}^{n} d_i}{d} d_1$$

Suppose that the r-th principal minor (r < n-1) is

$$(4.1) \qquad |\Omega_r| = \frac{\sum_{i=r+1}^{n} d_i}{d} \prod_{i=1}^{r} d_i$$

If we prove that

$$|\Omega_{r+1}| = \frac{\sum_{i=r+2}^{n} d_i}{d} \prod_{i=1}^{r+1} d_i$$

it follows from induction that**

$$(4.2) \qquad |\Omega_{n-1}| = \frac{d_n}{d} \cdot \prod_{i=1}^{n-1} d_i = \frac{\prod_{i=1}^{n} d_i}{d}$$

PROOF: The ratio $|\Omega_r|/|\Omega_{r+1}|$ is equal to element (r+1, r+1) of

$$\Omega_{r+1}^{-1} = [D_{r+1} - \frac{\delta_{r+1}\delta'_{r+1}}{d}]^{-1} =$$

$$D_{r+1}^{-1} - \iota_{r+1}[\iota'_{r+1}\delta_{r+1} - d]^{-1}\iota'_{r+1}$$

Hence

$$\frac{|\Omega_r|}{|\Omega_{r+1}|} = d_{r+1}^{-1} + (\sum_{i=r+2}^{n} d_i)^{-1} = \frac{\sum_{i=r+1}^{n} d_i}{d_{r+1} \sum_{i=r+2}^{n} d_i}$$

* In this section we drop the "hats" for reasons of notational convenience.
** (4.2) can be proved alternatively by using the results of Appendix A6 of Dhrymes (1970) - as pointed out by Mr. Ten Cate of the Netherlands Central Planning Bureau - but since we need (4.1) in the sequel we prefer to give the full proof.

or

$$|\Omega_{r+1}| = \frac{d_{r+1} \sum_{i=r+2}^{n} d_i}{\sum_{i=r+1}^{n} d_i} |\Omega_r| = \frac{\sum_{i=r+2}^{n} d_i}{d} \prod_{i=1}^{r+1} d_i$$

Proposition 2.

From (4.1) and (4.2) it is obvious that Ω_{n-1} will be positive-definite if either all d_i's are positive or at most one d_i is negative with d being negative as well. Next, suppose 2 d_i's are negative and take without any loss of generality d_{n-1} and d_n. It follows from (4.2) that $|\Omega_{n-1}|$ can only be positive when $d > 0$. But then it follows from (4.1) that $|\Omega_{n-2}|$ is negative and consequently Ω_{n-1} would not be positive-definite. Suppose 3 d_i's are negative, say d_{n-2}, d_{n-1}, and d_n. $|\Omega_{n-1}|$ can only be positive when $d < 0$, but then $|\Omega_{n-2}|$ is negative. The argument can easily be extended to more than 3 d_i's. This proves the necessity of the conditions.

Proposition 3.

The stationary points of f in S are obtained by solving the following system of first-order conditions (compare (2.6))

$$d_i \sum_{j \neq i} d_j = \alpha_i d \qquad i = 1, \ldots, n \tag{4.3}$$

Summing (4.3) over $i \neq n$ and subtracting (4.3) for $i = n$, one easily verifies that a stationary point can only exist when

$$2\sum_{1 \leq j < i \leq n-1} d_j d_i = d\left(\sum_{i=1}^{n-1} \alpha_i - \alpha_n\right) \tag{4.4}$$

From the definition of S_1 and S_2 in (3.5) it follows that $d_i > 0$ for $i \neq n$. Therefore a stationary point of f in S_1 can only exist when $\alpha_n < \Sigma_{i=1}^{n-1} \alpha_i$ and likewise a stationary point of f in S_2 can only exist when $\alpha_n > \Sigma_{i=1}^{n-1} \alpha_i$.

Proposition 4.

