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TESTING NONNESTED LINEAR HYPOTHESES II:
SOME INVARIANT EXACT TESTS

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Abstract

In this paper we derive a number of invariant tests for the problem of testing linear hypotheses.
The power functions of these tests are studied and it turns out to depend on the value of $r = \text{rank}(X'Z)$ (where X and Z are the given regressor matrices) whether the tests have level α, are unbiased and possess certain other desirable properties.
The required computations in order to use the tests in practice are given.
We also derive large sample approximations to the critical values and the p-values of the tests.

Contents

1. Introduction
2. The transformed problem and reduction through invariance
3. $\mathcal{C}_\alpha$-class tests
4. The distribution function of the general test statistic
5. Properties of the $\mathcal{C}_\alpha$-class tests
6. The tests ϕ_1, ϕ_2 and ϕ_3
7. The distribution functions of the test statistics
8. The computation of the test statistics and power functions
9. Interpretation of the tests in terms of the original hypotheses
10. A large sample approximation to the critical values and p-values of the tests
11. Required computations for the applications
12. Summary and conclusion

References

Appendices

A. The characteristic function $\psi(t_1, t_2)$

B. The characteristic function and distribution function of a linear combination of mutually independent χ^2 random variables

C. The distribution function of Q_s considered as a function of the parameters

D. A large sample approximation to the distribution function of the test statistics

1. Introduction

This study is the sequel to Bouman [1], where the problem of testing linear hypotheses is reduced through invariance considerations.
The purpose of this paper is to derive reasonable invariant exact tests by building on the above mentioned previous results.
The problem of testing linear hypotheses has the following form:
Let y be a n-vector of observations from a multinormal distribution, i.e.,

$$(1.1) \qquad y \sim n(\mu, \sigma^2 I),$$

where $\mu \in \mathbb{R}^n$ and $\sigma > 0$ are unknown.
On the basis of y we want to test the hypotheses

$$(1.2) \qquad H_0: \mu = X\beta \quad \text{vs.} \quad H_1: \mu = Z\gamma,$$

where X and Z are given nonstochastic regressor matrices of the order $n \times k$ and $n \times \ell$, respectively, and where $\beta \in \mathbb{R}^k$ and $\gamma \in \mathbb{R}^\ell$ are unknown.
In order to make H_0 and H_1 mutually exclusive, we assume that under H_1 the vector $\gamma \in \mathbb{R}^\ell$ is such that $Z\gamma \neq X\beta$ for $\beta \in \mathbb{R}^k$.
Further we assume that $\text{rank}(X) = k < n$ and $\text{rank}(Z) = \ell < n$.
If, in general, $M(A)$ denotes the linear (vector-) subspace (of $\mathbb{R}^n$) spanned by the columnvectors of the $n \times m$ matrix A, the above problem can also be stated as follows.
On the basis of the observable random vector y, the distribution of which is assumed to be $n(\mu, \sigma^2 I)$, we want to test

$$(1.3) \qquad H_0: \mu \in M(X) \quad \text{vs.} \quad H_1: \mu \in M(Z) \backslash M(X).$$

In general, hypotheses H_0 and H_1 are said to be nested when every parameter point under $H_0(H_1)$ is a limit point of $H_1(H_0)$. If this is not the case the hypotheses are called nonnested or separate. When H_0 and H_1 have no common limit points we speak of strictly separate hypotheses.
For our problem this means that the hypotheses are nested if and only if $M(X) \subset M(Z)$ or $M(Z) \subset M(X)$, which occurs if and only if

$$p = \dim(M(X) \cap M(Z)) = \min(k, \ell).$$

In the nested case $M(Z) \subset M(X)$ it seems more realistic to test

$$H_0: \mu \in M(X)\backslash M(Z) \quad \text{vs.} \quad H_1: \mu \in M(Z),$$

instead of (1.3), since $M(Z)\backslash M(X)$ is the empty set when $M(Z) \subset M(X)$. Note that in our problem the hypotheses cannot be strictly separate since the points belonging to the set

$$\{(\mu, \sigma) | \mu = 0, \sigma > 0\}$$

are always common limit points of H_0 and H_1.
As said before this paper is concerned with the derivation of invariant exact tests for the problem of testing linear hypotheses.
In Section 2 some of the main results of Bouman [1] are presented. In Section 3 we construct a whole class of invariant tests. The distribution function of the general test statistic is derived in Section 4. Section 5 is concerned with the question whether the tests posses certain desirable properties, such as having level α and being unbiased. In Section 6 we consider three specific tests and the distribution functions of the corresponding test statistics are derived in Section 7.
The computation of the test statistics from the data (y, X, Z) and the computation of the power functions in terms of the parameters β, γ and σ are considered in Section 8.
Section 9 is concerned with the interpretation of the tests.
When n is large, the critical values and p-values of the tests can easily be approximated as is shown in Section 10.
The required computations are given in Section 11 and a summary of the results is presented in Section 12.
In this paper only a small number of references is made. For more general literature on the topic of testing linear hypotheses we refer to Bouman [1]. Finally, in the Appendices (A) - (D) a number of special results are derived.

2. The transformed problem and reduction through invariance

Before applying invariance considerations to our testing problem it is convenient to transform the original problem into a problem with a more simple structure.
As we saw in the preceding section, we assume that the vector of observations y has a $n(\mu, \sigma^2 I)$ distribution and we want to test

$$(2.1) \qquad H_0: \mu \in M(X) \quad \text{vs.} \quad H_1: \mu \in M(Z)\backslash M(X).$$

Now it is shown in Bouman [1] that this problem can be transformed into a more simple problem through the linear transformation $w = R'y$, where the nonsingular $n \times n$ matrix R has the following structure

$$(2.2) \qquad R = [\underset{(\ell-p)}{R_1} \vdots \underset{(k-p)}{R_2} \vdots \underset{(p)}{R_3} \vdots \underset{(m)}{R_4}]\ (n),$$

where $p = \dim(M(X) \cap M(Z))$ and $m = n+p-k-1$.
The columnvectors of the submatrices R_i form an orthonormal basis for the following linear subspaces V_i:

$$(2.3) \qquad \begin{aligned} V_1 &= M(X)^{\perp} \cap (M(X) + M(Z)) \\ V_2 &= M(Z)^{\perp} \cap (M(X) + M(Z)) \\ V_3 &= M(X) \cap M(Z) \\ V_4 &= M(X)^{\perp} \cap M(Z)^{\perp}, \end{aligned}$$

where, in general, $M(A)^{\perp}$ denotes the orthogonal complement (with respect to $\mathbb{R}^n$) of the linear subspace $M(A)$ and where + denotes the sum of 2 linear subspaces.
The above subspaces satisfy:

$$
(2.4) \qquad \begin{aligned}
&\dim(V_1) = \ell-p \\
&\dim(V_2) = k-p \\
&\dim(V_3) = p \\
&\dim(V_4) = m \\
&V_i \cap V_j = \{0\},\ i \neq j \\
&V_1 \oplus V_2 \oplus V_3 \oplus V_4 = \mathbb{R}^n,
\end{aligned}
$$

where 0 is the null vector and where the symbol $\oplus$ denotes the direct sum of 2 linear subspaces.
From $w = R'y$ and $y \sim n(\mu, \sigma^2 I)$ it easily follows that

$$
(2.5) \qquad w \sim n(R'\mu, \sigma^2 R'R).
$$

If $\theta = R'\mu$, $w_i = R_i'y$, $\theta_i = R_i'\mu$, $i = 1, 2, 3, 4$, it is not difficult to see that $w' = (w_1' \ \ w_2' \ \ w_3' \ \ w_4')$, $\theta' = (\theta_1' \ \ \theta_2' \ \ \theta_3' \ \ \theta_4')$ and that we always have

$$
(2.6) \qquad \begin{aligned}
&\theta_4 = 0 \\
&\Omega = R'R = \begin{bmatrix} I & D & 0 & 0 \\ D' & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},
\end{aligned}
$$

where $D = R_1'R_2$.
Moreover it can be shown that $\theta_1 = 0$ if and only if $\mu \in M(X)$ and also $\theta_2 = 0$ if and only if $\mu \in M(Z)$.
Hence, it is seen from (2.1), (2.5) and (2.6) that the original problem (2.1) is equivalent to the following transformed problem:
On the basis of the vector of observations w which has a $n(\theta, \sigma^2\Omega)$ distribution, we want to test

$$
(2.7) \qquad H_0: \theta_1 = 0 \quad \text{vs.} \quad H_1: \theta_1 \neq 0,\ \theta_2 = 0.
$$

It should be observed that there exists a whole class of matrices R of the above type which give rise to a transformed problem of the form

described above. Since for our purposes it does not matter which particular matrix R from this class is chosen, we extend the above model to the situation where R is supposed to be an unknown matrix. However we always know that $D = R_1'R_2$ where the columnvectors of R_1 and R_2 form an orthonormal basis for the subspaces V_1 and V_2, respectively.
Now it is not difficult to verify that the transformed problem remains invariant under the following group of transformations of the sample space (the space of w) onto itself:

$$(2.8) \qquad G: w \to g(w) = cHw + a,$$

for all $c \in \mathbb{R}^1$, $c \neq 0$, all vectors $a' = (0' \quad 0' \quad a_3' \quad 0')$ with $a_3 \in \mathbb{R}^p$ and all orthogonal matrices H of the form

$$(2.9) \qquad H = \begin{bmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{bmatrix},$$

where H_1 is an orthogonal $(\ell-p)\times(\ell-p)$ matrix, H_2 is an orthogonal $(k-p)\times(k-p)$ matrix, H_3 is an orthogonal $p\times p$ matrix and H_4 is an orthogonal $m\times m$ matrix.
The transformations (2.8) can be interpreted as changes of the coordinate system in which the data (w) are expressed. When a problem is independent of the particular coordinate system chosen, it is natural to restrict attention to tests which satisfy the same property, since otherwise the acceptance or rejection of the hypothesis under consideration would depend on the choice of the coordinates, which is quite arbitrary and has no bearing on the problem.
That is, we restrict attention to tests (functions of w) which are invariant with respect to (2.8):

$$(2.10) \qquad \phi(g(w)) = \phi(w) \text{ for all } g \in G \text{ and } w \in \mathbb{R}^n,$$

where ϕ is a critical function,i.e., $\phi(w)$ is a statistic and $0 \leq \phi(w) \leq 1$ for all $w \in \mathbb{R}^n$.*)

*) For any value of w a test or critical function ϕ specifies the probability of rejecting H_0 when the sample outcome w is observed.

Now a test ϕ is invariant under G (i.e., satisfies (2.10)) if and only if ϕ is a function of a maximal invariant statistic $T = t(w)$, i.e., $\phi(w) = \psi(t(w))$, where ψ is a critical function.
A statistic $T = t(w)$ is maximal invariant if and only if

a) $t(g(w)) = t(w)$ for all $g \in G$ and $w \in \mathbb{R}^n$.
b) $t(w_*) = t(w)$ implies that $w_* = g(w)$ for some $g \in G$.

It follows that the class of invariant tests can be represented by $\phi(T)$, where ϕ is a critical function and $T = t(w)$ is a maximal invariant.
As is shown in Bouman [1], when

$$T_1 = t_1(w) = \frac{w_1'w_1}{w_4'w_4}$$

(2.11)

$$T_2 = t_2(w) = \frac{w_2'w_2}{w_4'w_4}$$

the statistic

(2.12) $$T = t(w) = (t_1(w), t_2(w)) = (T_1, T_2)$$

is maximal invariant under G.
The above discussion shows that the principle of invariance reduces the sample space $\mathbb{R}^n$ (the space of w) to the space of T, which is a subspace of $\mathbb{R}^2$ since $T_1 \geq 0$ and $T_2 \geq 0$.
Usually, invariance not only reduces the sample space but also the parameter space since, as is typically the case, the probability distribution of the maximal invariant depends only on a function of the parameters. In order to see this for our problem we consider the probability distribution of T.
We shall use the fact that T is invariant under the transformations $g \in G: w \to g(w) = cHw + a$, with c, a and H as indicated above (see (2.8) and (2.9)).
Now it can be proved, see Bouman [1], that for any D there exists an orthogonal matrix of the type H, say

$$K = \begin{bmatrix} K_1 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 \\ 0 & 0 & K_3 & 0 \\ 0 & 0 & 0 & K_4 \end{bmatrix},$$

with arbitrary orthogonal submatrices K_3 and K_4 and orthogonal submatrices K_1 and K_2 depending on D such that

(2.13) $\quad K_1DK_2' = C,$

where the $(\ell-p) \times (k-p)$ matrix C has the form

(2.14) $$C = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\rho_1}I_{(m_1)} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sqrt{\rho_M}I_{(m_M)} \end{bmatrix} \begin{matrix} (\ell-r) \\ (m_1) \\ \vdots \\ (m_M) \end{matrix}$$

$$\qquad (k-r) \quad (m_1) \quad \cdots \quad (m_M)$$

Here $\rho_1, \rho_2, \ldots, \rho_M$ are the M different eigenvalues with $0 < \rho_j < 1$ and multiplicities $m_1, m_2, \ldots, m_M$ of the $k\times k$ matrix $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$ (or equivalently, of the $\ell\times\ell$ matrix $(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z$) and r is defined by $r = p + \sum_{j=1}^{M} m_j$.

It can easily be seen that $r = \text{rank}(X'Z)$, $0 \leq p \leq r \leq \min(k, \ell)$ and that the hypotheses in (2.1) or (2.7) are nested if and only if $p = \min(k, \ell)$.

Let $c = \frac{1}{\sigma}$, $a = -\frac{1}{\sigma}Kb$ with $b' = (0' \;\; 0' \;\; \theta_3' \;\; 0')$ and $H = K$, then $u = g(w) = cHw + a = \frac{1}{\sigma}K(w-b)$ or equivalently $u' = (u_1' \;\; u_2' \;\; u_3' \;\; u_4')$ with

$$u_1 = \frac{1}{\sigma}K_1w_1$$

$$u_2 = \frac{1}{\sigma}K_2w_2$$

$$u_3 = \frac{1}{\sigma}K_3(w_3 - \theta_3)$$

$$u_4 = \frac{1}{\sigma}K_4w_4$$

Since T is invariant under G we have

$$(2.15) \qquad T_1 = \frac{u_1'u_1}{u_4'u_4}, \qquad T_2 = \frac{u_2'u_2}{u_4'u_4}$$

and from $w \sim n(\theta, \sigma^2\Omega)$ it follows that

$$(2.16) \qquad u \sim n(\delta, \Gamma),$$

where

$$\delta' = (\delta_1' \quad \delta_2' \quad 0' \quad 0')$$

$$(2.17) \qquad \delta_1 = \frac{1}{\sigma} K_1\theta_1, \qquad \delta_2 = \frac{1}{\sigma} K_2\theta_2$$

$$\Gamma = \frac{1}{\sigma} K(\sigma^2\Omega)(\frac{1}{\sigma} K)' = K\Omega K' = \begin{bmatrix} I & C & 0 & 0 \\ C' & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

This shows that the probability distribution of T depends on the unknown parameters through δ_1 and δ_2. Note that the hypotheses in terms of δ become

$$(2.18) \qquad H_0\colon \delta_1 = 0 \quad \text{vs.} \quad H_1\colon \delta_1 \neq 0,\ \delta_2 = 0.$$

When the subvectors u_1, u_2, δ_1 and δ_2 are partitioned as follows

$$u_1 = \begin{bmatrix} u_{10} \\ u_{11} \\ \vdots \\ u_{1M} \end{bmatrix} \begin{matrix} (\ell-r) \\ (m_1) \\ \vdots \\ (m_M) \end{matrix}, \quad u_2 = \begin{bmatrix} u_{20} \\ u_{21} \\ \vdots \\ u_{2M} \end{bmatrix} \begin{matrix} (k-r) \\ (m_1) \\ \vdots \\ (m_M) \end{matrix},$$

$$\delta_1 = \begin{bmatrix} \delta_{10} \\ \delta_{11} \\ \vdots \\ \delta_{1M} \end{bmatrix} \begin{matrix} (\ell-r) \\ (m_1) \\ \vdots \\ (m_M) \end{matrix}, \quad \delta_2 = \begin{bmatrix} \delta_{20} \\ \delta_{21} \\ \vdots \\ \delta_{2M} \end{bmatrix} \begin{matrix} (k-r) \\ (m_1) \\ \vdots \\ (m_M) \end{matrix},$$

it can be seen from (2.16), (2.17) and (2.14) that u_{10}, u_{20}, (u_{11}, u_{21}), ..., (u_{1M}, u_{2M}) and u_4 are mutually stochastically independent and have the following probability distributions

(2.19)

$$u_{10} \sim n(\delta_{10}, I_{(\ell-r)})$$

$$u_{20} \sim n(\delta_{20}, I_{(k-r)})$$

$$\begin{bmatrix} u_{1j} \\ u_{2j} \end{bmatrix} \sim n\left(\begin{bmatrix} \delta_{1j} \\ \delta_{2j} \end{bmatrix}, \begin{bmatrix} I_{(m_j)} & \sqrt{\rho_j} I_{(m_j)} \\ \sqrt{\rho_j} I_{(m_j)} & I_{(m_j)} \end{bmatrix} \right), \; j = 1, 2, \ldots, M$$

$$u_4 \sim n(0, I_{(m)}).$$

Next we define

$$U_{1j} = u'_{1j} u_{1j}$$

$$U_{2j} = u'_{2j} u_{2j}$$

(2.20) $$U_1 = u'_1 u_1$$

$$U_2 = u'_2 u_2$$

$$U_4 = u'_4 u_4$$

Then it easily follows that

$$T_1 = \frac{U_1}{U_4}$$

(2.21)

$$T_2 = \frac{U_2}{U_4},$$

with $U_1 = \sum_{j=0}^{M} U_{1j}$, $U_2 = \sum_{j=0}^{M} U_{2j}$ and also that
U_{10}, U_{20}, (U_{11}, U_{21}), ..., (U_{1M}, U_{2M}), U_4 are mutually independent.

The above results enable us to characterize the probability distribution of T.
The latter distribution is uniquely determined by the distribution function $F(t_1, t_2)$ as well as by the characteristic function $\psi(t_1, t_2)$, which are defined by

$$F(t_1, t_2) = P(T_1 \leq t_1, T_2 \leq t_2)$$

$$\psi(t_1, t_2) = E(e^{it_1T_1 + it_2T_2}),$$

where i denotes the imaginary unit.
Since in our case it is more easy to find $\psi(t_1, t_2)$ than $F(t_1, t_2)$, we shall characterize the probability distribution of T by $\psi(t_1, t_2)$.
We get

$$\psi(t_1, t_2) = E(e^{it_1T_1+it_2T_2}) = E\{E(e^{it_1T_1+it_2T_2}|U_4)\}$$

$$= E\{E(e^{it_1U_1/U_4+it_2U_2/U_4} \mid U_4)\}$$

$$= \int_0^{\infty} E(e^{it_1U_1/u+it_2U_2/u} \mid U_4 = u)\, f(u)du,$$

where f(u) denotes the probability density function of the random variable U_4.
From (2.19) it follows that $U_4 = u_4'u_4 \sim \chi^2(m)$ and this shows that:

$$f(u) = \frac{u^{\frac{m}{2}-1}\, e^{-\frac{1}{2}u}}{\Gamma(\frac{m}{2})\, 2^{\frac{m}{2}}}, \quad u > 0 \tag{2.22}$$

Further we know that (U_1, U_2) and U_4 are independent which implies that

$$E(e^{it_1U_1/u + it_2U_2/u} \mid U_4 = u) = E(e^{it_1U_1/u + it_2U_2/u})$$

and therefore we have

$$\text{(2.23)} \qquad \psi(t_1, t_2) = \int_0^\infty E(e^{i(\frac{t_1}{u})U_1 + i(\frac{t_2}{u})U_2})\, f(u)du.$$

It remains to find $E(e^{it_1U_1+it_2U_2})$, i.e., the characteristic function of (U_1, U_2).

From the mutual independence of $(U_{10}, U_{20}), (U_{11}, U_{21}), \ldots, (U_{1M}, U_{2M})$ it follows that

$$E(e^{it_1U_1+it_2U_2}) = E(e^{it_1 \sum_{j=0}^{M} U_{1j}+it_2 \sum_{j=0}^{M} U_{2j}})$$

$$= E(\prod_{j=0}^{M} e^{it_1U_{1j}+it_2U_{2j}}) = \prod_{j=0}^{M} E(e^{it_1U_{1j}+it_2U_{2j}}) = \sum_{j=0}^{M} \Psi_j(t_1, t_2),$$

where

$$\text{(2.24)} \qquad \Psi_j(t_1, t_2) = E(e^{it_1U_{1j}+it_2U_{2j}}),$$

$j = 0, 1, \ldots, M.$

Since (2.19) implies that $U_{10} = u_{10}'u_{10} \sim \chi^2(\ell-r, v_{10})$, $U_{20} = u_{20}'u_{20} \sim \chi^2(k-r, v_{20})$, where $v_{10} = \delta_{10}'\delta_{10}$ and $v_{20} = \delta_{20}'\delta_{20}$, and since U_{10} and U_{20} are independent we get

$$\text{(2.25)} \qquad \Psi_0(t_1, t_2) = (1-2it_1)^{-\frac{\ell-r}{2}} \exp\{\frac{it_1v_{10}}{1-2it_1}\}\, (1-2it_2)^{-\frac{k-r}{2}} \exp\{\frac{it_2v_{20}}{1-2it_2}\}$$

When (u_{1j}, u_{2j}) have the joint probability distribution as given in (2.19), it is shown in Appendix A that the joint characteristic function of $U_{1j} = u_{1j}'u_{1j}$ and $U_{2j} = u_{2j}'u_{2j}$ is of the following form

$$\text{(2.26)} \qquad \Psi_j(t_1, t_2) = (1 - 2it_1 - 2it_2 - 4(1 - \rho_j)t_1t_2)^{-\frac{m_j}{2}}$$

$$\exp\{\frac{(it_1 + 2t_1t_2)v_{1j} + (it_2 + 2t_1t_2)v_{2j}}{1 - 2it_1 - 2it_2 - 4(1-\rho_j)t_1t_2}\},$$

$j = 1, 2, \ldots, M$, where $v_{1j} = \delta'_{1j}\delta_{1j}$ and $v_{2j} = \delta'_{2j}\delta_{2j}$.
Hence, (2.23) can be written as

$$(2.27) \qquad \psi(t_1, t_2) = \int_0^\infty \prod_{j=0}^{M} \Psi_j\left(\frac{t_1}{u}, \frac{t_2}{u}\right) f(u)du,$$

where $f(u)$ and $\Psi_j(t_1, t_2)$ are as given in (2.22) and (2.25), (2.26), respectively.
Since the characteristic function $\psi(t_1, t_2)$ of T depends on the parameters through

$$(2.28) \qquad v = v(\delta) = (v_1, v_2),$$

where $v_1 = (v_{10}, v_{11}, \ldots, v_{1M})$, $v_2 = (v_{20}, v_{21}, \ldots, v_{2M})$ and

$$(2.29) \qquad \begin{aligned} v_{1j} &= \delta'_{1j}\delta_{1j} \\ v_{2j} &= \delta'_{2j}\delta_{2j}, \end{aligned}$$

$j = 0, 1, \ldots, M$, it follows that the probability distribution of T has the same property.
In other words invariance reduces the parameter space to the space of v.
It should be noted from (2.29) that we always have $v \geq 0$.
Further it is een from (2.18) that we can write the hypotheses in terms of v as follows

$$(2.30) \qquad H_0: v_1 = 0 \quad \text{vs.} \quad H_1: v_1 \neq 0, v_2 = 0.$$

The above discussion shows that the principle of invariance reduces the testing problem (2.7) to the problem of testing (2.30) on the basis of the observation $T = (T_1, T_2)$, which is known to have the characteristic function $\psi(t_1, t_2)$.
If we define

$$(2.31) \qquad \begin{aligned} \omega_0 &= \{v | v = (v_1, v_2), v_1 = 0, v_2 \geq 0\} \\ \omega_1 &= \{v | v = (v_1, v_2), v_1 \geq 0, v_1 \neq 0, v_2 = 0\}, \end{aligned}$$

the hypotheses (2.30) of the reduced problem can be rewritten as

$$(2.32) \qquad H_0\colon v \in \omega_0 \quad \text{vs.} \quad H_1\colon v \in \omega_1.$$

Note that the point $v = 0$ (which belongs to ω_0) forms the boundary between H_0 and H_1 and that $v = 0$ if and only if the original parameter point (μ, σ) in (2.1) satisfies $\mu \in M(X) \cap M(Z)$ and $\sigma > 0$.
Having reduced the problem through invariance considerations we now try to find the "best" test among the invariant tests, or equivalently, we shall try to find the "best" test for the reduced problem.
This problem will be considered in the next section and as a measure of quality of a test we shall use the power function of the test. In our case, when $\phi(T)$ is an invariant test, the power function of the test is defined by

$$(2.33) \qquad \pi(\phi, v) = E_v(\phi(T)), \; v \in \omega_0 \cup \omega_1.$$

Note that, since any test or critical function ϕ specifies the conditional probability of rejecting H_0 given that $T = t$, i.e.,

$$\phi(t) = P(H_0 \text{ is rejected} \mid T = t),$$

we have

$$\pi(\phi, v) = E_v(\phi(T)) = E_v(P(H_0 \text{ is rejected} \mid T))$$
$$= P_v(H_0 \text{ is rejected}).$$

Since the rejection of H_0 is a wrong decision for $v \in \omega_0$ and a correct decision when $v \in \omega_1$, it is therefore desirable to find a test which makes $\pi(\phi, v)$ small for $v \in \omega_0$ and large for $v \in \omega_1$.

At the end of this section we consider the special case where the linear hypotheses are nested.
This situation occurs if $M(X) \subset M(Z)$ or $M(Z) \subset M(X)$, or equivalently, if $p = \min(k, \ell)$. We have 3 subcases:
a) $M(X) = M(Z)$, or equivalently, $p = k = \ell$
b) $M(X) \subset M(Z)$ and $M(X) \neq M(Z)$, which corresponds to $p = k < \ell$
c) $M(Z) \subset M(X)$ and $M(X) \neq M(Z)$, which is equivalent to $p = \ell < k$.

Obviously, case a) is trivial. It is easily seen that w, θ and Ω become equal to

$$w' = (w_3' \;\; w_4'),\; \theta' = (\theta_3' \;\; 0'),\; \Omega = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

It follows that there are no maximal invariants and the only invariant functions are the constant functions.
In case b) we get:

$$w' = (w_1' \;\; w_3' \;\; w_4'),\; \theta' = (\theta_1' \;\; \theta_3' \;\; 0'),\; \Omega = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and the hypotheses (2.7) become

$$H_0\colon \theta_1 = 0 \quad \text{vs.} \quad H_1\colon \theta_1 \neq 0,$$

which are obviously nested hypotheses.
As a maximal invariant we obtain

$$T = t_1(w) = T_1 = \frac{w_1'w_1}{w_4'w_4}$$

and v becomes

$$v = v_1 = v_{10} = \delta_{10}'\delta_{10}.$$

Hence the spaces ω_0 and ω_1 in the reduced problem take the form

$$\omega_0 = \{v | v = v_{10} = 0\},\; \omega_1 = \{v | v = v_{10} > 0\}.$$

Since in the case c) it is seen that $M(Z)\backslash M(X)$ is the empty set, we have to modify the original hypotheses in (2.1). It is natural to take

$$\text{(2.34)} \qquad H_0\colon \mu \in M(X)\backslash M(Z) \quad \text{vs.} \quad H_1\colon \mu \in M(Z).$$

For the transformed problem we get

$$w' = (w_2' \;\; w_3' \;\; w_4'),\; \theta' = (\theta_2' \;\; \theta_3' \;\; 0'),\; \Omega = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and the hypotheses (2.7) now become (see (2.34))

$$H_0\colon\ \theta_2 \neq 0 \quad \text{vs.} \quad H_1\colon\ \theta_2 = 0,$$

which are again nested hypotheses.
The maximal invariant takes the form

$$T = t_2(w) = T_2 = \frac{w_2'w_2}{w_4'w_4}$$

and for v we obtain

$$v = v_2 = v_{20} = \delta_{20}'\delta_{20},$$

which shows that ω_0 and ω_1 can be written as

$$\omega_0 = \{v \mid v = v_{20} > 0\},\ \omega_1 = \{v \mid v = v_{20} = 0\}.$$

3. $\mathcal{C}_\alpha$-class tests

A test ϕ^* would be a uniformly best test for the reduced problem if for any other test ϕ we have:

$$\text{(3.1)} \quad \begin{aligned} &\pi(\phi^*, v) \leq \pi(\phi, v) \text{ for all } v \in \omega_0 \\ &\pi(\phi^*, v) \geq \pi(\phi, v) \text{ for all } v \in \omega_1. \end{aligned}$$

However, it is well known that, not only for our problem, but in general, such a uniformly best test does not exist.
The usual procudure then is to restrict attention to the subclass of level α tests. A test ϕ is said to have level α if the size of the test is equal to a preassigned significance level α, i.e.,

$$\sup_{\omega_0} \pi(\phi, v) = \alpha.$$

Now we try to find a best test within this restricted class.
A level α test ϕ^* is uniformly most powerful (UMP) for the reduced problem if for any other level α test ϕ we have

$$\text{(3.2)} \quad \pi(\phi^*, v) \geq \pi(\phi, v) \text{ for all } v \in \omega_1.$$

If there exists a UMP level α test ϕ^* for the reduced problem (2.32) in terms of T, we can use the fact that $T = t(w)$ and define the test ψ^* through

$$\psi^*(w) = \phi^*(t(w)).$$

Then the test ψ^* is a UMP invariant level α test for the original problem (2.7) in terms of w.
It is not difficult to show, see Bouman [1], that in the case of nested linear hypotheses ($p = \min(k, \ell)$) there exists a UMP invariant level α test for the problem. This test turns out to be the classical F test. Whether or not there exists a UMP invariant level α test for the nonnested case ($p < \min(k, \ell)$) is an open problem.
In order to find a reasonable test for the nonnested case we shall narrow the class of tests still further.

First of all we always require that the tests have level α, i.e.,

(i) $$\sup_{\omega_0} \pi(\phi, v) = \alpha.$$

That is, we restrict attention to tests ϕ, whose probability of a type I error is at most equal to a preassigned level α. Tests with this property are often called exact tests.

Another useful criterion is unbiasedness.

A test ϕ is said to be unbiased if

(ii) $$\sup_{\omega_0} \pi(\phi, v) \leq \inf_{\omega_1} \pi(\phi, v).$$

This is a reasonable requirement, since if a test is biased, at least for some parameter points under H_1 the probability of accepting H_0 is larger than at some parameter points under H_0.

In general, when H_0 and H_1 have common boundary points, a test is called α-similar if the power function has the value α at all these boundary points.

Since in our case $v = 0$ is the only common boundary point of ω_0 and ω_1, a test ϕ is α-similar if

(3.3) $$\pi(\phi, 0) = \alpha.$$

When the power function $\pi(\phi, v)$ of a test ϕ is continuous in v the conditions (i) and (ii) imply property (3.3). In other words, in that case any unbiased level α test is α-similar.

This can be seen as follows. Since $0 \in \omega_0$, we have $\sup_{\omega_0} \pi(\phi, v) \geq \pi(\phi, 0)$.
If $\pi(\phi, v)$ is continuous in v it follows that

$$\inf_{\omega_1} \pi(\phi, v) = \inf_{\bar{\omega}_1} \pi(\phi, v),$$

where $\bar{\omega}_1$ is the closure of ω_1, i.e., $\bar{\omega}_1 = \omega_1 \cup \{0\}$. From $0 \in \bar{\omega}_1$ we get $\inf_{\omega_1} \pi(\phi, v) = \inf_{\bar{\omega}_1} \pi(\phi, v) \leq \pi(\phi, 0)$ and therefore

(3.4) $$\inf_{\omega_1} \pi(\phi, v) \leq \pi(\phi, 0) \leq \sup_{\omega_0} \pi(\phi, v).$$

Suppose that ϕ satisfies (i) and (ii), then we have

(3.5) $$\alpha = \sup_{\omega_0} \pi(\phi, v) \leq \inf_{\omega_1} \pi(\phi, v).$$

If we combine (3.4) and (3.5) we obtain

$$\sup_{\omega_0} \pi(\phi, v) = \inf_{\omega_1} \pi(\phi, v) = \pi(\phi, 0) = \alpha,$$

which shows the stated property.
So in our search for unbiased level α tests we can restrict attention to the class of α-similar tests, provided that the power functions are continuous.
From the practical point of view it is important to consider tests with computable power functions, since the value of the power function at parameter points of interest gives us an idea about the quality of the test.
We therefore require that

(iii) $\pi(\phi, v) = E_v(\phi(T))$ is numerically computable for any $v \in \omega_0 \cup \omega_1$.