After substituting $\Sigma_{j\neq i}\, d_j = d - d_i$, we obtain the following solution for (4.3) in terms of α_i and d

$$d_i = \frac{d}{2} \pm \frac{d}{2}\left(1 - \frac{4\alpha_i}{d}\right)^{\frac{1}{2}} \qquad i = 1, \ldots, n \tag{4.5}$$

with $d \geq 4\alpha_n$ or $d < 0$ because we are only interested in real-valued solutions.For stationary points in S_1, it is certainly true that

$$d_i = \frac{d}{2} - \frac{d}{2}\left(1 - \frac{4\alpha_i}{d}\right)^{\frac{1}{2}} \qquad \text{for } i \neq n$$

For suppose that for some $j \neq n$, it would be true that

$$d_j = \frac{d}{2} + \frac{d}{2}\left(1 - \frac{4\alpha_j}{d}\right)^{\frac{1}{2}}$$

Then it follows that

$$d = \sum_{i=1}^{n} d_i \geq \sum_{i\neq j,n} d_i + d + \frac{d}{2}\left[\left(1 - \frac{4\alpha_j}{d}\right)^{\frac{1}{2}} - \left(1 - \frac{4\alpha_n}{d}\right)^{\frac{1}{2}}\right] > d$$

because $\alpha_j \leq \alpha_n$ for $j \neq n$ and $d_k > 0$ $(k = 1, \ldots, n)$. Consequently, we have to consider two possible solutions in S_1, viz., either

$$d_i = \frac{d}{2} - \frac{d}{2}\left(1 - \frac{4\alpha_i}{d}\right)^{\frac{1}{2}} \qquad i = 1, \ldots, n \tag{4.6}$$

or

$$d_i = \frac{d}{2} - \frac{d}{2}\left(1 - \frac{4\alpha_i}{d}\right)^{\frac{1}{2}} \qquad i = 1, \ldots, n-1 \tag{4.7}$$

$$d_n = \frac{d}{2} + \frac{d}{2}\left(1 - \frac{4\alpha_n}{d}\right)^{\frac{1}{2}}$$

It should be noted that in the case of multiple maxima, a solution must be of the form (4.6). For suppose that $\alpha_j = \alpha_n$ for some $j \neq n$.

Evidently, $\alpha_n < \sum_{i=1}^{n-1} \alpha_i$ and so a solution must be of the form (4.6) or (4.7). Summing (4.7) over i, we obtain

$$d = \sum_{i=1}^{n} d_i = \sum_{i \neq j,n} d_i + d > d$$

Consequently, if a solution exists, it must be of the form (4.6). In the analysis of (4.7) we may confine ourselves therefore to the case where the strict inequality sign holds true

$$\alpha_n > \alpha_i \qquad i = 1, \ldots, n-1 \tag{4.8}$$

a) Let us now first consider (4.6). Summing over i and applying some rearrangements yields

$$f_1(d) = \sum_{i=1}^{n} \left(1 - \frac{4\alpha_i}{d}\right)^{\frac{1}{2}} - (n-2) = 0 \tag{4.9}$$

It is easy to show that $f_1(d)$ has the following properties

(i) $\lim_{d \to \infty} f_1(d) = 2$

(ii) $f_1'(d) > 0$

Evidently, there is a unique solution to (4.9) provided that

$$f_1(4\alpha_n) = \sum_{i=1}^{n-1} \left(1 - \frac{\alpha_i}{\alpha_n}\right)^{\frac{1}{2}} - (n-2) \overset{\text{def}}{=} \gamma \leq 0 \tag{4.10}$$

Note that in the case of multiple maxima, (4.10) is always met with. If $\gamma < 0$, we have to solve (4.9) numerically; (see I in Figure 1). If $\gamma = 0$, the solution of (4.9) is $d = 4\alpha_n$. The solution for d should be substituted into (4.6) in order to obtain the solution for $d_1,\ldots,d_n$.

b) Consider (4.7). Summing (4.7) over i leads to

$$f_2(d) = \sum_{i=1}^{n-1} \left(1 - \frac{4\alpha_i}{d}\right)^{\frac{1}{2}} - \left(1 - \frac{4\alpha_n}{d}\right)^{\frac{1}{2}} - (n-2) = 0 \tag{4.11}$$

It is easily seen that

(4.12) $$\lim_{d \to \infty} f_2(d) = 0$$

Let us now define the following functions

$$c_i(d) = \frac{(1 - \frac{4\alpha_n}{d})^{\frac{1}{2}}}{(1 - \frac{4\alpha_i}{d})^{\frac{1}{2}}} \qquad i = 1, \ldots, n-1$$

which have the properties (see (4.8))