Tests with this property will shortly be called computable tests.
At this point it should be observed that imposing the restrictions (i), (ii) and (iii) does not automatically yield a satisfactory test. There are several tests which satisfy these requirements, but as we shall see below some of them do not make much sense. In other words not every computable, α-similar test which satisfies (i) and (ii) is a good test.
For instance, consider the test $\phi(T) \equiv \alpha$. This is a purely randomized test which rejects H_0 with probability α regardless of the observation $T = (T_1, T_2)$.
It will be clear that in general this is not a good test. However, since $\pi(\phi, v) = \alpha$ for all $v \in \omega_0 \cup \omega_1$, it is seen that the test is computable.
Moreover, from $\pi(\phi, 0) = \alpha$ it follows that the test is α-similar.
Obviously we have

$$\alpha = \sup_{\omega_0} \pi(\phi, v) \leq \inf_{\omega_1} \pi(\phi, v),$$

which shows that the test is unbiased with level α. In other words the α-similar test $\phi(T) \equiv \alpha$ satisfies the properties (i), (ii) and (iii).
In the trivial case of nested models, i.e., when $p = k = \ell$ ($M(X) = M(Z)$), this test is UMP, but in all other situations the test $\phi(T) \equiv \alpha$ is a bad test.
As a second example consider the test

$$(3.6) \qquad \phi(T) = 1 \text{ if } T_1 \frac{m}{\ell-p} \geq c_1$$

$$= 0 \text{ if } T_1 \frac{m}{\ell-p} < c_1,$$

where c_1 is chosen in such a way that $\pi(\phi, 0) = \alpha$ and where we assume that $\ell > p$, since otherwise the test is not defined.
Note that this is a purely nonrandomized test which rejects H_0 if $T_1 \frac{m}{\ell-p} \geq c_1$.
By construction the test is α-similar and we shall show that it also satisfies the properties (i), (ii) and (iii).
From Section 2, see (2.15), we know that

$$(3.7) \qquad T_1 = \frac{u_1'u_1}{u_4'u_4}$$

and from (2.16) and (2.17) it can be concluded that $u_1'u_1$ and $u_4'u_4$ are independent random variables with the following distributions

$$(3.8) \qquad \begin{aligned} u_1'u_1 &\sim \chi^2(\ell-p,\ d_1) \\ u_4'u_4 &\sim \chi^2(m), \end{aligned}$$

where

$$(3.9) \qquad d_1 = \delta_1'\delta_1 = \sum_{j=0}^{M} v_{1j}.$$

Hence, it follows that $T_1 \frac{m}{\ell-p}$ has a noncentral F distribution with $\ell-p$ and m degrees of freedom and noncentrality parameter d_1, i.e.,

$$(3.10) \qquad T_1 \frac{m}{\ell-p} \sim F(\ell-p,\ m,\ d_1).$$

Since under H_0: $v \in \omega_0$ we have $v_{1j} = 0$, $j = 0, 1, \ldots, M$, it is seen that $d_1 = 0$ under H_0 and this shows that

(3.11) $\quad T_1 \frac{m}{\ell-p} \sim F(\ell-p, m)$ under H_0,

i.e., under H_0 the random variable $T_1 \frac{m}{\ell-p}$ has a central F distribution with ℓ-p and m degrees of freedom.
The power function of ϕ becomes

$$(3.12) \qquad \pi(\phi, v) = E_v(\phi(T)) = P_v(T_1 \frac{m}{\ell-p} \geq c_1)$$
$$= 1 - G(c_1; \ell-p, m, d_1),$$

for any $v \in \omega_0 \cup \omega_1$, where $G(x; r_1, r_2, \lambda)$ is the distribution function of the $F(r_1, r_2, \lambda)$ distribution.
Since $v = 0$ implies that $d_1 = 0$, it follows that $\pi(\phi, 0) = \alpha$ is equivalent to

$$G(c_1; \ell-p, m, 0) = 1-\alpha,$$

which shows that the critical value c_1 can be found from the table of the $F(\ell-p, m)$ distribution. Once c_1 is known we can compute $\pi(\phi, v)$ in (3.12) for any $v \in \omega_0 \cup \omega_1$ from tables of the $F(\ell-p, m, d_1)$ distribution. Hence, the test ϕ is computable.
Further it is easily seen from the fact that $d_1 = 0$ under H_0 that

$$(3.13) \qquad \pi(\phi, v) = 1-G(c_1; \ell-p, m, 0) = \alpha \text{ for all } v \in \omega_0.$$

Next consider H_1: $v \in \omega_1$. Then we have $v_{1j} \geq 0$, $j = 0, 1, \ldots, M$ and at least one $v_{1j} > 0$. This implies that $d_1 > 0$ under H_1.
Now it can be shown, see Lehmann [7], p. 316, that $G(x; r_1, r_2, \lambda)$ is a strictly decreasing function of $\lambda \geq 0$ for any x. It follows that $1-G(c_1; \ell-p, m, d_1)$ is strictly increasing in $d_1 \geq 0$ and therefore we have

$$(3.14) \qquad \pi(\phi, v) = 1-G(c_1; \ell-p, m, d_1) > \alpha \text{ for all } v \in \omega_1.$$

From (3.13) and (3.14) it is easily seen that

$$\alpha = \sup_{\omega_0} \pi(\phi, v) \leq \inf_{\omega_1} \pi(\phi, v),$$

which shows that ϕ is unbiased with level α.
Hence, the α-similar test ϕ as defined in (3.6) satisfies the properties (i), (ii) and (iii).
It is not difficult to verify that this test is UMP for the nested case with $M(X) \subset M(Z)$ and $M(X) \neq M(Z)$, i.e., when $p = k < \ell$. For the other (nonnested) situations it seems to be a reasonable test, especially since it has the desirable properties that under H_1:

$$\pi(\phi, v) > \alpha \text{ for all } v \in \omega_1$$

and

$$\pi(\phi, v) \to 1 \text{ if at least one of the } v_{1j}\text{'s} \to \infty.$$

The latter property is equivalent to

$$\lim_{d_1 \to \infty} \pi(\phi, v) = 1,$$

since $d_1 \to \infty$ if and only if at least one of the v_{1j}'s $\to \infty$.
Note that $d_1 = \sum_{j=0}^{M} v_{1j}$ can be considered as a measure of distance of a point $v \in \omega_1$ to H_0.

On the other hand, in the nonnested cases (i.e., when $p < \min(k, \ell)$) the above test has the drawback that under H_0

$$\pi(\phi, v) = \alpha \text{ for all } v \in \omega_0$$

and

$$\pi(\phi, v) \not\to 0 \text{ if at least one of the } v_{2j}\text{'s} \to \infty.$$

In a way, this is what could be expected, since the test completely ignores the information contained in the variable T_2.

As a final example we consider the test

$$(3.15)\qquad \begin{aligned}\phi(T) &= 1 \text{ if } T_2 \frac{m}{k-p} \leq c_2\\ &= 0 \text{ if } T_2 \frac{m}{k-p} > c_2,\end{aligned}$$

provided that $k > p$, where c_2 is chosen in such a way that $\pi(\phi, 0) = \alpha$. Again this is a purely nonrandomized test with critical region $T_2 \frac{m}{k-p} \leq c_2$.

Obviously, the test is α-similar.

In the same way as before it follows from the results of Section 2 that

$$(3.16)\qquad T_2 \frac{m}{k-p} \sim F(k-p, m, d_0),$$

where

$$(3.17)\qquad d_0 = \delta_2'\delta_2 = \sum_{j=0}^{M} v_{2j}.$$

Note that d_0 can be considered as a measure of the distance of a point $v \in \omega_0$ to H_1. Under H_0 we have $v_{2j} \geq 0$, $j = 0, 1, \ldots, M$ and it is seen that $d_0 \geq 0$ under H_0. Since under H_1 we have $v_{2j} = 0$, $j = 0, 1, \ldots, M$, it follows that $d_0 = 0$ under H_1 and this shows that

$$(3.18)\qquad T_2 \frac{m}{k-p} \sim F(k-p, m) \text{ under } H_1.$$

The power function of ϕ becomes

$$(3.19)\qquad \begin{aligned}\pi(\phi, v) = E_v(\phi(T)) &= P_v(T_2 \frac{m}{k-p} \leq c_2)\\ &= G(c_2; k-p, m, d_0),\end{aligned}$$

for any $v \in \omega_0 \cup \omega_1$.

Now $v = 0$ implies that $d_0 = 0$ and it is seen that $\pi(\phi, 0) = \alpha$ is equivalent to

$$G(c_2; k-p, m, 0) = \alpha,$$

showing that the critical value c_2 can be found from the table of the $F(k-p, m)$ distribution.
When c_2 is known we can compute $\pi(\phi, v)$ in (3.19) for any $v \in \omega_0 \cup \omega_1$ from the tables of the $F(k-p, m, d_0)$ distribution. Hence, the test ϕ is computable.
Since $G(c_2; k-p, m, d_0)$ is strictly decreasing in $d_0 \geq 0$, it is easily seen that

$$(3.20) \qquad \pi(\phi, v) = G(c_2; k-p, m, d_0) \leq \alpha \text{ for all } v \in \omega_0.$$

Further, the fact that $d_0 = 0$ under H_1 implies that

$$(3.21) \qquad \pi(\phi, v) = G(c_2; k-p, m, 0) = \alpha \text{ for all } v \in \omega_1.$$

Together with $\pi(\phi, 0) = \alpha$ it follows from (3.20) and (3.21) that

$$\alpha = \sup_{\omega_0} \pi(\phi, v) \leq \inf_{\omega_1} \pi(\phi, v),$$

i.e., the test is unbiased with level α.
Hence, we have shown that the α-similar test ϕ as defined in (3.15) satisfies the properties (i), (ii) and (iii).
This test turns out to be UMP for the nested case with $M(Z) \subset M(X)$ and $M(X) \neq M(Z)$, i.e., when $p = \ell < k$.
Although the test has the desirable properties that under H_0

$$\pi(\phi, v) < \alpha \text{ for all } v \in \omega_0 \backslash \{0\}$$

and

$$\lim_{d_0 \to \infty} \pi(\phi, v) \to 0,$$

it is not very attractive for the nonnested situations, since under H_1

$$\pi(\phi, v) = \alpha \text{ for all } v \in \omega_1$$

and

$$\pi(\phi, v) \not\to 1 \text{ if } d_1 \to \infty.$$

Here it should be noted that the test ϕ as given in (3.15) ignores the information contained in T_1.
In order to exclude tests as considered in the above examples for the problem of testing nonnested linear hypotheses, we impose the following additional requirement

(iv) $\pi(\phi, v) \to 0$ if $d_0 \to \infty$ and $\pi(\phi, v) \to 1$ if $d_1 \to \infty$.

Such a test will be called usable.
We recall that $d_0 = \sum_{j=0}^{M} v_{2j}$ and $d_1 = \sum_{j=0}^{M} v_{1j}$ and that d_0 can be thought of the distance of a point $v \in \omega_0$ to ω_1 and d_1 as the distance of a point $v \in \omega_1$ to ω_0.
Further we have

$$
\begin{aligned}
& v \in \omega_0 \text{ if and only if } d_0 \geq 0,\ d_1 = 0 \\
& v \in \omega_1 \text{ if and only if } d_0 = 0,\ d_1 > 0 \\
(3.22) \quad & v = 0 \text{ if and only if } d_0 = 0,\ d_1 = 0 \\
& d_0 \to \infty \text{ if and only if at least one } v_{2j} \to \infty \\
& d_1 \to \infty \text{ if and only if at least one } v_{1j} \to \infty.
\end{aligned}
$$

Another desirable property for α-similar tests is

(v) $\pi(\phi, v) < \alpha$ for all $v \in \omega_0 \setminus \{0\}$ and $\pi(\phi, v) > \alpha$ for all $v \in \omega_1$.

Such tests will be called strictly discriminating.

Now it is not difficult to see that property (v) implies the properties (i) and (ii) for α-similar tests with continuous power function.

The above discussion makes it clear that the "best" we can do in the given situation is to concentrate on α-similar tests which are computable and usable (i.e., satisfy (iii) and (iv)) and then verify whether these tests are strictly discriminating (i.e., satisfy (v)).
For many tests it turns out to be rather difficult to find out whether or not property (v) is satisfied.

Often the following stronger property can be more easily handled:

(vi) For $v \in \omega_0$ the function $\pi(\phi, v)$ is strictly decreasing in each of the variables v_{2j} ($j = 0, 1, \ldots, M$) and for $v \in \omega_1$ the function $\pi(\phi, v)$ is strictly increasing in each of the variables v_{1j} ($j = 0, 1, \ldots, M$).

Obviously, for α-similar tests, property (vi) implies the properties (iv) and (v).
In order to find α-similar tests which are computable and usable we shall investigate the behavior of the observation $T = (T_1, T_2)$ under the hypotheses H_0 and H_1, respectively.
In general, if X is a random variable with probability distribution P_θ, where θ denotes a parameter, we say that X is stochastically strictly increasing in θ if $P_\theta(X > x)$ is a strictly increasing function of θ for any x.
Thus, if X is stochastically increasing in θ, X tends to have larger values as θ increases.
As we saw above T_1 and T_2 have the following probability distributions:

$$T_1 \frac{m}{\ell-p} \sim F(\ell-p, m, d_1)$$

$$T_2 \frac{m}{k-p} \sim F(k-p, m, d_0).$$

We also saw that, if $G(x; r_1, r_2, \lambda)$ is the distribution function of a $F(r_1, r_2, \lambda)$ distribution, $G(x; r_1, r_2, \lambda)$ is strictly decreasing in $\lambda \geq 0$ for any value of x.
This shows that T_1 is stochastically strictly increasing in d_1 and T_2 is stochastically strictly increasing in d_0.
In view of (3.22) this means that under H_0 the random variable T_1 tends to be small, whereas T_1 tends to have larger values under H_1 as d_1 increases. On the other hand, T_2 tends to be small under H_1, but under H_0 the random variable T_2 tends to have larger values as d_0 increases.

Schematically, we have:

Table 1: Behavior of $T = (T_1, T_2)$

	H_0	H_1
T_1	small	large
T_2	large	small

Since a test ϕ is given by specifying the probability $\phi(t)$ that H_0 is rejected for every possible outcome t of T, it is natural to select tests with large values of $\phi(t)$ when T_1 is large and T_2 is small, and with small values of $\phi(t)$ when T_1 is small and T_2 is large.
From the practical point of view it is desirable to have nonrandomized tests and we therefore consider tests of the type given by

$$(3.23) \qquad \phi(T) = 1 \text{ if } S = \frac{a_2 + b_2 T_2}{a_1 + b_1 T_1} \leq c$$

$$= 0 \text{ if } S = \frac{a_2 + b_2 T_2}{a_1 + b_2 T_2} > c,$$

where $a_1 \geq 0$, $a_2 \geq 0$, $b_1 > 0$ and $b_2 > 0$.
That is, we consider tests with critical region $S \leq c$ and it is easily seen from the above scheme that the test statistic S tends to be large under H_0 and small under H_1.
With respect to (3.23) we make the following remarks.
In the first place we note that (3.23) specifies a whole class of tests, since with every choice of a_1, a_2, b_1, b_2 and c there corresponds a test.
Secondly, the reason for defining S as the ratio of two linear functions is that this choice yields computable tests.
In the third place it should be noted that we do not consider tests with $b_1 = 0$ or $b_2 = 0$, because this would mean that we ignore the information contained in T_1 or T_2 (or both).

The restrictions $b_1 > 0$ and $b_2 > 0$ imply that in specifying tests of the type (3.23), we can take without loss of generality $b_1 = b_2 = 1$.
This follows from the fact that the event

$$S = \frac{a_2 + b_2T_2}{a_1 + b_1T_1} \leq c,$$

with $a_1 \geq 0$, $a_2 \geq 0$, $b_1 > 0$ and $b_2 > 0$ is equivalent to

$$S^* = \frac{a_2^* + T_2}{a_1^* + T_1} \leq c^*,$$

with $a_1^* \geq 0$, $a_2^* \geq 0$, where

$$S^* = S\,\frac{b_1}{b_2},\quad a_1^* = \frac{a_1}{b_1},\quad a_2^* = \frac{a_2}{b_2} \text{ and } c^* = c\,\frac{b_1}{b_2}.$$

Hence we can redefine (3.23) as

$$(3.24) \qquad \phi(T) = 1 \text{ if } S = \frac{a_2 + T_2}{a_1 + T_1} \leq c$$

$$= 0 \text{ if } S = \frac{a_2 + T_2}{a_1 + T_1} > c,$$

where $a_1 \geq 0$, $a_2 \geq 0$.
That is, we consider tests with critical region $S \leq c$ where the test statistic S is defined by

$$(3.25) \qquad S = \frac{a_2 + T_2}{a_1 + T_1},$$

with $a_1 \geq 0$, $a_2 \geq 0$.
When $p = k$ we take $a_2 > 0$, whereas $a_1 > 0$ in case $p = \ell$, since otherwise S is not defined.

Let $F_v(s)$ be the distribution function of the test statistic S, i.e. $F_v(s) = P_v(S \leq s)$, then it follows from (3.24) that the power function $\pi(\phi, v)$ of ϕ becomes

(3.26) $\pi(\phi, v) = E_v(\phi(T)) = P_v(S \leq c) = F_v(c).$

This means that the properties of the power function can be derived from the distribution function of the test statistic S.

In the next section we shall show that for arbitrary $a_1 \geq 0$ and $a_2 \geq 0$ the distribution function $F_v(s)$ satisfies the following properties:

(a) $F_v(s) \to 0$ if $d_0 \to \infty$ and $F_v(s) \to 1$ if $d_1 \to \infty$, for all $s > 0$.

(b) $F_v(s)$ is continuous in v for any $s > 0$.

(c) $F_v(s)$ is numerically computable for all v and all $s > 0$.

(d) $F_v(s)$ is continuous in $s > 0$ for any v.

At this point we recall that we were looking for α-similar tests with the property of being computable and usable.

We shall now show that all tests belonging to the following class meet the above requirements.

For any α $(0 < \alpha < 1)$ we define the class of tests $\mathcal{C}_\alpha$ by

(3.27) $$\mathcal{C}_\alpha = \{\phi \mid \phi(t) = 1 \text{ if } \frac{a_2 + t_2}{a_1 + t_1} \leq c,\ \phi(t) = 0 \text{ elsewhere},$$

$$\text{where c satisfies } F_0(c) = \alpha;\ a_1 \geq 0,\ a_2 \geq 0\},$$

where $t = (t_1, t_2)$ with $t_1 \geq 0$, $t_2 \geq 0$.

Note that for any fixed value of α there corresponds a test $\phi \in \mathcal{C}_\alpha$ with every choice of a_1 and a_2. This can be seen as follows. From (d) we see that $F_0(s)$ is continuous in $s > 0$ for any (a_1, a_2), which implies the existence of a value of $c > 0$ satisfying $F_0(c) = \alpha$.

When $\phi \in \mathcal{C}_\alpha$ the power function of ϕ becomes

(3.28) $$\pi(\phi, v) = E_v(\phi(T)) = P_v\left(\frac{a_2 + T_2}{a_1 + T_1} \leq c\right)$$

$$= P_v(S \leq c) = F_v(c),\ v \in \omega_0 \cup \omega_1,$$

where c is chosen in such a way that $F_0(c) = \alpha$.

Therefore it follows at once from (b) that $\pi(\phi, v)$ is continuous in v.

Next we shall show that for any α the tests $\phi \in \mathcal{C}_\alpha$ are α-similar, computable and usable.