(4.13)
$$0 \leq c_i(d) < 1$$
$$\lim_{d \to \infty} c_i(d) = 1$$

and

(4.14) $$c_i'(d) > 0$$

The derivative of $f_2(d)$ with respect to d is easily shown to be equal to

(4.15) $$f_2'(d) = \frac{2}{d^2(1 - \frac{4\alpha_n}{d})^{\frac{1}{2}}} \left[\sum_{i=1}^{n-1} c_i(d)\alpha_i - \alpha_n\right]$$

From $\alpha_n < \Sigma_{i=1}^{n-1} \alpha_i$, (4.13), and (4.14) it follows that there exists only one $d_0 \in [4\alpha_n, \infty)$ such that

$$\sum_{i=1}^{n-1} c_i(d_0)\alpha_i - \alpha_n = 0$$

or, by virtue of (4.15), that

$$f_2'(d_0) = 0$$

Obviously, $f_2'(d) < 0$ for $d < d_0$ and $f_2'(d) > 0$ for $d > d_0$. The latter implies that $\lim_{d \to \infty} f_2(d) = 0$ from below. Consequently, there is a unique solution to (4.11) provided that

(4.16) $$f_2(4\alpha_n) = \gamma \geq 0$$

For $\gamma > 0$, we have to solve (4.11) numerically; (see II in Figure 1), while $\gamma = 0$, yields once again $d = 4\alpha_n$ as a solution. The solution for d should be substituted into (4.7) in order to obtain the solution for $d_1, \ldots, d_n$.

Proposition 5.

For $d < 0$, it is certainly true that $(1 - 4\alpha_i d^{-1})^{\frac{1}{2}} > 1$. Therefore, we have to consider only one possible solution of (4.3) in S_2, viz.,

(4.17)
$$d_i = \frac{d}{2} - \frac{d}{2}\left(1 - \frac{4\alpha_i}{d}\right)^{\frac{1}{2}} \qquad i = 1, \ldots, n-1$$
$$d_n = \frac{d}{2} + \frac{d}{2}\left(1 - \frac{4\alpha_n}{d}\right)^{\frac{1}{2}}$$

because all other possibilities generate wrong signs. As for proposition 4b we have $\alpha_n > \alpha_i$, i.c. (4.8).

Summing (4.17) over i, we once again obtain (4.11), but now:

(4.18) $$\lim_{d \to -\infty} f_2(d) = 0$$

Of course, the derivative of $f_2(d)$ with respect to d is given by (4.15) as before, but now the functions $c_i(d)$ have the following properties (see (4.8))

(4.19)
$$1 < c_i(d) < (\alpha_n/\alpha_i)^{\frac{1}{2}}$$
$$\lim_{d \to -\infty} c_i(d) = 1 \text{ and } \lim_{d \to 0} c_i(d) = (\alpha_n/\alpha_i)^{\frac{1}{2}}$$

and once again

(4.20) $$c_i'(d) > 0$$

From (4.19) it is obvious that

(4.21) $$\sum_{i=1}^{n-1} \alpha_i - \alpha_n < \sum_{i=1}^{n-1} c_i(d)\alpha_i - \alpha_n < \sum_{i=1}^{n-1} (\alpha_n\alpha_i)^{\frac{1}{2}} - \alpha_n$$

Hence, for $\sum_{i=1}^{n} \alpha_i < \alpha_n < (\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}})^2$, (4.21) and (4.20) imply that there is only one $d_0 \in (-\infty, 0)$ such that

$$\sum_{i=1}^{n-1} c_i(d_0)\alpha_i - \alpha_n = 0$$

entailing according to (4.15) that $f_2'(d_0) = 0$. Obviously, $f_2'(d) < 0$ for $d < d_0$ and $f_2'(d) > 0$ for $d > d_0$. From the definition of the functions $c_i(d)$ we derive that

$$f_2(d) = \left(1 - \frac{4\alpha_n}{d}\right)^{\frac{1}{2}} \left[\sum_{i=1}^{n-1} \{c_i(d)\}^{-1} - 1\right] - (n-2)$$

$$> \left(1 - \frac{4\alpha_n}{d}\right)^{\frac{1}{2}} \frac{1}{\alpha_n} \left[\sum_{i=1}^{n-1} (\alpha_n\alpha_i)^{\frac{1}{2}} - \alpha_n\right] - (n-2)$$

because of (4.19). Since the term between brackets is positive, it follows that

$$\lim_{d \to 0} f_2(d) = \infty$$

Together with (4.18) this implies that there is a unique solution to (4.11) (see III in Figure 1). The solution for d should be substituted into (4.17) in order to obtain the solution for $d_1, \ldots, d_n$.