Since c satisfies $F_0(c) = \alpha$, it is seen from (3.28) that $\pi(\phi, 0) = F_0(c) = \alpha$ for any $\phi \in C_\alpha$, which shows that the tests $\phi \in C_\alpha$ are α-similar. Further it follows from (c) that $F_0(s)$ is computable for all $s > 0$. This enables us to compute a value of c which satisfies $F_0(c) = \alpha$. Once c is known, according to property (c) we can compute $\pi(\phi, v) = F_v(c)$ for all $v \in \omega_0 \cup \omega_1$. That is, any $\phi \in C_\alpha$ is computable.
Finally, it is seen from (a) that $\pi(\phi, v) = F_v(c) \to 0$ if $d_0 \to \infty$ and $\pi(\phi, v) = F_v(c) \to 1$ if $d_1 \to \infty$, which shows that any $\phi \in C_\alpha$ is usable.

Having constructed for any α a class of α-similar tests with continuous power function and satisfying the properties (iii) and (iv), the question arises whether this class contains tests which satisfy the property (v) (and consequently also (i) and (ii)).
However, before we shall investigate this question, it remains to prove that the distribution function $F_v(s)$ of the general test statistic S satisfies the poperties (a) - (d). This will be done in the next section.

4. The distribution function of the general test statistic

In this section we consider the distribution function $F_v(s)$ of the general test statistic S as defined in (3.25) of the preceding section. We first write S in the following form

$$(4.1) \qquad S = \frac{a_2 + T_2}{a_1 + T_1} = \frac{a_2 + U_2/U_4}{a_1 + U_1/U_4} = \frac{a_2 + u_2'u_2/u_4'u_4}{a_1 + u_1'u_1/u_4'u_4},$$

where use has been made of (2.15), (2.10) and (2.21) of Section 2. From Section 2, see (2.16) and (2.17), we also know that the vector u, defined by $u' = (u_1' \;\; u_2' \;\; u_3' \;\; u_4')$, satisfies

$$(4.2) \qquad u \sim n(\delta, \Gamma),$$

where $\delta' = (\delta_1' \;\; \delta_2' \;\; 0' \;\; 0')$ with δ_1 and δ_2 unknown and where Γ is a known matrix (see (2.17)).
Let $z \sim n(0, \Gamma)$ where $z' = (z_1' \;\; z_2' \;\; z_3' \;\; z_4')$, then the distribution of z does not depend on δ. Now if $u = z + \delta$, it follows that $u \sim n(\delta, \Gamma)$. That is, u in (4.1) and (4.2) can be written as $u = z + \delta$ with $z \sim n(0, \Gamma)$.
This shows that S can be written as

$$(4.3) \qquad S = s_\delta(z) = \frac{a_2 + (z_2 + \delta_2)'(z_2 + \delta_2)/z_4'z_4}{a_1 + (z_1 + \delta_1)'(z_1 + \delta_1)/z_4'z_4}$$

with $z \sim n(0, \Gamma)$.
We note that, since the distribution of $T = (T_1, T_2)$ depends on δ through $v = v(\delta) = (v_1, v_2)$ as defined in (2.28) and (2.29), it follows from (4.1) that also $F_v(s)$ only depends on δ through $v = v(\delta) = (v_1, v_2)$.
Further we recall that $d_0 = \sum_{j=0}^{M} v_{2j} = \delta_2'\delta_2$ and $d_1 = \sum_{j=0}^{M} v_{1j} = \delta_1'\delta_1$.
We shall now prove the properties (a) - (d) of the foregoing section for arbitrary fixed values of $a_1 \geq 0$, $a_2 \geq 0$.

(a) $\qquad F_v(s) \to 0$ if $d_0 \to \infty$ and $F_v(s) \to 1$ if $d_1 \to \infty$ for all $s > 0$.

Proof:
First consider H_0. Then $d_0 \geq 0$ and $d_1 = 0$ or, equivalently, $\delta_1 = 0$ and from (4.3) we get

$$S = s_\delta(z) = \frac{a_2 + (z_2 + \delta_2)'(z_2 + \delta_2)/z_4'z_4}{a_1 + z_1'z_1/z_4'z_4}.$$

Since $d_0 = \delta_2'\delta_2$ it follows that $d_0 \to \infty$ if and only if at least one element of the vector $\delta_2 \to \pm\infty$.

Now the random vector z has a $n(0, \Gamma)$ distribution which is independent of δ. So for every realization of z it is seen that $(z_2 + \delta_2)'(z_2 + \delta_2) \to \infty$ if at least one element of $\delta_2 \to \pm\infty$. Hence if $d_0 \to \infty$ then $s_\delta(z) \to \infty$ for every realization of z. This implies that $S \to \infty$ with probability 1 if $d_0 \to \infty$, which in turn shows that $F_v(s) = P_v(S \leq s) \to 0$ for all s if $d_0 \to \infty$.

In the second place consider H_1. Now $d_0 = 0$ and $d_1 > 0$, or equivalently, $\delta_1 \neq 0$ and $\delta_2 = 0$.

From (4.3) we have:

$$S = s_\delta(z) = \frac{a_2 + z_2'z_2/z_4'z_4}{a_1 + (z_1 + \delta_1)'(z_1 + \delta_1)/z_4'z_4}.$$

In this case it follows from $d_1 = \delta_1'\delta_1$ that $d_1 \to \infty$ if and only if at least one element of $\delta_1 \to \pm\infty$.

Again for every realization of z it is seen that $(z_1 + \delta_1)'(z_1 + \delta_1) \to \infty$ if at least one element of $\delta_1 \to \pm\infty$.

Therefore, if $d_1 \to \infty$ it follows that $s_\delta(z) \to 0$ for every realization of z. This implies that $S \to 0$ with probability 1 if $d_1 \to \infty$, which shows that $F_v(s) = P_v(S \leq s) \to 0$ for $s < 0$ and $F_v(s) = P_v(S \leq s) \to 1$ for $s > 0$, if $d_1 \to \infty$ and this completes the proof of (a).

(b) $F_v(s)$ is continuous in v for any $s > 0$.

<u>Proof:</u>

From $v = v(\delta) = (v_1, v_2)$ it follows that $v \to v_*$ if and only if $\delta \to \delta_*$ for some δ_* which satisfies $v_* = v(\delta_*)$. Now for any realization of the random vector z it is seen from (4.3) that

$$\lim_{\delta\to\delta_*} s_\delta(z) = s_{\delta_*}(z).$$

Therefore, if $S_* = s_{\delta_*}(z)$, it follows that

$$\lim_{\delta\to\delta_*} S = S_* \text{ with probability 1.}$$

The latter result implies that

$$\lim_{v \to v_*} F_v(s) = F_{v_*}(s),$$

for all s where $F_{v_*}(s)$ is continuous.

Under (d) we shall prove that $F_v(s)$ is continuous in $s > 0$ for any v. Hence $F_{v_*}(s)$ is continuous in $s > 0$ and this shows that

$$\lim_{v \to v_*} F_v(s) = F_{v_*}(s) \text{ for any } s > 0,$$

which completes the proof of (b).

(c) $F_v(s)$ is computable for all v and all $s > 0$.

Proof:

From (4.1) we get

$$S = \frac{a_2 + T_2}{a_1 + T_1} = \frac{a_2 + U_2/U_4}{a_1 + U_1/U_4} = \frac{U_2 + a_2U_4}{U_1 + a_1U_4}, \tag{4.4}$$

where U_1, U_2 and U_4 are as given in (2.20) of Section 2. Since $a_1 \geq 0$, $a_2 \geq 0$, $P_v(U_1 > 0) = P_v(U_2 > 0) = P_v(U_4 > 0) = 1$ it follows that $P_v(S > 0) = 1$ and consequently $F_v(s) = P_v(S \leq s) = 0$ for $s \leq 0$. Thus we can restrict attention to the points $s > 0$. The event $S \leq s$ is equivalent to

$$\frac{U_2 + a_2U_4}{U_1 + a_1U_4} \leq s,$$

and since the latter event is equivalent to

$$-s\, U_1 + U_2 + (-a_1s + a_2)\, U_4 \leq 0,$$

it follows that

$$S \leq s \text{ if and only if } Q_s \leq 0, \tag{4.5}$$

where the random variable Q_s is defined by

(4.6) $\quad Q_s = -s\ U_1 + U_2 + (-a_1 s + a_2)\ U_4,\ s > 0.$

From (4.5) we have $P_v(S \leq s) = P_v(Q_s \leq 0)$ and if $G_v(x, s) = P_v(Q_s \leq x)$ it is seen that

(4.7) $\quad F_v(s) = G_v(0, s),$

for $v \in \omega_0 \cup \omega_1$ and $s > 0$.
In order to compute $G_v(0, s)$ we shall first derive the characteristic function $\psi_v(t, s)$ of Q_s.
We get from (4.6)

(4.8) $$\psi_v(t, s) = E(e^{itQ_s}) = E(e^{i(-s)tU_1+itU_2+i(-a_1s+a_2)tU_4})$$

$$= E(e^{i(-s)tU_1+itU_2})E(e^{i(-a_1s+a_2)tU_4}),$$

where i denotes the imaginary unit and where use has been made of the independence of (U_1, U_2) and U_4, see Section 2.
The first term at the right-hand side of (4.8) is the characteristic function of (U_1, U_2) developed at the point $(-st, t)$, whereas the second term is the characteristic function of U_4 at the point $(-a_1s+a_2)t$.
Now we know from Section 2 that $U_4 \sim \chi^2(m)$ and therefore we have

(4.9) $$E(e^{i(-a_1s+a_2)U_4}) = (1-2i\lambda(s)t)^{-\frac{m}{2}},$$

where $\lambda(s)$ is defined by

(4.10) $\quad \lambda(s) = -a_1 s + a_2.$

We also know from Section 2 that

(4.11) $$E(e^{it_1U_1+it_2U_2}) = \prod_{j=0}^{M} \Psi_j(t_1, t_2),$$

where $\Psi_j(t_1, t_2)$, $j = 0, 1, \ldots, M$ are as given in (2.25) and (2.26).
Substitution of (4.9) and (4.11) (with $t_1 = -st$ and $t_2 = t$) into (4.8) yields:

$$(4.12)\qquad \psi_V(t, s) = [\prod_{j=0}^{M} \Psi_j(-st, t)]\,(1-2i\lambda(s)t)^{-\frac{m}{2}}.$$

From (2.25) we get

$$(4.13)\qquad \Psi_0(-st, t) = (1-2i(-s)t)^{-\frac{\ell-r}{2}} \exp\{it \frac{-sv_{10}}{1-2i(-s)t}\}$$

$$(1-2it)^{-\frac{k-r}{2}} \exp\{it \frac{v_{20}}{1-2it}\}.$$

Since, in general,

$$(1-2it)^{-\frac{r}{2}} \exp\{\frac{it\theta}{1-2it}\}$$

is the characteristic function of a $\chi^2(r, \theta)$distribution, and since the characteristic function of the sum of 2 independent random variables is equal to the product of the characteristic functions, it follows that (4.13) is the characteristic function of

$$-sV_{10}(s) + V_{20}(s),$$

where $V_{10}(s)$ and $V_{20}(s)$ are independent random variables with the following distributions

$$(4.14)\qquad \begin{aligned} V_{10}(s) &\sim \chi^2(\ell-r, v_{10}) \\ V_{20}(s) &\sim \chi^2(k-r, v_{20}). \end{aligned}$$

In a similar way from (2.26) we have

$$(4.15)\qquad \Psi_j(-st, t) = (1-2i(1-s)t + 4s(1-\rho_j)t^2)^{-\frac{m_j}{2}}$$

$$\exp\{it \frac{(1+2ist)v_{2j} - s(1-2it)v_{1j}}{1-2i(1-s)t + 4s(1-\rho_j)t^2}\},$$

for $j = 1, 2, \ldots, M$, where ρ_j and m_j are given numbers, see Section 2. The terms at the right-hand side of (4.15) can be factorized as follows.

$$(4.16)\qquad 1-2i(1-s)t + 4s(1-\rho_j)t^2 = (1-2i\lambda_{1j}(s)t)(1-2i\lambda_{2j}(s)t),$$

where

(4.17)
$$\lambda_{1j}(s) = \tfrac{1}{2}(1-s) - \tfrac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)}$$
$$\lambda_{2j}(s) = \tfrac{1}{2}(1-s) + \tfrac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)},$$

$j = 1, 2, \ldots, M.$

We also have

(4.18)
$$\frac{(1+2ist)v_{2j} - s(1-2it)v_{1j}}{1-2i(1-s)t + 4s(1-\rho_j)t^2} = \frac{\lambda_{1j}(s)\ \tau_{1j}(s)}{1-2i\lambda_{1j}(s)t} + \frac{\lambda_{2j}(s)\ \tau_{2j}(s)}{1-2i\lambda_{2j}(s)t},$$

where

(4.19)
$$\tau_{1j}(s) = c_{1j}(s)v_{2j} + d_{1j}(s)v_{1j}$$
$$\tau_{2j}(s) = c_{2j}(s)v_{2j} + d_{2j}(s)v_{1j}$$

and

(4.20)
$$c_{1j}(s) = \frac{\lambda_{1j}(s) + s}{\lambda_{1j}(s)[\lambda_{1j}(s) - \lambda_{2j}(s)]}$$
$$c_{2j}(s) = \frac{\lambda_{2j}(s) + s}{\lambda_{2j}(s)[\lambda_{2j}(s) - \lambda_{1j}(s)]}$$
$$d_{1j}(s) = \frac{-s[\lambda_{1j}(s) - 1]}{\lambda_{1j}(s)[\lambda_{1j}(s) - \lambda_{2j}(s)]}$$
$$d_{2j}(s) = \frac{-s[\lambda_{2j}(s) - 1]}{\lambda_{2j}(s)[\lambda_{2j}(s) - \lambda_{1j}(s)]},$$

$j = 1, 2, \ldots, M.$

Substitution of (4.16) and (4.18) into (4.15) yields

(4.21)
$$\Psi_j(-st,\ t) = (1 - 2i\lambda_{1j}(s)t)^{-\frac{m_j}{2}} \exp\{it\,\frac{\lambda_{1j}(s)\tau_{1j}(s)}{1 - 2i\lambda_{1j}(s)t}\}$$
$$(1 - 2i\lambda_{2j}(s)t)^{-\frac{m_j}{2}} \exp\{it\,\frac{\lambda_{2j}(s)\tau_{2j}(s)}{1 - 2i\lambda_{2j}(s)t}\},$$

$j = 1, 2, \ldots, M.$

Using the same argument as before it is seen that (4.21) is the characteristic function of

$$\lambda_{1j}(s)V_{1j}(s) + \lambda_{2j}(s)V_{2j}(s),$$

where the random variables $V_{1j}(s)$ and $V_{2j}(s)$ are independent and has the following distributions

$$V_{1j}(s) \sim \chi^2(m_j, \tau_{1j}(s))$$

(4.22)

$$V_{2j}(s) \sim \chi^2(m_j, \tau_{2j}(s)),$$

for $j = 1, 2, \ldots, M$.

If we define $\rho_0 = 0$ we can extend (4.17) to the case $j = 0$. This yields $\lambda_{10}(s) = -s$ and $\lambda_{20}(s) = 1$. By doing the same for (4.19) and (4.20) we obtain $c_{10}(s) = 0$, $c_{20}(s) = 1$, $d_{10}(s) = 1$, $d_{20}(s) = 0$ and consequently $\tau_{10}(s) = v_{10}$ and $\tau_{20}(s) = v_{20}$.

Hence, if we define m_{1j} and m_{2j} for $j = 0, 1, \ldots, M$ as

$$m_{10} = \ell - r$$

(4.23) $$m_{20} = k - r$$

$$m_{1j} = m_{2j} = m_j,\ j = 1, 2, \ldots, M,$$

it is seen from (4.13), (4.14) and (4.21), (4.22) that

(4.24) $$\Psi_j(-st, t) = (1 - 2i\lambda_{1j}(s)t)^{-\frac{m_{1j}}{2}} \exp\{it \frac{\lambda_{1j}(s)\tau_{1j}(s)}{1 - 2i\lambda_{1j}(s)t}\}$$

$$(1 - 2i\lambda_{2j}(s)t)^{-\frac{m_{2j}}{2}} \exp\{it \frac{\lambda_{2j}(s)\tau_{2j}(s)}{1 - 2i\lambda_{2j}(s)t}\},$$

for $j = 0, 1, \ldots, M$, is the characteristic function of

$$\lambda_{1j}(s)V_{1j}(s) + \lambda_{2j}(s)V_{2j}(s),$$

where the independent random variables $V_{1j}(s)$ and $V_{2j}(s)$ are distributed as follows

$$(4.25)\qquad \begin{aligned} &V_{1j}(s) \sim \chi^2(m_{1j}, \tau_{1j}(s)) \\ &V_{2j}(s) \sim \chi^2(m_{2j}, \tau_{2j}(s)), \end{aligned}$$

$j = 0, 1, \ldots, M$.
With respect to the coefficients $\lambda_{1j}(s)$ and $\lambda_{2j}(s)$ it should be noted that

$$(4.26)\qquad \begin{aligned} &\lambda_{1j}(s) < 0,\ \lambda_{2j}(s) > 0 \\ &\lambda_{1j}(s) + \lambda_{2j}(s) = 1-s \\ &\lambda_{1j}(s)\lambda_{2j}(s) = -s(1-\rho_j), \end{aligned}$$

for $j = 0, 1, \ldots, M$.
Further it is easily verified from (4.20) that

$$c_{1j}(s) > 0,\ c_{2j}(s) > 0,\ d_{1j}(s) > 0 \text{ and } d_{2j}(s) > 0$$

for $j = 1, 2, \ldots, M$.