Proposition 6 *

We start by proving part A of the proposition. Choose any number $M > 1$ such that $\forall\, x \notin [\frac{1}{M}, M]$

(4.22) $$\log x + \frac{\alpha_1}{x} > \max\{f(1, \ldots, 1), 0\} + \log n + \sum_{j=1}^{n} |\log \alpha_j|$$

implying that

(4.23) $$\alpha_i \in (\frac{1}{M}, M) \qquad i = 1, \ldots, n.$$

Let the set B_R be defined by

(4.24) $$B_R = \{(d_1, \ldots, d_n) \in R^n \mid \frac{1}{R} \leq d_j \leq R \quad \forall\, 1 \leq j \leq n\}$$

and let C_R be such that

(4.25) $$f(C_R) = \min_{x \in B_R} f(x)$$

From the continuity of f on the compact set B_R, it follows that C_R always exists, though it need not be unique.

LEMMA 1. If $R > M$, then either C_R is a stationary point of f or $C_R = (C_{R1}, \ldots, C_{Rn})$ can be taken such that $C_{Rj} \in [\frac{1}{M}, M]$ $(j = 1,\ldots,n-1)$ and $C_{Rn} = R$.

PROOF. If C_R is not a stationary point, it must be a boundary point. Evidently, C_R can be taken such that C_{Rn} is at least as large as C_{Rj} $(j \neq n)$. For suppose that $C_{Rj} > C_{Rn}$ for some $j \neq n$. Then the value of f will certainly not increase when the values of C_{Rj} and C_{Rn} are interchanged. Hence, C_R can be taken such that $C_{Rn} \geq C_{Rj}$ $(j \neq n)$. Suppose now $C_{Rn} \neq R$. Then it must be true that $C_{Rj} = R^{-1}$ for some $j \neq n$. From (3.4), however, one easily verifies that

* Without any loss of generality, we assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \alpha_n$.

$$\frac{\partial f}{\partial d_j} = \frac{d_j - \alpha_j}{d_j^2} - \frac{1}{d} = \frac{R^{-1} - \alpha_j}{R^{-2}} - \frac{1}{d} < 0$$

because of (4.23). Consequently, an infinitesimal increase of C_{Rj} would give rise to a smaller value of f. Hence, $C_{Rn} = R$.

Finally, suppose that $C_{Rj} \notin [\frac{1}{M}, M]$ for some $j \neq n$. Obviously,

$$(4.26) \qquad \log d_i + \frac{\alpha_i}{d_i} \geq \min_{d_i}\{\log d_i + \frac{\alpha_i}{d_i}\} = \log \alpha_i + 1 \qquad \forall\, d_i \in (0,\infty)$$

Substituting (4.26) into (3.4), we obtain

$$f(C_R) = \sum_{i=1}^{n} \left(\log C_{Ri} + \frac{\alpha_i}{C_{Ri}}\right) - \log\left(\sum_{i=1}^{n} C_{Ri}\right)$$

$$\geq \sum_{i\neq j,n} (\log \alpha_i + 1) + \log C_{Rj} + \frac{\alpha_j}{C_{Rj}} + \log R + \frac{\alpha_n}{R} - \log nR$$

$$> \sum_{i\neq j,n} (\log \alpha_i + 1) + \log R + \frac{\alpha_n}{R} - \log nR +$$

$$+ \max\{f(1, \ldots, 1), 0\} + \log n + \sum_{j=1}^{n} |\log \alpha_j|$$

$$> \max\{f(1, \ldots, 1), 0\} + n - 2 > f(1, \ldots, 1)$$

where the first strict inequality sign follows from (4.22). Consequently, C_R can not give rise to a minimum of f and hence $C_{Rj} \in [\frac{1}{M}, M]$ for all $j \neq n$. This completes the proof of Lemma 1.