Let us now return to the characteristic function $\psi_V(t, s)$ of the random variable Q_s in (4.12).
Since the characteristic function of a sum of mutually independent random variables is equal to the product of the characteristic functions it follows from the above results that Q_s can be written as

$$(4.27)\qquad Q_s = \sum_{j=0}^{M} [\lambda_{1j}(s)V_{1j}(s) + \lambda_{2j}(s)V_{2j}(s)] + \lambda(s)V(s),$$

where the random variables $V_{10}(s), V_{11}(s), \ldots, V_{1M}(s), V_{20}(s), \ldots, V_{2M}(s), V(s)$ are mutually independent and where

$$(4.28)\qquad \left.\begin{aligned} &V_{1j}(s) \sim \chi^2(m_{1j}, \tau_{1j}(s)) \\ &V_{2j}(s) \sim \chi^2(m_{2j}, \tau_{2j}(s)) \end{aligned}\right\} j = 0, 1, \ldots, M$$
$$V(s) \sim \chi^2(m).$$

It also follows that $\psi_v(t, s)$ becomes

(4.29) $$\psi_v(t, s) = [\prod_{j=0}^{M} (1 - 2i\lambda_{1j}(s)t)^{-\frac{m_{1j}}{2}} (1 - 2i\lambda_{2j}(s)t)^{-\frac{m_{2j}}{2}}]$$
$$(1 - 2i\lambda(s)t)^{-\frac{m}{2}} \exp\{it \sum_{j=0}^{M} [\frac{\lambda_{1j}(s)\tau_{1j}(s)}{1 - 2i\lambda_{1j}(s)t} + \frac{\lambda_{2j}(s)\tau_{2j}(s)}{1 - 2i\lambda_{2j}(s)t}]\}.$$

Note $\psi_v(t, s)$ is completely known for every choice of $a_1 \geq 0$, $a_2 \geq 0$, $v \in \omega_0 \cup \omega_1$ and $s > 0$.

As is shown is Appendix B, see (B.25), for a random variable of the type considered in (4.27) with characteristic function (4.29), we have the following inversion formula

(4.30) $$G_v(x, s) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \{\mathrm{Im}(\psi_v(t, s)e^{-itx})/t\}dt,$$

where in general, Im(z) denotes the imaginary part of the complex number z, i.e., $\mathrm{Im}(z) = (z - \bar{z})/(2i)$ and where $G_v(x, s)$ is the distribution function of Q_s.

The value of the integrand t = 0 is given by

$$\mathrm{Im}(\psi_v(t, s)e^{-itx})/t\Big|_{t=0} \overset{\text{def.}}{=} \lim_{t\downarrow 0}\{\mathrm{Im}(\psi_v(t, s)e^{-itx})/t\} = E(Q_s) - x.$$

With the aid of (4.7) it follows from (4.30) that

(4.31) $$F_v(s) = G_v(0, s) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \{\mathrm{Im}(\psi_v(t, s))/t\}dt,$$

for any $a_1 \geq 0$, $a_2 \geq 0$, $v \in \omega_0 \cup \omega_1$ and $s > 0$.

Also,

$$\mathrm{Im}(\psi_v(t, s))/t\Big|_{t=0} = E(Q_s),$$

where it is not difficult to verify that

(4.32) $$E(Q_s) = \sum_{j=0}^{M} [\lambda_{1j}(s)(m_{1j} + \tau_{1j}(s)) + \lambda_{2j}(s)(m_{2j} + \tau_{2j}(s))] + \lambda(s)m$$
$$= -s(\ell-p+d_1) + (k-p+d_0) + (-a_1 s + a_2)m.$$

We recall that

$$d_0 = \sum_{j=0}^{M} v_{2j},\ d_1 = \sum_{j=0}^{M} v_{1j} \text{ and } m = n+p-k-\ell.$$

It should also be noted that $\psi_v(t, s)$ has a simple form at the point $v = 0$ since for $v = 0$ we get $\tau_{1j}(s) = \tau_{2j}(s) = 0$, $j = 0, 1, \ldots, M$. This yields

$$\psi_0(t, s) = [\prod_{j=0}^{M} (1-2i\lambda_{1j}(s)t)^{-\frac{m_{1j}}{2}} (1-2i\lambda_{2j}(s)t)^{-\frac{m_{2j}}{2}}]$$

$$(1-2i\lambda(s)t)^{-\frac{m}{2}}.$$

Now the function $\mathrm{Im}(\psi_v(t, s))$ can be determined and as is shown in (B.26) of Appendix B, the formula (4.31) can be written in the following way

$$\text{(4.33)} \qquad F_v(s) = G_v(0, s) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin \varepsilon_v(u, s)}{u\, \gamma_v(u, s)}\, du,$$

where $\gamma_v(u, s)$ and $\varepsilon_v(u, s)$ are given by

$$\text{(4.34)} \qquad \gamma_v(u, s) = [\prod_{j=0}^{M} (1 + \lambda_{1j}^2(s)u^2)^{\frac{m_{1j}}{4}} (1 + \lambda_{2j}^2(s)u^2)^{\frac{m_{2j}}{4}}]\, (1 + \lambda^2(s)u^2)^{\frac{m}{4}}$$

$$\exp\{\frac{1}{2} \sum_{j=0}^{M} [\frac{\tau_{1j}(s)\lambda_{1j}^2(s)u^2}{1 + \lambda_{1j}^2(s)u^2} + \frac{\tau_{2j}(s)\lambda_{2j}^2(s)u^2}{1 + \lambda_{2j}^2(s)u^2}]\}.$$

and

$$\text{(4.35)} \qquad \varepsilon_v(u, s) = \frac{1}{2} \sum_{j=0}^{M} [m_{1j} \operatorname{arctg}(\lambda_{1j}(s)u) + m_{2j} \operatorname{arctg}(\lambda_{2j}(s)u)$$

$$+ \frac{\tau_{1j}(s)\lambda_{1j}(s)u}{1 + \lambda_1^2(s)u^2} + \frac{\tau_{2j}(s)\lambda_{2j}(s)u}{1 + \lambda_{2j}^2(s)u^2}]$$

$$+ \frac{1}{2}\, m \operatorname{arctg}(\lambda(s)u).$$

The value of the integrand at u = 0 is given by

$$\left.\frac{\sin \varepsilon_v(u, s)}{u\ \gamma_v(u, s)}\right|_{u=0} = \frac{1}{2} E(Q_s),$$

where $E(Q_s)$ is as given in (4.32).
For any $a_1 \geq 0$, $a_2 \geq 0$, $v \in \omega_0 \cup \omega_1$ and $s > 0$ we can compute $F_v(s)$ through numerical integration of the right-hand side of (4.33).
The computerprogram FQUAD computes $F_v(s)$ to any desired degree of accuracy. This program is developed by Louter and can be found in Koerts and Abrahamse [6]. The above method of computing the distribution function of S (or Q_s) is usually called Imhof's method and for more details on this numerical integration we refer to Appendix B. This completes the proof of (c).
Before we shall prove property (d), we make the following additional remarks.
In the first place we note that the computations are much simpler when v = 0. In this case we have

$$(4.36) \qquad F_0(s) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin \varepsilon_0(u, s)}{u\ \gamma_0(u, s)}\, du,$$

where

$$(4.37) \qquad \gamma_0(u, s) = \Big[\prod_{j=0}^{M} (1 + \lambda_{1j}^2(s)u^2)^{\frac{m_{1j}}{4}} (1 + \lambda_{2j}^2(s)u^2)^{\frac{m_{2j}}{4}}\Big](1 + \lambda^2(s)u^2)^{\frac{m}{4}}$$

$$\varepsilon_0(u, s) = \frac{1}{2} \sum_{j=0}^{M} [m_{1j} \operatorname{arctg}(\lambda_{1j}(s)u) + m_{2j} \operatorname{arctg}(\lambda_{2j}(s)u)]$$

$$+ \frac{1}{2} m \operatorname{arctg}(\lambda(s)u).$$

For the value of the integrand at u = 0 we get

$$\left.\frac{\sin \varepsilon_0(u, s)}{u\ \gamma_0(u, s)}\right|_{u=0} = \frac{1}{2}[-s(\ell-p) + (k-p) + (-a_1 s + a_2)m].$$

In the second place, for the applications it is important to see that

$$0 \leq M \leq \min(k, \ell).$$

Now, usually k and ℓ are small and this shows that, in most cases, M and therefore also the number of terms in (4.33), (4.34) and (4.35) will be small.

(d) $F_v(s)$ is continuous in $s > 0$ for any v.

Proof:
Consider the random variable Q_s as defined in (4.6). With the aid of (2.20), (2.16) and (2.17) of Section 2 we can rewrite Q_s in the following way

(4.38) $$Q_s = q_s(u) = -su_1'u_1 + u_2'u_2 + (-a_1 s + a_2)u_4'u_4,$$

where $u' = (u_1' \;\; u_2' \;\; u_3' \;\; u_4')$ and $u \sim n(\delta, \Gamma)$.

Consider an arbitrary $s_* > 0$ and let $Q_{s_*} = q_{s_*}(u)$.
For every δ and every realization of the random vector u it follows from (4.38) that

$$\lim_{s \to s_*} q_s(u) = q_{s_*}(u),$$

which shows that $Q_s \to Q_{s_*}$ with probability 1.
If $v = v(\delta)$, the latter property implies that

$$\lim_{s \to s_*} G_v(x, s) = G_v(x, s_*)$$

for all x where $G_v(x, s_*)$ is continuous.
Now it is shown in Appendix B (see (B.17)) that $G_v(x, s)$ is continuous in all x (for any v and any $s > 0$). It follows that $G_v(x, s_*)$ is continuous in $x = 0$ and therefore we have

$$\lim_{s \to s_*} G_v(0, s) = G_v(0, s_*).$$

Since $F_v(s) = G_v(0, s)$ (see 4.7)) we obtain

$$\lim_{s \to s_*} F_v(s) = F_v(s_*)$$

for any v, which proves the desired result.
At the end of this section we recall from Section 3 that a particular C_α-class test is specified by first fixing a value of α and then chosing a test $\phi \in C_\alpha$ by fixing $a_1 \geq 0$ and $a_2 \geq 0$.

This test has critical region $S \leq c$, where c has to be taken in such a way that $F_0(c) = \alpha$.
So it follows from (4.36) that we have to solve the equation (i.e., find the value of c which satisfies)

$$\frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin \varepsilon_0(u, c)}{u \, \gamma_0(u, c)} \, du = \alpha.$$

It will be clear that this requires an iteration procedure, where within each iteration we have to perform a numerical integration.
Since in general the critical values c cannot be tabulated it seems preferable to compute the p-value (critical level) of the test, instead of the critical value.
The p-value is defined by

$$(4.39) \qquad F_0(S) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin \varepsilon_0(u, S)}{u \, \lambda_0(u, S)} du,$$

where S is the test statistic.
Obviously, in order to compute $F_0(S)$ no iteration procedure is required.
The procedure where H_0 is rejected when $S \leq c$ with c satisfying $F_0(c) = \alpha$ is precisely equivalent to the procedure where H_0 is rejected when $F_0(S) \leq \alpha$.
That is, the p-value $F_0(S)$ can be thought of as a standardized test statistic which for $v = 0$ has a uniform distribution over the interval (0, 1).

5. Properties of the C_α-class tests

In Section 3 we saw that every test $\phi \in C_\alpha$ is α-similar and has the properties (iii) and (iv) (i.e., it is computable and usable). It is important to note that ϕ possesses these properties for any problem of testing linear hypotheses, that is, regardless of the particular regressor matrices X and Z.
Now the question arises whether ϕ has level α or is unbiased, and, if the answer is yes, whether ϕ is strictly discriminating. In other words, we may ask whether $\phi \in C_\alpha$ satisfies the properties (i), (ii) and (v) as formulated in Section 3.
The answer to this question turns out to depend on the specific testing problem, that is, it depends on the particular matrices X and Z.
To be more specific, we shall see below that when the quantity $r = \text{rank}(X'Z)$ satisfies certain conditions, every test $\phi \in C_\alpha$ possesses the properties (i), (ii) and (v) for any value of α.
As we saw in Section 3, when the α-similar test ϕ has a continuous power function, property (v) implies the properties (i) and (ii). Since the power function of any $\phi \in C_\alpha$ is continuous in v it suffices to verify whether ϕ satisfies property (v).
Now, as was already noticed before the following stronger property (which implies (v) for any $\phi \in C_\alpha$) is often more easy to verify:

(vi) For $v \in \omega_0$ the function $\pi(\phi, v)$ is strictly decreasing in each of the variables v_{2j} $(j = 0, 1, \ldots, M)$ and for $v \in \omega_1$ the function $\pi(\phi, v)$ is strictly increasing in each of the variables v_{1j} $(j = 0, 1, \ldots, M)$.

We shall investigate how the power function $\pi(\phi, v)$ depends on the parameters $v_{20}, v_{21}, \ldots, v_{2M}$ under H_0 and on the parameters $v_{10}, v_{11}, \ldots, v_{1M}$ under H_1.
In order to do this we shall derive the partial derivatives

$$\frac{\partial \pi(\phi, v)}{\partial v_{2j}}, \quad j = 0, 1, \ldots, M$$

for $v \in \omega_0$ and

$$\frac{\partial\pi(\phi,\ v)}{\partial v_{1j}},\ j = 0,\ 1,\ \ldots,\ M$$

for $v \in \omega_1$.

We know from Section 3 that a test $\phi \in \mathcal{C}_\alpha$ has the following power function

$$(5.1)\qquad \pi(\phi,\ v) = F_v(c),\ v \in \omega_0 \cup \omega_1,$$

where $F_v(s)$ is the distribution function of the general test statistic S (as defined in (3.25)) and where c satisfies $F_0(c) = \alpha$.

Moreover, in Section 4 we have shown that

$$(5.2)\qquad F_v(s) = G_v(0,\ s),$$

where $G_v(x,\ s)$ is the distribution function of the auxiliary random variable Q_s (as defined in (4.6) or (4.27)); and also that (see (4.30))

$$(5.3)\qquad G_v(x,\ s) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \{\mathrm{Im}(\psi_v(t,\ s)e^{-itx})/t\}dt,$$

where $\psi_v(t,\ s)$ is the characteristic function of Q_s as given in (4.29).

The above results show that the partial derivatives of $\pi(\phi,\ v)$ with respect to v_{2j} and v_{1j} ($j = 0,\ 1,\ \ldots,\ M$) can be found from (5.3), since

$$(5.4)\qquad \begin{aligned} \frac{\partial\pi(\phi,\ v)}{\partial v_{2j}} &= \frac{\partial G_v(0,\ c)}{\partial v_{2j}} \\ \frac{\partial\pi(\phi,\ v)}{\partial v_{1j}} &= \frac{\partial G_v(0,\ c)}{\partial v_{1j}}, \end{aligned}$$

$j = 0,\ 1,\ \ldots,\ M$.

For this reason, from (5.3) we shall first derive expressions for

$$\frac{\partial G_v(x,\ s)}{\partial v_{2j}} \text{ and } \frac{\partial G_v(x,\ s)}{\partial v_{1j}},$$

for $j = 0,\ 1,\ \ldots,\ M$, any x ($-\infty < x < \infty$) and any $s > 0$.

We start with $v \in \omega_0$. As is shown in Appendix C we may differentiate the right-hand side of (5.3) under the integral and this yields, see (C.25):

$$(5.5)\qquad \frac{\partial G_v(x,\ s)}{\partial v_{20}} = -\ g_{0,v}(x,\ s)$$

for $-\infty < x < \infty$, all $s > 0$, all $v \in \omega_0$ and any $a_1 \geq 0$, $a_2 \geq 0$, where $g_{0,v}(x, s)$ is the probability density function of the random variable $Q_{0,s}$ as defined in (C.18), (C.19) and (C.20) of Appendix C.
Since $g_{0,v}(x, s)$ is a probability density function it follows that $g_{0,v}(x, s) \geq 0$ for $-\infty < x < \infty$.
However we shall show that in this case we have $g_{0,v}(x, s) > 0$ for $-\infty < x < \infty$.
In order to see this, we observe that $Q_{0,s}$ is a sum of mutually independent random variables. Except for trivial cases this sum contains at least one random variable, say W_1, with probability density function

$$(5.6) \qquad \begin{aligned} f_1(x) &> 0, \ x > 0 \\ f_1(x) &= 0, \ x \leq 0 \end{aligned}$$

and at least one random variable W_2 with probability density function

$$(5.7) \qquad \begin{aligned} f_2(x) &> 0, \ x < 0 \\ f_2(x) &= 0, \ x \geq 0. \end{aligned}$$

That is, $Q_{0,s}$ can be written as

$$(5.8) \qquad Q_{0,s} = W_1 + W_2 + W_3,$$

where W_1, W_2 and W_3 are mutually independent and where W_3 has an arbitrary density function $f_3(x)$.
Let $W = W_1 + W_2$, then the probability density $f(x)$ of W is the convolution of the densities $f_1(x)$ and $f_2(x)$, i.e.,

$$(5.9) \qquad f(x) = \int_{-\infty}^{\infty} f_2(x-w)f_1(w)dw, \ -\infty < x < \infty.$$

From (5.6) it follows that (5.9) becomes

$$(5.10) \qquad f(x) = \int_{0}^{\infty} f_2(x-w)f_1(w)dw, \ -\infty < x < \infty.$$

Suppose that $x > 0$, then according to (5.7) we have $f_2(x-w) = 0$ for

$w \leq x$ and it is seen from (5.10) that

$$f(x) = \int_x^\infty f_2(x-w)f_1(w)dw, \quad x > 0.$$

On the other hand, when $x \leq 0$ it follows that $f_2(x-w) > 0$ for $w > 0$ and we get

$$f(x) = \int_0^\infty f_2(x-w)f_1(w)dw, \quad x \leq 0.$$

This shows that W has the following density

$$(5.11) \qquad \begin{aligned} f(x) &= \int_x^\infty f_2(x-w)f_1(w)dw, \quad x > 0 \\ f(x) &= \int_0^\infty f_2(x-w)f_1(w)dw, \quad x \leq 0. \end{aligned}$$

Since the integrand $f_2(x-w)f_1(w)$ in (5.11) is always strictly positive it is seen that

$$(5.12) \qquad f(x) > 0 \text{ for } -\infty < x < \infty.$$

From (5.8) it follows that $Q_{0,s} = W + W_3$ with W and W_3 independent and in a similar way we get

$$(5.13) \qquad g_{0,v}(x, s) = \int_{-\infty}^{\infty} f(x-w)f_3(w)dw, \quad -\infty < x < \infty.$$

Now suppose that $f_3(x) > 0$ for $x \in A$, where A is arbitrary and $f_3(x) = 0$ elsewhere.
This yields

$$(5.14) \qquad g_{0,v}(x, s) = \int_A f(x-w)f_3(w)dw, \quad -\infty < x < \infty.$$

From (5.12) it is seen that the integrand $f(x-w)f_3(w)$ in (5.14) is always strictly positive and this shows that

$$(5.15) \qquad g_{0,v}(x, s) > 0, \quad -\infty < x < \infty,$$

for all $s > 0$, all $v \in \omega_0$ and any $a_1 \geq 0$, $a_2 \geq 0$.
Together the results (5.4), (5.5) and (5.15) imply

$$(5.16) \qquad \frac{\partial\pi(\phi, v)}{\partial v_{20}} = -g_{0,v}(0, c) < 0, \ v \in \omega_0,$$

for every test $\phi \in \mathcal{C}_\alpha$ and any value of α.
Next we consider $\partial G_v(x, s)/\partial v_{2j}$ for $j = 1, 2, \ldots, M$.
As is shown in Appendix C, see (C.39), in this case differentiating (5.3) with respect to v_{2j} yields

$$(5.17) \qquad \frac{\partial G_v(x, s)}{\partial v_{2j}} = -[g_{j,v}(x, s) - 2sg'_{j,v}(x, s)],$$

for $j = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v \in \omega_0$ and any $a_1 \geq 0$, $a_2 \geq 0$.
Here $g_{j,v}(x, s)$ is the probability density function of the random variable $Q_{j,s}$ as given in (C.33), (C.34) and (C.35) of Appendix C and $g'_{j,v}(x, s)$ is defined by

$$g'_{j,v}(x, s) = \frac{\partial g_{j,v}(x, s)}{\partial x}.$$

It follows from (5.4) that

$$(5.18) \qquad \frac{\partial\pi(\phi, v)}{\partial v_{2j}} = -[g_{j,v}(0, c) - 2cg'_{j,v}(0, c)],$$

for $j = 1, 2, \ldots, M$, $v \in \omega_0$, every test $\phi \in \mathcal{C}_\alpha$ and any value of α.
In this case we cannot conclude that $\partial\pi(\phi, v)/\partial v_{2j} < 0$ for $v \in \omega_0$, every test $\phi \in \mathcal{C}_\alpha$ and any value α.
From (5.18) it is seen that, in general, it will depend on a_1, a_2, c and the point $v \in \omega_0$ whether or not $\frac{\partial\pi(\phi, v)}{\partial v_{2j}} < 0$. That is, it depends on the choice of α, the particular $\phi \in \mathcal{C}_\alpha$ and the point $v \in \omega_0$.
However, when $m_{10} = \ell - r \geq 2$ it is shown in Appendix C, see (C.46), that (5.17) can be rewritten as

$$(5.19) \qquad \frac{\partial G_v(x, s)}{\partial v_{2j}} = -g^*_{j,v}(x, s),$$

for $j = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v \in \omega_0$ and any $a_1 \geq 0$, $a_2 \geq 0$, where $g^*_{j,v}(x, s)$ is the probability density function of the random variable $Q^*_{j,s}$ as defined in (C.41), (C.42) and (C.43) of Appendix C.

By using the same argument as before it is seen that

$$(5.20) \qquad g^*_{j,v}(x, s) > 0, \ -\infty < x < \infty,$$

for j = 1, 2, ..., M, all $s > 0$, all $v \in \omega_0$ and any $a_1 \geq 0$, $a_2 \geq 0$.

Hence, it follows from (5.4) that, under the condition $r \leq \ell-2$, we have

$$(5.21) \qquad \frac{\partial \pi(\phi, v)}{\partial v_{2j}} = -g^*_{j,v}(0, c) < 0, \ v \in \omega_0,$$

j = 1, 2, ..., M, for every test $\phi \in C_\alpha$ and any value of α.

Moreover, as is proved in Appendix C, see (C.51), when $m = n+p-k-\ell \geq 2$ and $\lambda(s) = -a_1 s + a_2 = -s$ for all $s > 0$, we have

$$(5.22) \qquad \frac{\partial G_v(x, s)}{\partial v_{2j}} = -g^*_{j,v}(x, s) < 0,$$

for j = 1, 2, ..., M, $-\infty < x < \infty$, all $s > 0$ and all $v \in \omega_0$, where now $g^*_{j,v}(x, s)$ is the probability density function of $Q^*_{j,s}$ with $\lambda(s) = -s$.

Except for trivial cases the condition $n+p-k-\ell \geq 2$ is always fulfilled.

On the other hand it is seen that $-a_1 s + a_2 = -s$ for all $s > 0$ if and only if $a_1 = 1$ and $a_2 = 0$.

As we know from Section 3, see (3.27), the choice of $a_1 = 1$ and $a_2 = 0$ corresponds to the test, say, $\phi_2 \in C_\alpha$ (for any α) given by

$$(5.23) \qquad \begin{aligned} \phi_2(T) &= 1 \text{ if } S = \frac{T_2}{1 + T_1} \leq c \\ &= 0 \text{ if } S = \frac{T_2}{1 + T_1} > c. \end{aligned}$$

where c is chosen in such a way that $F_0(c) = \alpha$.

It follows therefore from (5.4) and (5.22) that the power function $\pi(\phi_2, v)$ of the test $\phi_2 \in C_\alpha$ given by (5.23) satisfies:

$$(5.24) \qquad \frac{\partial \pi(\phi_2, v)}{\partial v_{2j}} = -g^*_{j,v}(0, c) < 0, \ v \in \omega_0,$$

for j = 1, 2, ..., M and any value of α.

This completes the case $v \in \omega_0$.
We proceed with the case where $v \in \bar{\omega}_1 = \omega_1 \cup \{0\}$.
In a similar way, with the aid of Appendix C, we can deduce the following results from (5.3).

$$(5.25) \qquad \frac{\partial G_v(x, s)}{\partial v_{10}} = s h_{0,v}(x, s),$$

for $-\infty < x < \infty$, all $s > 0$, all $v \in \bar{\omega}_1$ and any $a_1 \geq 0$, $a_2 \geq 0$, where $h_{0,v}(x, s)$ is a probability density function which satisfies $h_{0,v}(x, s) > 0$ for $-\infty < x < \infty$.
Thus it is seen from (5.4) that

$$(5.26) \qquad \frac{\partial \pi(\phi, v)}{\partial v_{10}} = c h_{0,v}(0, c) > 0, \; v \in \omega_1,$$

for every test $\phi \in \mathcal{C}_\alpha$ and any value of α.
Moreover we have

$$(5.27) \qquad \frac{\partial G_v(x, s)}{\partial v_{1j}} = s[h_{j,v}(x, s) + 2h'_{j,v}(x, s)],$$

for $j = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v \in \bar{\omega}_1$ and any $a_1 \geq 0$, $a_2 \geq 0$, where $h_{j,v}(x, s)$ is a probability density function and where

$$h'_{j,v}(x, s) = \frac{\partial h_{j,v}(x, s)}{\partial x}.$$

It follows from (5.4) that

$$(5.28) \qquad \frac{\partial \pi(\phi, v)}{\partial v_{1j}} = c[h_{j,v}(0, c) + 2h'_{j,v}(0, c)], \; v \in \omega_1,$$

for $j = 1, 2, \ldots, M$, every test $\phi \in \mathcal{C}_\alpha$ and any value of α.
Again it depends on the choice of α, the particular test $\phi \in \mathcal{C}_\alpha$ and the point $v \in \omega_1$ whether or not $\pi(\phi, v)/\partial v_{1j} > 0$.
However, when $m_{20} = k-r \geq 2$ we can rewrite (5.27) as follows:

$$(5.29) \qquad \frac{\partial G_v(x, s)}{\partial v_{1j}} = s h^*_{j,v}(x, s),$$

for $j = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, $v \in \bar{\omega}_1$ and any $a_1 \geq 0$,

$a_2 \geq 0$, where $h^*_{j,v}(x, s)$ is a probability density function with $h^*_{j,v}(x, s) > 0$ for $-\infty < x < \infty$.

Hence, it follows from (5.4) that, under the condition $r \leq k-2$, we have:

$$(5.30) \qquad \frac{\partial \pi(\phi, v)}{\partial v_{1j}} = ch^*_{j,v}(0, c) > 0, \; v \in \omega_1,$$

$j = 1, 2, \ldots, M$, for every $\phi \in \mathcal{C}_\alpha$ and any value of α.

Moreover, when $m = n+p-k-\ell \geq 2$ and $\lambda(s) = -a_1 s + a_2 = 1$ for all $s > 0$, we get

$$(5.31) \qquad \frac{\partial G_v(x, s)}{\partial v_{1j}} = sh^*_{j,v}(x, s) > 0,$$

for $j = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$ and all $v \in \bar{\omega}_1$, where $h^*_{j,v}(x, s)$ is the probability density function from (5.29) with $\lambda(s) = 1$.

In this case it is easily seen that $-a_1 s + a_2 = 1$ for all $s > 0$ if and only if $a_1 = 0$ and $a_2 = 1$.

We know from Section 3 that the choice of $a_1 = 0$ and $a_2 = 1$ corresponds to the test, say, $\phi_3 \in \mathcal{C}_\alpha$ (for any α) given by

$$(5.32) \qquad \begin{aligned} \phi_3(T) &= 1 \text{ if } S = \frac{1 + T_2}{T_1} \leq c \\ &= 0 \text{ if } S = \frac{1 + T_2}{T_1} > c, \end{aligned}$$

where c satisfies $F_0(c) = \alpha$.

It is seen from (5.4) and (5.31) that the power function $\pi(\phi_3, v)$ of the test $\phi_3 \in \mathcal{C}_\alpha$ as defined in (5.32) has the property

$$(5.33) \qquad \frac{\partial \pi(\phi_3, v)}{\partial v_{1j}} = ch^*_{j,v}(0, c) > 0, \; v \in \omega_1,$$

for $j = 1, 2, \ldots, M$ and any value of α.

This completes the case $v \in \omega_1$.

Before we summarize the results of this section we recall that the following quantities are given or can be deduced from the regressor matrices X and Z:

n = the number of observations

k = the number of columns of X

ℓ = the number of columns of Z

$0 = \rho_0 < \rho_1 < \dots < \rho_M < \rho_{M+1} = 1$ are the $M+2$ different eigenvalues of the matrix $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$ (or equivalently, $(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z$)

$m_1, m_2, \dots, m_M$ are the multiplicities of the eigenvalues $\rho_1, \rho_2, \dots, \rho_M$

p = multiplicity of the eigenvalue $\rho_{M+1} = 1$

$r = p + \sum_{j=1}^{M} m_j$

It is not difficult to verify that:

$k = \dim(\mathcal{M}(X))$

$\ell = \dim(\mathcal{M}(Z))$

$p = \dim(\mathcal{M}(X) \cap \mathcal{M}(Z))$

$r = \text{rank}(X'Z)$

$0 \leq p \leq r \leq \min(k, \ell)$

$p = r$ if and only if $M = 0$

The following conclusions can be drawn from the results of this section. For any α and every test $\phi \in \mathcal{C}_\alpha$ we have:

(a) $\pi(\phi, v)$ is strictly decreasing in v_{20} when $v \in \omega_0$.

(b) If $r \leq \ell-2$ the function $\pi(\phi, v)$ is strictly decreasing in each v_{2j} for $j = 1, 2, \dots, M$ when $v \in \omega_0$.

(c) $\pi(\phi, v)$ is strictly increasing in v_{10} when $v \in \omega_1$.

(d) If $r \leq k-2$ the function $\pi(\phi, v)$ is strictly increasing in each v_{1j} for $j = 1, 2, \dots, M$ when $v \in \omega_1$.

Further we have for any α:

(e) The power function $\pi(\phi_2, v)$ of the test ϕ_2 is strictly decreasing in each v_{2j} for $j = 1, 2, \dots, M$ when $v \in \omega_0$.

(f) The power function $\pi(\phi_3, v)$ of the test ϕ_3 is strictly increasing in each v_{1j} for $j = 1, 2, \dots, M$ when $v \in \omega_1$.

It should be emphasized that the latter two properties are valid regardless of the value of r.
From (a), (b), (c) and (d) it follows that when $r = p$ or $r \leq \min(k, \ell)-2$, for any value of α all the tests $\phi \in \mathcal{C}_\alpha$ have the property (vi).
Further it is seen from (a) and (e) that $\pi(\phi_2, v)$ is always strictly decreasing in each v_{2j} for $j = 0, 1, \ldots, M$ when $v \in \omega_0$.
In other words, for any α the test $\pi(\phi_2, v)$ has level α (i.e., is an exact test), and moreover, it has the desirable property that $\pi(\phi_2, v) < \alpha$ for all $v \in \omega_0\backslash\{0\}$.
It also follows from (c) and (f) that $\pi(\phi_3, v)$ is strictly increasing in each v_{1j} for $j = 0, 1, \ldots, M$ when $v \in \omega_1$.
That is, for any α the test $\phi_3 \in \mathcal{C}_\alpha$ has a guaranteed power, i.e., $\pi(\phi_3, v) > \alpha$ for all $v \in \omega_1$.
Whether the tests ϕ_2 and ϕ_3 satisfy property (vi) depends on the value of r and in the next section we shall try to find for any r a suitable test $\phi \in \mathcal{C}_\alpha$ which satisfies property (vi).

6. The tests ϕ_1, ϕ_2 and ϕ_3

In the foregoing section we saw that it depends on the value of $r = \text{rank}(X'Z)$ whether the tests $\phi \in C_\alpha$ satisfy property (vi). We therefore consider the following situations:

(A) $r = p$

(B) $r > p$

Case (B) is divided into the following subcases:

(B.1) $r \leq \min(k, \ell) - 2$

(B.2) $\ell-2 < r \leq k-2$ } $k \neq \ell$

(B.3) $k-2 < r \leq \ell-2$

(B.4) $r > \max(k, \ell) - 2$

Note that the above possibilities are mutually exclusive and exhaustive. With the aid of the results (a) - (f) from Section 5 we shall try to find in each of the above cases a test $\phi \in C_\alpha$ which satisfies property (vi).

Case (A): We already saw in Section 5 that for any value of α all tests $\phi \in C_\alpha$ have property (vi).
Note that in this case we have $v = (v_1, v_2) = (v_{10}, v_{20})$ since $M = 0$ when $r = p$.
Case (B.1): Again for any value of α all the tests $\phi \in C_\alpha$ satisfy property (vi). As was already remarked in Section 5, this easily follows from the properties (a), (b), (c) and (d).
Case (B.2): Since $r \leq k-2$ it follows from (c) and (d) that any test $\phi \in C_\alpha$ has a power function $\pi(\phi, v)$ which is strictly increasing in each of the variables v_{1j}, $j = 0, 1, \ldots, M$ when $v \in \omega_1$. This implies that in particular the test $\phi_2 \in C_\alpha$ has this property. On the other hand it is seen from (a) and (e) that $\pi(\phi_2, v)$ is strictly decreasing in each v_{2j} for $j = 0, 1, \ldots, M$ when $v \in \omega_0$. This shows that for any α the test $\phi_2 \in C_\alpha$ has property (vi).
The test ϕ_2 has critical region (see (5.23))

(6.1) $$S_2 = \frac{T_2}{1 + T_1} \leq c_2,$$

where c_2 is chosen in such a way that $F_{2,0}(c_2) = \alpha$. Here $F_{2,v}(s)$ is the distribution function of the test statistic S_2, i.e.,

$$F_{2,v}(s) = P_v(S_2 \leq s).$$

Case (B.3): Since $r \leq \ell-2$ it is seen from (a) and (b) that the power function $\pi(\phi, v)$ of any test $\phi \in \mathcal{C}_\alpha$ is strictly decreasing in each v_{2j}, $j = 0, 1, \ldots, M$ when $v \in \omega_0$. Thus in particular the test $\phi_3 \in \mathcal{C}_\alpha$ has this property. Now it follows from (c) and (f) that $\pi(\phi_3, v)$ is strictly increasing in each v_{1j} for $j = 0, 1, \ldots, M$ when $v \in \omega_1$. Hence, for any α the test $\phi_3 \in \mathcal{C}_\alpha$ has property (vi).
We know from (5.32) that the test ϕ_3 has the following critical region

(6.2) $$S_3 = \frac{1 + T_2}{T_1} \leq c_3,$$

where c_3 has to be taken in such a way that $F_{3,0}(c_3) = \alpha$. The function $F_{3,v}(s)$ is the distribution function of the test statistic S_3, that is,

$$F_{3,v}(s) = P_v(S_3 \leq s).$$

Case (B.4): On the basis of the properties (a) - (f) from Section 5 we cannot find a test $\phi \in \mathcal{C}_\alpha$ which satisfies property (vi). In this case it seems reasonable to take the test ϕ_2. As we saw in Section 5 this test has always level α (that is, it is always an exact test) and satisfies the desirable property that $\pi(\phi_2, v) < \alpha$ for all $v \in \omega_0$.
Moreover, since $\phi_2 \in \mathcal{C}_\alpha$, it follows that this test is α-similar, computable and usable, where we recall that the latter property means that

$$\pi(\phi_2, v) \to 0 \text{ if } d_0 = \sum_{j=0}^{M} v_{2j} \to \infty \text{ and}$$

$$\pi(\phi_2, v) \to 1 \text{ if } d_1 = \sum_{j=0}^{M} v_{1j} \to \infty.$$

Next we consider the cases (A) and (B.1).
Since in these cases every test $\phi \in \mathcal{C}_\alpha$ has property (vi), it is not clear which particular test we should take.
In this situation we prefer the test $\phi_1 \in \mathcal{C}_\alpha$ which corresponds to the choice of $a_1 = a_2 = 1$ (see (3.27) of Section 3), i.e.,

$$\begin{aligned} \phi_1(T) &= 1 \text{ if } S = \frac{1+T_2}{1+T_1} \leq c \\ &= 0 \text{ if } S = \frac{1+T_2}{1+T_1} > c, \end{aligned} \tag{6.3}$$

where c satisfies $F_0(c) = \alpha$.
Equivalently, we can say that the test ϕ_1 has critical region

$$S_1 = \frac{1+T_2}{1+T_1} \leq c_1, \tag{6.4}$$

where c_1 is chosen in such a way that $F_{1,0}(c_1) = \alpha$.
Here $F_{1,v}(s)$ is the distribution function of the test statistic S_1, i.e., $F_{1,v}(s) = P_v(S_1 \leq s)$.
The reason for choosing ϕ_1 is that this test turns out to be the generalized likelihood-ratio (GLR) test with level α for the problem of testing linear hypotheses, as is shown in Bouman [1].