Letting m run through the set of positive integers larger than M, we obtain sequences $\{C_m\}$ and $\{f(C_m)\}$. As $\{f(C_m)\}$ is a nonincreasing sequence and for every point $x \in S_1$ there exists a positive integer m_0 such that $x \in B_m$ for all $m \geq m_0$, it certainly holds true that

(4.27) $$\lim_{m\to\infty} f(C_m) = \inf_{x\in S_1} f(x)$$

In propositon 4, we have proved that f can only have one stationary point in S_1. So there are two possibilities for the sequence $\{C_m\}$, viz., (i) from certain N onwards $C_m = x^0$, the unique stationary point of f in S_1, and obviously

$$\inf_{x\in S_1} f(x) = \lim_{m\to\infty} f(C_m) = f(x^0)$$

(ii) from certain N onwards C_m is always a boundary point.
In the latter case, it follows from Lemma 1 that $C_{mj} \in [\frac{1}{M}, M]$ for $j \neq n$. Therefore, the sequence $\{C_m\}$ has a subsequence $\{C_{m'}\}$ such that

$$\lim_{m'\to\infty} C_{m'j} = C'_j \in [\frac{1}{M}, M] \qquad \text{for } j \neq n$$

because of the compactness of $[\frac{1}{M}, M]$.
Hence,

$$\lim_{m\to\infty} f(C_m) = \lim_{m'\to\infty} f(C_{m'}) = \lim_{m'\to\infty} f(C_{m'1}, \ldots, C_{m'n}) =$$

$$= \lim_{m'\to\infty} f(C'_1, \ldots, C'_{n-1}, m') = \sum_{i=1}^{n-1} \left(\log C'_i + \frac{\alpha_i}{C'_i}\right)$$

From (4.26) it follows that the last expression is minimized for $C'_i = \alpha_i$ $(i = 1, \ldots, n-1)$ and consequently (4.27) implies

$$\inf_{x\in S_1} f(x) = \lim_{m\to\infty} f(C_m) = \sum_{i=1}^{n-1} \log \alpha_i + n - 1$$

Finally, we have to determine whether

$$\inf_{x\in S_1} f(x) = f(x^0) \text{ or } \inf_{x\in S_1} f(x) = \sum_{i=1}^{n-1} \log \alpha_i + n - 1$$

In proposition 3, we have proved that f does not have stationary points in S_1 when $\alpha_n \geq \Sigma_{i=1}^{n-1} \alpha_i$. Therefore, (4.28) $$\inf_{x\in S_1} f(x) = \sum_{i=1}^{n-1} \log \alpha_i + n - 1 \text{ if } \sum_{i=1}^{n-1} \alpha_i \leq \alpha_n \leq \Big(\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}}\Big)^2$$

When $\alpha_n < \sum_{i=1}^{n-1} \alpha_i$, we have

$$f\left(\alpha_1, \ldots, \alpha_{n-1}, \frac{\alpha_n \sum_{i=1}^{n-1} \alpha_i}{\sum_{i=1}^{n-1} \alpha_i - \alpha_n}\right) =$$

$$= \sum_{i=1}^{n-1} \log \alpha_i + n - 1 + \log\left(\frac{\alpha_n}{\sum_{i=1}^{n-1} \alpha_i}\right) + 1 - \frac{\alpha_n}{\sum_{i=1}^{n-1} \alpha_i} < \sum_{i=1}^{n-1} \log \alpha_i + n - 1$$

Consequently, $\Sigma_{i=1}^{n-1} \log \alpha_i + n - 1$ can not be the infimum of f in S_1 and so we have

(4.29) $$\inf_{x\in S_1} f(x) = f(x^0) < \sum_{i=1}^{n-1} \log \alpha_i + n-1 \text{ if } \alpha_n < \sum_{i=1}^{n-1} \alpha_i$$

This completes the proof of part A of the proposition.

In order to prove part B of the proposition, we first restrict ourselves to the case $\alpha_n < (\Sigma_{i=1}^{n-1} \alpha_i^{\frac{1}{2}})^2$. Choose $\gamma_i = \alpha_i - \xi_i$ $(i = 1, \ldots, n-1)$ such that

(4.30) $$\xi_i > 0 \text{ and } \Big(\sum_{i=1}^{n-1} \gamma_i^{\frac{1}{2}}\Big)^2 > \alpha_n$$

and choose any number $M > n+1$ such that $\forall R \in [M, \infty)$

(4.31) $$- R^2\alpha_1 + 2R < 0$$

and such that* $\forall\, x \notin [\frac{1}{M}, M]$

(4.32) $$\log x + \frac{\alpha_1}{x} > \max\{f(1, \ldots, 1, -n-1), 0\} + \alpha_n + \sum_{j=1}^{n} |\log \alpha_j|$$