In general, when λ is the GLR statistic, the GLR test with level α has critical region

$$\lambda \leq \lambda_0, \tag{6.5}$$

where λ_0 has to be taken in such a way that

$$\sup_{H_0} \Pr(\lambda \leq \lambda_0) = \alpha. \tag{6.6}$$

In the case of testing linear hypotheses it is not difficult to show (see Bouman [1]) that

$$\lambda = S_1^{\frac{n}{2}}. \tag{6.7}$$

Hence the level α GLR test has critical region

(6.8) $$S_1 \leq \lambda_0^{\frac{2}{n}} = c_1,$$

where c_1 has to be chosen in such a way that

(6.9) $$\sup_{\omega_0} P_v(S_1 \leq c_1) = \alpha.$$

Now, in general it is not clear how we can find the critical value c_1. However, as we saw above, in the cases (A), (B.1) and (B.3) we have

(6.10) $$\sup_{\omega_0} P_v(S_1 \leq s) = P_0(S_1 \leq s),$$

for all $s > 0$.

This shows that in these cases c_1 can be found from

(6.11) $$F_{1,0}(c_1) = P_0(S_1 \leq c_1) = \alpha.$$

In other words, in the cases (A), (B.1) and (B.3) the test ϕ_1 is the GLR test with level α.

However, in case (B.3) we choose the test ϕ_3 instead of ϕ_1, since ϕ_3 has the property (vi).

Although the GLR principle is not based on optimum considerations, it has been very successful in leading to satisfactory procedures in many specific problems.

Moreover, under fairly general conditions the GLR test possesses optimum asymptotic properties.

Another, more practical, reason for choosing the test ϕ_1 in the cases (A) and (B.1) is the fact that the test statistic S_1 can very easily be computed form the original data set (y, X, Z), as we shall see in Section 8.

On the other hand, also from the practical point of view, in case (A) we could perhaps better choose the tests ϕ_2 and ϕ_3, since the critical values c_2 and c_3 of these tests can be found from the central F distribution. This can be seen as follows. In case (A) we have $r = p$ and therefore $M = 0$. It is seen from (2.19), (2.20) and (2.21) of Section 2 that

$$T_1 = \frac{U_{10}}{U_4}$$

(6.12)

$$T_2 = \frac{U_{20}}{U_4}$$

where U_{10}, U_{20} and U_4 are mutually independent and posses the following distributions

$$U_{10} \sim \chi^2(\ell-p,\ d_1)$$

(6.13) $$U_{20} \sim \chi^2(k-p,\ d_0)$$

$$U_4 \sim \chi^2(m).$$

Note that $d_0 = v_{20}$ and $d_1 = v_{10}$, since $M = 0$.
The test statistics S_2 and S_3 as defined in (6.1) and (6.2), respectively, can be written as

$$S_2 = \frac{U_{20}}{U_{10} + U_4}$$

(6.14)

$$S_3 = \frac{U_{20} + U_4}{U_{10}}$$

From $m = n+p-k-\ell$ and the above results it easily follows that

$$U_{10} + U_4 \sim \chi^2(n-k,\ d_1)$$

(6.15)

$$U_{20} + U_4 \sim \chi^2(n-\ell,\ d_0).$$

Under H_0 we have $d_0 \geq 0$ and $d_1 = 0$ and it is seen from (6.13), (6.14) and (6.15) that

$$S_2\,\frac{n-k}{k-p} \sim F(k-p,\ n-k,\ d_0)$$

(6.16)

$$S_3\,\frac{\ell-p}{n-\ell} \sim F(n-\ell,\ \ell-p,\ d_0)$$

When H_1 is true we have $d_0 = 0$ and $d_1 > 0$ and we get in a similar way

$$\frac{1}{S_2}\frac{k-p}{n-k} \sim F(n-k,\ k-p,\ d_1)$$

(6.17)

$$\frac{1}{S_3}\frac{n-\ell}{\ell-p} \sim F(\ell-p,\ n-\ell,\ d_1)$$

In order to find the critical values c_2 and c_3 we need the distributions of S_2 and S_3, respectively, when $v = 0$. Now $v = 0$ if and only if $d_0 = d_1 = 0$ and it is easily seen from (6.16) that for $v = 0$ we get

$$S_2 \frac{n-k}{k-p} \sim F(k-p,\ n-k)$$

(6.18)

$$S_3 \frac{\ell-p}{n-\ell} \sim F(n-\ell,\ \ell-p),$$

which shows that in case (A) the critical values c_2 and c_3 can be found from the tables of the central F distribution.
The results of this section can be summarized in the following table.

Table 2.

Situation					
	A	B			
		B.1	B.2	B.3	B.4
Test	ϕ_1 ϕ_2, ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_2

The test ϕ_1, ϕ_2 and ϕ_3 have critical regions as defined in (6.4), (6.1) and (6.2), respectively.
Finally we make the following remarks.

When $|k-\ell| \geq 2$ there always exists a test with property (vi). For if $|k-\ell| \geq 2$, then either $k-\ell \geq 2$ or $\ell-k \geq 2$. That is, either $\ell \leq k-2$ or $k \leq \ell-2$.

Since $r \leq \min(k, \ell)$ it follows that $|k-\ell| \geq 2$ implies that $r \leq \max(k, \ell) - 2$. In other words, when $|k-\ell| \geq 2$ the situation (B.4), i.e., $r > \max(k, \ell) - 2$, never occurs and it is seen from the above results that there always exists a test with property (vi).

We recall that property (vi) implies the properties (i), (ii) and (v). That is, when a test possesses property (vi) it has level α (i.e., it is an exact test) and it is unbiased and strictly discriminating.

The test ϕ_2 is not defined in the case $p = k < \ell$, i.e., the nested case with $M(X) \subset M(Z)$ and $M(X) \neq M(Z)$.

Obviously, this is a subcase of (A) and the critical regions of the tests ϕ_1 and ϕ_3 become

$$S_1 = \frac{1}{1 + T_1} \leq c_1 \tag{6.19}$$

and

$$S_3 = \frac{1}{T_1} \leq c_3, \tag{6.20}$$

respectively.

Both (6.19) and (6.20) are equivalent to the test with critical region

$$T_1 \frac{n-\ell}{\ell-k} \geq c, \tag{6.21}$$

which is precisely the UMP invariant level α test for this nested case (see (3.6) of Section 3).

Similarly, the test ϕ_3 is not defined when $p = \ell < k$, i.e., the nested case with $M(Z) \subset M(X)$ and $M(X) \neq M(Z)$.

Again this is a subcase of (A) and the critical regions of the tests ϕ_1 and ϕ_2 are

$$S_1 = 1 + T_2 \leq c_1 \tag{6.22}$$

and

(6.23) $S_2 = T_2 \leq c_2$,

respectively.
In this case (6.22) and (6.23) are equivalent to the test with critical region

(6.24) $T_2 \frac{n-k}{k-\ell} \leq c$,

being the UMP invariant level α test (see (3.15) of Section 3).
Further it should be emphasized that under all circumstances, that is, regardless of the value of r, the test ϕ_2 is a level α test, whereas ϕ_3 has guaranteed power greater than α.

7. The distribution functions of the test statistics

The distribution functions of the test statistics S_1, S_2 and S_3 can easily be derived from the general formula of the distribution function $F_v(s)$ of the test statistic S as given in (4.33), (4.34) and (4.35) of Section 4.

Since

$$S_1 = \frac{1 + T_2}{1 + T_1},$$

the distribution function $F_{1,v}(s)$ of S_1 can be obtained from $F_v(s)$ through the substitution of $a_1 = a_2 = 1$.

By making use of $\lambda(s) = -a_1 s + a_2 = 1-s$ we get

$$(7.1) \qquad F_{1,v}(s) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin \varepsilon_{1,v}(u, s)}{u\gamma_{1,v}(u, s)}du,$$

where

$$(7.2) \qquad \gamma_{1,v}(u, s) = [\prod_{j=0}^{M} (1 + \lambda_{1j}^2(s)u^2)^{\frac{m_{1j}}{4}} (1 + \lambda_{2j}^2(s)u^2)^{\frac{m_{2j}}{4}}](1+(1-s)^2u^2)^{\frac{m}{4}}$$

$$\exp\{\frac{1}{2}\sum_{j=0}^{M}[\frac{\tau_{1j}(s)\lambda_{1j}^2(s)u^2}{1 + \lambda_{1j}^2(s)u^2} + \frac{\tau_{2j}(s)\lambda_{2j}^2(s)u^2}{1 + \lambda_{2j}^2(s)u^2}]\}$$

and

$$(7.3) \qquad \varepsilon_{1,v}(u, s) = \frac{1}{2}\sum_{j=0}^{M}[m_{1j}\text{arctg}(\lambda_{1j}(s)u) + m_{2j}\text{arctg}(\lambda_{2j}(s)u)$$

$$+ \frac{\tau_{1j}(s)\lambda_{1j}(s)u}{1 + \lambda_{1j}^2(s)u^2} + \frac{\tau_{2j}(s)\lambda_{2j}(s)u}{1 + \lambda_{2j}^2(s)u^2}] + \frac{1}{2}m\ \text{arctg}((1-s)u).$$

The value of the integrand at u = 0 is given by

$$(7.4) \qquad \left.\frac{\sin \varepsilon_{1,v}(u, s)}{u\ \gamma_{1,v}(u, s)}\right|_{u=0} = \frac{1}{2}[(n-\ell)-s(n-k) + (d_0 - sd_1)].$$

The coefficients m_{10} and m_{20} are defined as

(7.5) $$m_{10} = \ell - r$$ $$m_{20} = k - r$$

The test statistic S_2 is given by

$$S_2 = \frac{T_2}{1 + T_1}.$$

Hence, the distribution function $F_{2,v}(s)$ of S_2 can be found from $F_v(s)$ through substituting $a_1 = 1$ and $a_2 = 0$.
Since $\lambda(s) = -a_1 s + a_2 = -s = \lambda_{10}(s)$, the terms $(1 + \lambda_{10}^2(s)u^2)$ and $(1 + \lambda^2(s)u^2)$ in (4.34) and $\frac{1}{2}\, m_{10}\operatorname{arctg}(\lambda_{10}(s)u)$ and $\frac{1}{2}\, m \operatorname{arctg}(\lambda(s)u)$ in (4.35) can be taken together. We get:

(7.6) $$F_{2,v}(s) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin \varepsilon_{2,v}(u, s)}{u\, \gamma_{2,v}(u, s)}du,$$

where

(7.7) $$\gamma_{2,v}(u, s) = [\prod_{j=0}^{M} (1 + \lambda_{1j}^2(s)u^2)^{\frac{m_{1j}}{4}} (1 + \lambda_{2j}^2(s)u^2)^{\frac{m_{2j}}{4}}]$$

$$\exp\{\frac{1}{2} \sum_{j=0}^{M} [\frac{\tau_{1j}(s)\lambda_{1j}^2(s)u^2}{1 + \lambda_{1j}^2(s)u^2} + \frac{\tau_{2j}(s)\lambda_{2j}^2(s)u^2}{1 + \lambda_{2j}^2(s)u^2}]\}$$

and

(7.8) $$\varepsilon_{2,v}(u, s) = \frac{1}{2} \sum_{j=0}^{M} [m_{1j}\operatorname{arctg}(\lambda_{1j}(s)u) + m_{2j}\operatorname{arctg}(\lambda_{2j}(s)u)$$

$$+ \frac{\tau_{1j}(s)\lambda_{1j}(s)u}{1 + \lambda_{1j}^2(s)u^2} + \frac{\tau_{2j}(s)\lambda_{2j}(s)u}{1 + \lambda_{2j}^2(s)u^2}].$$

The value of the integrand at $u = 0$ is given by

(7.9) $$\left.\frac{\sin \varepsilon_{2,v}(u, s)}{u\, \gamma_{2,v}(u, s)}\right|_{u=0} = \frac{1}{2}[(k-p) - s(n-k) + (d_0 - sd_1)]$$

and m_{10} and m_{20} are defined as

$$(7.10)\qquad \begin{aligned} m_{10} &= \ell-r+m = n+p-k-r \\ m_{10} &= k-r. \end{aligned}$$

The test statistic S_3 is defined as

$$S_3 = \frac{1 + T_2}{T_1}.$$

This shows that the distribution function $F_{3,v}(s)$ of S_3 can be obtained from $F_v(s)$ by substituting $a_1 = 0$ and $a_2 = 1$. Now we have $\lambda(s) = -a_1 s + a_2 = 1 = \lambda_{20}(s)$ and in this case the terms $(1 + \lambda_{20}^2(s)u^2)$ and $(1 + \lambda^2(s)u^2)$ in (4.34) and $\frac{1}{2}\, m_{20}\mathrm{arctg}(\lambda_{20}(s)u)$ and $\frac{1}{2}\, m\ \mathrm{arctg}(\lambda(s)u)$ in (4.35) can be taken together. This yields:

$$(7.11)\qquad F_{3,v}(s) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin \varepsilon_{3,v}(u, s)}{u\ \gamma_{3,v}(u, s)}du,$$

where

$$(7.12)\qquad \gamma_{3,v}(u, s) = \gamma_{2,v}(u, s)$$

and

$$(7.13)\qquad \varepsilon_{3,v}(u, s) = \varepsilon_{2,v}(u, s).$$

In this case the value of the integrand at $u = 0$ is given by

$$(7.14)\qquad \left.\frac{\sin \varepsilon_{3,v}(u, s)}{u\ \gamma_{3,v}(u, s)}\right|_{u=0} = \frac{1}{2}\,[(n-\ell) - s(\ell-p) + (d_0 - sd_1)]$$

The coefficients m_{10} and m_{20} become

$$(7.15)\qquad \begin{aligned} m_{10} &= \ell-r \\ m_{20} &= k-r+m = n+p-\ell-r. \end{aligned}$$

In the above formulae the following coefficients are used:

$$m = n+p-k-\ell$$

$$r = p + \sum_{j=1}^{M} m_j$$

$$m_{1j} = m_{2j} = m_j, \; j = 1, 2, \ldots, M$$

$$\left.\begin{aligned} \lambda_{1j}(s) &= \frac{1}{2}(1-s) - \frac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)} \\ \lambda_{2j}(s) &= \frac{1}{2}(1-s) + \frac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)} \end{aligned}\right\}, \; j = 0, 1, \ldots, M,$$

where $\rho_0 = 0$

$$\left.\begin{aligned} \tau_{1j}(s) &= c_{1j}(s)v_{2j} + d_{1j}(s)v_{1j} \\ \tau_{2j}(s) &= c_{2j}(s)v_{2j} + d_{2j}(s)v_{1j} \end{aligned}\right\}, \; j = 0, 1, \ldots, M,$$

where

$$c_{1j}(s) = \frac{\lambda_{1j}(s) + s}{\lambda_{1j}(s)[\lambda_{1j}(s) - \lambda_{2j}(s)]}$$

$$c_{2j}(s) = \frac{\lambda_{2j}(s) + s}{\lambda_{2j}(s)[\lambda_{2j}(s) - \lambda_{1j}(s)]}$$

and

$$d_{1j}(s) = \frac{-s[\lambda_{1j}(s) - 1]}{\lambda_{1j}(s)[\lambda_{1j}(s) - \lambda_{2j}(s)]}$$

$$d_{2j}(s) = \frac{-s[\lambda_{2j}(s) - 1]}{\lambda_{2j}(s)[\lambda_{2j}(s) - \lambda_{1j}(s)]}$$

$$d_0 = \sum_{j=0}^{M} v_{2j}$$

$$d_1 = \sum_{j=0}^{M} v_{1j}.$$

Note that under H_0 ($v \in \omega_0$) we have $v_{1j} = 0$, $v_{2j} \geq 0$, $j = 0, 1, \ldots, M$.