(4.33) $$\log x + \frac{\xi_i}{x} > \max\{f(f1, \ldots, 1, -n-1), 0\} + \sum_{j=1}^{n-1} |\log \xi_j|$$

$$i = 1, \ldots, n-1$$

Let the set B'_R be defined by

(4.34) $$B'_R = \{(d_1, \ldots, d_n) \in R^n \mid \frac{1}{R} \leq d_j \leq R \quad \forall\, 1 \leq j \leq n-1,\ |d_n| \leq R,\ \sum_{j=1}^{n} d_j \leq -\frac{1}{R}\}$$

and let C_R be such that

(4.35) $$f(C_R) = \min_{x \in B'_R} f(x)$$

LEMMA 2. If $R > M$, then either C_R is a stationary point of f or $C_R = (C_{R1}, \ldots, C_{Rn})$ is such that $C_{Rj} \in [\frac{1}{M}, M]$ $(j = 1, \ldots, n-1)$ and $C_{Rn} = -R$ or $\Sigma_{j=1}^{n} C_{Rj} = -R^{-1}$.

PROOF. If C_R is not a stationary point, it must be a boundary point. Suppose $C_{Rn} > -R$ and $\Sigma_{j=1}^{n} C_{Rj} < -R^{-1}$. Because $C_{Rn} > -R$, it is certainly true that $C_{Rj} < R$ $(j = 1, \ldots, n-1)$ and hence C_R must be a boundary point such that $C_{Rj} = R^{-1}$ for some $j \neq n$. From (3.4), however, it is easily seen that

$$\frac{\partial f}{\partial d_j} = \frac{d_j - \alpha_j}{d_j^2} - \frac{1}{d} = \frac{R^{-1} - \alpha_j}{R^{-2}} - \frac{1}{\sum_{j=1}^{n} C_{Rj}} < -R^2\alpha_1 + 2R < 0$$

* Note that (4.32) once again implies (4.23).

because of (4.31). Consequently, an infinitesimal increase of C_{Rj} would give rise to a smaller value of f without violating the restriction $\Sigma_{j=1}^{n} C_{Rj} \leq -R^{-1}$. Hence, $C_{Rn} = -R$ or $\Sigma_{j=1}^{n} C_{Rj} = -R^{-1}$. Suppose next that $C_{Rn} = -R$ and $C_{Rj} \notin [\frac{1}{M}, M]$ for some $j \neq n$. As in the proof of Lemma 1, we obtain

$$f(C_R) = \sum_{i\neq j,n} (\log C_{Ri} + \frac{\alpha_i}{C_{Ri}}) + \log C_{Rj} + \frac{\alpha_j}{C_{Rj}} + \log(\frac{C_{Rn}}{\sum_{j=1}^{n} C_{Rj}}) + \frac{\alpha_n}{C_{Rn}}$$

$$\geq \sum_{i\neq j,n} (\log \alpha_i + 1) + \log C_{Rj} + \frac{\alpha_j}{C_{Rj}} + \log(\frac{C_{Rn}}{\sum_{j=1}^{n} C_{Rj}}) + \frac{\alpha_n}{C_{Rn}}$$

$$> \sum_{i\neq j,n} (\log \alpha_i + 1) + \log C_{Rj} + \frac{\alpha_j}{C_{Rj}} - \frac{\alpha_n}{R}$$

$$> \sum_{i\neq j,n} (\log \alpha_i + 1) - \frac{\alpha_n}{R} + \max\{f(1, \ldots, 1, -n-1), 0\} +$$

$$+ \alpha_n + \sum_{j=1}^{n} |\log \alpha_j|$$

$$> \max\{f(1, \ldots, 1, -n-1), 0\} + n - 2 > f(1, \ldots, 1, -n-1)$$

where the first strict inequality sign follows from $C_{Rn}(\Sigma_{j=1}^{n} C_{Rj})^{-1} > 1$ and the second one from (4.32). Consequently, C_R can not give rise to a minimum of f and hence $C_{Rj} \in [\frac{1}{M}, M]$ for all $j \neq n$.

Before proceeding, we first prove the following lemma, that will be used in the sequel.

LEMMA 3. Let g be defined by

$$g(x_1, \ldots, x_{n-1}, \theta) = \sum_{i=1}^{n-1} \frac{\alpha_i}{x_i} - \frac{\alpha_n}{\sum_{i=1}^{n-1} x_i + \theta}$$

and let $\alpha_n < (\Sigma_{i=1}^{n-1} \alpha_i^{\frac{1}{2}})^2$. Then $g > 0$ for all $(x_1, \ldots, x_{n-1}, \theta) \in R_+^n$.

PROOF. Choose any point $(x_1^0, \ldots, x_{n-1}^0, \theta^0) \in R_+^n$ and let $\mu = \Sigma_{i=1}^{n-1} x_i^0 + \theta^0$. Define the set L_μ by

$$L_\mu = \{(x_1, \ldots, x_{n-1}) \in R_+^{n-1} | \sum_{i=1}^{n-1} x_i + \theta^0 = \mu\}$$

Applying Lagrange's method, one easily verifies that $g(x_1, \ldots, x_{n-1}, \theta^0)$ attains its minimum in L_μ when

$$x_i = \frac{(\mu - \theta^0)\alpha_i^{\frac{1}{2}}}{(\sum_{j=1}^{n-1} \alpha_j^{\frac{1}{2}})} \qquad i = 1, \ldots, n-1 \tag{4.36}$$

Substituting (4.36) into $g(x_1, \ldots, x_{n-1}, \theta^0)$, we obtain

$$\min_{x \in L_\mu} g(x_1, \ldots, x_{n-1}, \theta^0) = \frac{(\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}})^2}{\mu - \theta^0} - \frac{\alpha_n}{\mu} > 0 \tag{4.37}$$

As $(x_1^0, \ldots, x_{n-1}^0) \in L_\mu$, it certainly holds true that $g(x_1^0, \ldots, x_{n-1}^0, \theta^0) > 0$, which proves the lemma.

In order to complete the proof of Lemma 2, suppose $\Sigma_{j=1}^n C_{Rj} = -R^{-1}$ and $C_{Rj} \notin [\frac{1}{M}, M]$ for some $j \neq n$. From the definition of γ_i (i = 1, ..., n-1) just above (4.30), it follows that

$$f(C_R) = \sum_{i\neq j,n} \left(\log C_{Ri} + \frac{\xi_i}{C_{Ri}}\right) + \log C_{Rj} + \frac{\xi_j}{C_{Rj}} +$$

$$+ \sum_{i=1}^{n-1} \frac{\gamma_i}{C_{Ri}} - \frac{\alpha_n}{\sum_{i=1}^{n-1} C_{Ri} + R^{-1}} + \log \left(\frac{C_{Rn}}{\sum_{i=1}^{n} C_{Ri}}\right)$$

$$> \sum_{i\neq j,n} \left(\log C_{Ri} + \frac{\xi_i}{C_{Ri}}\right) + \log C_{Rj} + \frac{\xi_j}{C_{Rj}} + \log\left(\frac{C_{Rn}}{\sum_{i=1}^{n} C_{Ri}}\right)$$

$$> \sum_{i\neq j,n} (\log \xi_i + 1) + \log C_{Rj} + \frac{\xi_j}{C_{Rj}}$$

$$> \sum_{i\neq j,n} (\log \xi_i + 1) + \max\{f(1, \ldots, 1, -n-1), 0\} +$$

$$+ \sum_{j=1}^{n-1} |\log \xi_j|$$

$$> \max\{f(1, \ldots, 1, -n-1), 0\} + n - 2 > f(1, \ldots, 1, -n-1)$$

where the first inequality sign follows from the application of Lemma 3, the second one from $C_{Rn}(\Sigma_{i=1}^{n} C_{Ri})^{-1} > 1$ and (4.26), and the third one from (4.33). Consequently, C_R can not give rise to a minimum of f and hence $C_{Rj} \in [\frac{1}{M}, M]$ for all $j \neq n$. This completes the proof of Lemma 2.

As in the proof of part A of the proposition, we may construct sequences $\{C_m\}$ and $\{f(C_m)\}$ by letting m run through the set of positive integers larger than M. In proposition 5, we have proved that f can only have one stationary point in S_2. So, once again, there are two possibilities for the sequence $\{C_m\}$, viz.,

(i) from certain N onwards $C_m = x^1$, the unique stationary point of f in S_2, and obviously

$$\inf_{x\in S_2} f(x) = \lim_{m\to\infty} f(C_m) = f(x^1)$$

(ii) from certain N onwards C_m is always a boundary point.

If C_m is a boundary point, it follows from Lemma 2 that $C_{mj} \in [\frac{1}{M}, M]$ $(j = 1, \ldots, n-1)$ and $C_{mn} = -m$ or $\Sigma_{j=1}^{n} C_{mj} = -m^{-1}$. In the latter case, however, we have

$$f(C_m) = \sum_{i=1}^{n-1} \left(\log C_{mi} + \frac{\alpha_i}{C_{mi}}\right) - \frac{\alpha_n}{\sum_{i=1}^{n-1} C_{mi} + m^{-1}} + $$

$$+ \log\left(\sum_{i=1}^{n-1} C_{mi} + m^{-1}\right) + \log m$$

and hence $\lim_{m\to\infty} f(C_m) = \infty$.

Consequently, from certain N onwards a boundary point C_m with $\Sigma_{j=1}^{n} C_{mj} = -m^{-1}$ can not give rise to a minimum of f. So we may restrict our attention to boundary points with $C_{mn} = -m$. In exactly the same way as in the proof of part A of the proposition, it then follows that

$$\inf_{x\in S_2} f(x) = \lim_{m\to\infty} f(C_m) = \sum_{i=1}^{n-1} \log \alpha_i + n - 1$$

In proposition 3, we have proved that f does not have stationary points in S_2, when $\alpha_n \leq \Sigma_{i=1}^{n-1} \alpha_i$. Therefore

$$\inf_{x\in S_2} f(x) = \sum_{i=1}^{n-1} \log \alpha_i + n - 1 \qquad \text{if } \alpha_n \leq \sum_{i=1}^{n-1} \alpha_i \tag{4.38}$$

When $\Sigma_{i=1}^{n-1} \alpha_i < \alpha_n < (\Sigma_{i=1}^{n-1} \alpha_i^{\frac{1}{2}})^2$, we observe as in part A of the proposition that

$$f\left(\alpha_1, \ldots, \alpha_{n-1}, \frac{\alpha_n \sum_{i=1}^{n-1} \alpha_i}{\sum_{i=1}^{n-1} \alpha_i - \alpha_n}\right) < \sum_{i=1}^{n-1} \log \alpha_i + n - 1$$

Consequently, $\Sigma_{i=1}^{n-1} \log \alpha_i + n - 1$ can not be the infimum of f in S_2 and so we have

(4.39) $$\inf_{x\in S_2} f(x) = f(x^1) < \sum_{i=1}^{n-1} \log \alpha_i + n-1 \text{ if } \sum_{i=1}^{n-1} \alpha_i < \alpha_n < \left(\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}}\right)^2$$

Finally, consider the case $\alpha_n = \left(\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}}\right)^2$. For $\lambda > 0$, we have

$$f\left(\lambda\alpha_1^{\frac{1}{2}}, \ldots, \lambda\alpha_{n-1}^{\frac{1}{2}}, -\lambda \sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}} - \lambda^2\right) =$$

$$= \sum_{i=1}^{n-1} \left(\log \lambda\alpha_i^{\frac{1}{2}} + \frac{\alpha_i^{\frac{1}{2}}}{\lambda}\right) - \frac{\alpha_n}{\lambda \sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}} + \lambda^2} + \log\left(\frac{\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}} + \lambda}{\lambda}\right)$$

$$= (n-2)\log \lambda + \sum_{i=1}^{n-1} \log \alpha_i^{\frac{1}{2}} + \log\left(\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}} + \lambda\right) + \frac{\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}}}{\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}} + \lambda}$$

Consequently,

$$\lim_{\lambda\downarrow 0} f\left(\lambda\alpha_1^{\frac{1}{2}}, \ldots, \lambda\alpha_{n-1}^{\frac{1}{2}}, -\lambda \sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}} - \lambda^2\right) = -\infty$$

and so we have

(4.40) $$\inf_{x\in S_2} f(x) = -\infty \qquad \text{if } \alpha_n = \left(\sum_{i=1}^{n-1} \alpha_i^{\frac{1}{2}}\right)^2$$

This completes the proof of proposition 6.

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