This implies that $\tau_{10}(s) = c_{10}(s)v_{20} = 0$, $\tau_{20}(s) = c_{20}(s)v_{20} = v_{20} \geq 0$, $\tau_{1j}(s) = c_{1j}(s)v_{2j} \geq 0$, $\tau_{2j}(s) = c_{2j}(s)v_{2j} \geq 0$ for $j = 1, 2, \ldots, M$ and $d_0 \geq 0$, $d_1 = 0$.

Under H_1 ($v \in \omega_1$) we get $v_{1j} \geq 0$, $v_{2j} = 0$, $j = 0, 1, \ldots, M$ and $v_{1j} > 0$ for at least one j. In this case it follows that $\tau_{10}(s) = d_{10}(s)v_{10} = v_{10} \geq 0$, $\tau_{20}(s) = d_{20}(s)v_{10} = 0$, $\tau_{1j}(s) = d_{1j}(s)v_{1j} \geq 0$, $\tau_{2j}(s) = d_{2j}(s)v_{1j} \geq 0$ for $j = 1, 2, \ldots, M$, where $\tau_{1j}(s) > 0$, $\tau_{2j}(s) > 0$ for at least one j and $d_0 = 0$, $d_1 > 0$.

In particular when $v = 0$ the above formulae are of a simple form, since in this case we have $v_{1j} = v_{2j} = 0$ for $j = 0, 1, \ldots, M$ and this implies that $\tau_{1j}(s) = \tau_{2j}(s) = 0$ for $j = 0, 1, \ldots, M$ and also $d_0 = d_1 = 0$.

The above results show that $F_{i,v}(s)$, $i = 1, 2, 3$ can be computed for any $s > 0$ and $v \in \omega_0 \cup \omega_1$ through numerical integration.

With the aid of he above formulae we can compute the critical values, the p-values and the power functions of the tests ϕ_1, ϕ_2 and ϕ_3.

The level α critical value c_i of the test ϕ_i satisfies $F_{i,0}(c_i) = \alpha$.

So we have to find the solution c_i of the equation

$$\alpha = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin \varepsilon_{i,0}(u, c_i)}{u\, \gamma_{i,0}(u, c_i)}du, \tag{7.16}$$

$i = 1, 2, 3$.

The p-value of the test ϕ_i is given by

$$F_{i,0}(S_i) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin \varepsilon_{i,0}(u, S_i)}{u\, \gamma_{i,0}(u, S_i)}du, \tag{7.17}$$

$i = 1, 2, 3$, where S_i is the test statistic.

Since the power function $\pi(\phi_i, v)$ of the test ϕ_i is given by $\pi(\phi_i, v) = F_{i,v}(c_i)$, we have

$$\pi(\phi_i, v) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin \varepsilon_{i,v}(u, c_i)}{u\, \gamma_{i,v}(u, c_i)}du, \tag{7.18}$$

$i = 1, 2, 3$, where $v \in \omega_0 \cup \omega_1$.

8. The computation of the test statistics and power functions

In order to apply the tests ϕ_1, ϕ_2 and ϕ_3 in practice we have to compute the values of the test statistics S_1, S_2 and S_3, respectively, from the original given data set (y, X, Z).
Moreover, if we want to compute the power function $\pi(\phi_i, v)$ of the test ϕ_i at a certain point $v \in \omega_0 \cup \omega_1$, it is important to know how the vector v can be expressed in terms of the original parameters (μ, σ), where $\sigma > 0$ and $\mu = X\beta$, $\beta \in \mathbb{R}^k$ under H_0 or $\mu = Z\gamma \neq X\beta$, $\gamma \in \mathbb{R}^\ell$ under H_1. That is, we want to compute v_{1j} and v_{2j} for $j = 0, 1, \ldots, M$ from the original parameter set (β, γ, σ). We start with the computation of S_1, S_2 and S_3.
Since the test statistics are simple functions of the maximal invariant statistic $T = (T_1, T_2)$, we shall first investigate how T can be expressed in terms of (y, X, Z).
We know from Section 2 that

(8.1)
$$T_1 = \frac{w_1'w_1}{w_4'w_4}$$
$$T_2 = \frac{w_2'w_2}{w_4'w_4},$$

where

(8.2)
$$w_1 = R_1'y$$
$$w_2 = R_2'y$$
$$w_4 = R_4'y.$$

The matrices R_1, R_2 and R_4 are submatrices of the transformation matrix R as defined in Section 2, i.e.,

(8.3)
$$R = [\underset{(\ell-p)}{R_1} \vdots \underset{(k-p)}{R_2} \vdots \underset{(p)}{R_3} \vdots \underset{(m)}{R_4}](n).$$

The substitution of (8.2) into (8.1) yields

(8.4)
$$T_1 = \frac{y'R_1R_1'y}{y'R_4R_4'y}$$
$$T_2 = \frac{y'R_2R_2'y}{y'R_4R_4'y}.$$

Now it is important to note that it is not necessary to construct a transformation matrix R, with the properties as stated in Section 2, in order to compute T_1 and T_2. This can be seen as follows.
As is shown in Bouman [1], the columnvectors of the n × (n-k) matrix $[R_1 \vdots R_4]$ form an orthonormal basis for the (n-k)-dimensional linear subspace $\mathcal{M}(X)^{\perp}$, which implies that

(8.5) $$R_1R_1' + R_4R_4' = M_X = I - X(X'X)^{-1}X'.$$

In a similar way it can be shown that

(8.6) $$R_2R_2' + R_4R_4' = M_Z = I - Z(Z'Z)^{-1}Z'.$$

Further we know from Section 2 that the columnvectors of the n×m matrix R_4 form an orthonormal basis for the m-dimensional linear subspace $\mathcal{M}(X)^{\perp} \cap \mathcal{M}(Z)^{\perp} = \mathcal{M}([X \vdots Z])^{\perp}$, where $m = n+p-k-\ell$. Hence, if the columnvectors of the n × (k+ℓ-p) matrix G form an arbitrary basis for the (k+ℓ-p)-dimensional linear subspace $\mathcal{M}([X \vdots Z])$, it follows that

(8.7) $$R_4R_4' = M_G = I - G(G'G)^{-1}G'.$$

How can we find such a matrix G?
We first observe that the number $p = \dim(\mathcal{M}(X) \cap \mathcal{M}(Z))$ is given, since p is equal to the multiplicity of the eigenvalue 1 of the matrix $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$ (or equivalently, $(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z$) and since we suppose that the eigenvalues and multiplicities of this matrix are known. We recall that $0 \leq p \leq \min(k, \ell)$.
In the second place we note that the number of columns that X and Z have in common is at most equal to p. Now it is not difficult to see that we

can always find a $n \times (k-p)$ submatrix X_2 of X such that the columnvectors of the matrix $[X_2 \vdots Z]$ form a basis for $M([X \vdots Z])$. If x_1, x_2, ..., x_k denote the columnvectors of X, we take $A_0 = Z$ and $A_1 = [x_1 \vdots A_0]$. Then we compute $\det(A_1'A_1)$. If $\det(A_1'A_1) = 0$ we delete x_1 and take $A_2 = [x_2 \vdots A_0]$, whereas if $\det(A_1'A_1) > 0$ we take $A_2 = [x_2 \vdots A_1]$. Next we compute $\det(A_2'A_2)$. When $\det(A_2'A_2) = 0$ we delete x_2 and take $A_3 = [x_3 \vdots A_0]$ or $A_3 = [x_3 \vdots A_1]$, and if $\det(A_2'A_2) > 0$ we take $A_3 = [x_3 \vdots A_2]$.
Proceeding in this way we obtain a $n \times (k+\ell-p)$ matrix $[X_2 \vdots Z]$ with rank $(k+\ell-p)$, the columns of which form a basis for $M([X \vdots Z])$.
Consequently, we can take

$$G = [X_2 \vdots Z]. \tag{8.8}$$

In a similar manner we can find a $n \times (\ell-p)$ submatrix Z_2 of Z such that the columns of the $n \times (k+\ell-p)$ matrix $[X \vdots Z_2]$ form a basis for $M([X \vdots Z])$, that is, we can also take

$$G = [X \vdots Z_2]. \tag{8.9}$$

It should be noted that only in the case when the number of common columnvectors of X and Z is smaller than p we have to follow the above procedure in order to find the matrix G.
In most applications however, the number of common columnvectors is precisely equal to p and in this case the matrix G can very easily be found through inspection. This follows from the fact that if the number of common columns of X and Z is p, we have

$$X = [X_1 \vdots X_2] \text{ and } Z = [Z_1 \vdots Z_2],$$

where X_1 and Z_1 are of the order $n \times p$ and satisfy $X_1 = Z_1$. Therefore we can simply take $G = [X_2 \vdots Z]$ or $G = [X \vdots Z_2]$. Note that these latter two matrices are essentially the same. That is, they contain the same columns, only the order of the columns is different. With the aid of (8.5), (8.6) and (8.7) we obtain from (8.4)

$$(8.10) \qquad T_1 = \frac{y'(R_1R_1' + R_4R_4')y - y'R_4R_4'y}{y'R_4R_4'y} = \frac{y'M_Xy - y'M_Gy}{y'M_Gy} .$$

and

$$(8.11) \qquad T_2 = \frac{y'(R_2R_2' + R_4R_4')y - y'R_4R_4'y}{y'R_4R_4'y} = \frac{y'M_Zy - y'M_Gy}{y'M_Gy} .$$

Next we consider the following three linear models

$$(8.12) \qquad \begin{aligned} y &= X\beta + u \\ y &= Z\gamma + u \\ y &= G\delta + u. \end{aligned}$$

The least-squares estimators of β, γ and δ are

$$(8.13) \qquad \begin{aligned} \hat{\beta} &= (X'X)^{-1}X'y \\ \hat{\gamma} &= (Z'Z)^{-1}Z'y \\ \hat{\delta} &= (G'G)^{-1}G'y \end{aligned}$$

and the residual vectors become

$$(8.14) \qquad \begin{aligned} \hat{u}_X &= y - X\hat{\beta} = M_Xy \\ \hat{u}_Z &= y - Z\hat{\gamma} = M_Zy \\ \hat{u}_G &= y - G\hat{\delta} = M_Gy \end{aligned}$$

This shows that

$$y'M_X y = \hat{u}_X'\hat{u}_X$$

(8.15) $$y'M_Z y = \hat{u}_Z'\hat{u}_Z$$

$$y'M_G y = \hat{u}_G'\hat{u}_G,$$

Substitution of (8.15) into (8.10) and (8.11) yields

$$T_1 = \frac{\hat{u}_X'\hat{u}_X - \hat{u}_G'\hat{u}_G}{\hat{u}_G'\hat{u}_G}$$

(8.16)

$$T_2 = \frac{\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G}{\hat{u}_G'\hat{u}_G},$$

which shows that T_1 and T_2 can easily be computed from least-squares regression of y on X, Z and G, respectively.

Finally, it follows from the definitions of the test statistics S_1, S_2 and S_3 that

$$S_1 = \frac{1 + T_2}{1 + T_1} = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X}$$

(8.17) $$S_2 = \frac{T_2}{1 + T_1} = \frac{\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G}{\hat{u}_X'\hat{u}_X}$$

$$S_3 = \frac{1 + T_2}{T_1} = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X - \hat{u}_G'\hat{u}_G}$$

Alternative expressions for the test statistics S_2 and S_3 can be obtained in the following way.
From $G = [X_2 \vdots Z]$ it is not difficult to verify that

(8.18) $$G(G'G)^{-1}G' = Z(Z'Z)^{-1}Z' + M_Z X_2(X_2'M_Z X_2)^{-1}X_2'M_Z.$$

It easily follows from (8.18) that

$$(8.19)\qquad M_Z - M_G = X_{2*}(X'_{2*}X_{2*})^{-1}X'_{2*},$$

where the n × (k-p) matrix X_{2*} is defined by

$$(8.20)\qquad X_{2*} = M_Z X_2.$$

The result (8.19) implies that

$$(8.21)\qquad \hat{u}'_Z\hat{u}_Z - \hat{u}'_G\hat{u}_G = y'(M_Z - M_G)y = \hat{\beta}'_{2*}X'_{2*}X_{2*}\hat{\beta}_{2*},$$

where

$$(8.22)\qquad \hat{\beta}_{2*} = (X'_{2*}X_{2*})^{-1}X'_{2*}y.$$

Note that $\hat{\beta}_{2*}$ is the least-squares estimator of the parameter vector β_{2*} in the linear model

$$(8.23)\qquad y = X_{2*}\beta_{2*} + u.$$

Substitution of (8.21) into the expression for S_2 as given in (8.17) yields

$$(8.24)\qquad S_2 = \frac{\hat{\beta}'_{2*}X'_{2*}X_{2*}\hat{\beta}_{2*}}{\hat{u}'_X\hat{u}_X},$$

where it should be noted that $\hat{\beta}'_{2*}X'_{2*}X_{2*}\hat{\beta}_{2*}$ is the so-called explained sum of squares in the linear model (8.23).
Similarly, from $G = [X \vdots Z_2]$ we obtain

$$(8.25)\qquad M_X - M_G = Z_{2*}(Z'_{2*}Z_{2*})^{-1}Z'_{2*},$$

where the n × (ℓ-p) matrix Z_{2*} is given by

$$(8.26)\qquad Z_{2*} = M_X Z_2.$$

This shows that

(8.27) $$\hat{u}_X'\hat{u}_X - \hat{u}_G'\hat{u}_G = y'(M_X - M_G)y = \hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*},$$

where

(8.28) $$\hat{\gamma}_{2*} = (Z_{2*}'Z_{2*})^{-1}Z_{2*}'y.$$

That is, $\hat{\gamma}_{2*}$ is the least-squares estimator of γ_{2*} in the linear model

(8.29) $$y = Z_{2*}\gamma_{2*} + u.$$

With the aid of (8.17) and (8.27) we get

(8.30) $$S_3 = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}}.$$

Here we see that $\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}$ is the explained sum of squares in the linear model (8.29).

Next we shall consider the computations which are required if we want to express the power function $\pi(\phi_i, v)$ of the test ϕ_i in terms of the original parameters (β, σ) under H_0 or (γ, σ) under H_1.

As is shown in Bouman [1], the parameters v_{1j} and v_{2j}, $j = 0, 1, \ldots, M$ can be expressed in terms of (β, γ, σ) as follows.

Under H_0 we have:

(8.31) $$v_{1j} = 0, \qquad j = 0, 1, \ldots, M$$
$$v_{2j} = \frac{\beta' H_j H_j' \beta}{\sigma^2}, \quad j = 0, 1, \ldots, M$$

where $\beta \in \mathbb{R}^k$, $\sigma > 0$ and

(8.32) $$H_j = \sqrt{1-\rho_j}\, X'\bar{X}U_j, \quad j = 0, 1, \ldots, M,$$

with $\rho_0 = 0$.

Under H_1 we have:

(8.33)
$$v_{1j} = \frac{\gamma' K_j K_j' \gamma}{\sigma^2}, \quad j = 0, 1, \ldots, M$$
$$v_{2j} = 0, \qquad j = 0, 1, \ldots, M,$$

where $\gamma \in \mathbb{R}^{\ell}$ is such that $Z\gamma \neq X\beta$, $\beta \in \mathbb{R}^k$, where again $\sigma > 0$ and

(8.34) $$K_j = \sqrt{1-\rho_j}\, Z'\bar{Z}V_j, \quad j = 0, 1, \ldots, M.$$

In the expressions (8.32) and (8.34) we have used the matrices $\bar{X}$, $\bar{Z}$, U_j and V_j ($j = 0, 1, \ldots, M$), these matrices can be found from X and Z in the following way.

The columnvectors of the $n\times k$ matrix $\bar{X}$ form an arbitrary orthonormal basis for the linear subspace $M(X)$ and the columnvectors of the $n\times\ell$ matrix $\bar{Z}$ form an arbitrary orthonormal basis for $M(Z)$.

There are several ways to compute $\bar{X}$ and $\bar{Z}$ (which are not unique) from X and Z, respectively, for instance, we can use the well-known Gram-Schmidt procedure.

With $\bar{X}$ and $\bar{Z}$ we form the $k\times k$ matrix $\bar{X}'\bar{Z}\bar{Z}'\bar{X}$ and the $\ell\times\ell$ matrix $\bar{Z}'\bar{X}\bar{X}'\bar{Z}$.

We first note that these matrices have the same nonzero eigenvalues.

In the second place, it is not difficult to see that $\bar{X}'\bar{Z}\bar{Z}'\bar{X}$ has the same eigenvalues as the matrix $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$, whereas $\bar{Z}'\bar{X}\bar{X}'\bar{X}$ has the same eigenvalues as $(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z$.

That is, $\bar{X}'\bar{Z}\bar{Z}'\bar{X}$ has an eigenvalue 1 with multiplicity p, M different eigenvalues $\rho_1, \rho_2, \ldots, \rho_M$ with $0 < \rho_j < 1$ and multiplicities $m_1, m_2, \ldots, m_M$ and an eigenvalue $\rho_0 = 0$ with multiplicity k-r, where $r = p + \sum_{j=1}^{M} m_j$.

The same holds true for $\bar{Z}'\bar{X}\bar{X}'\bar{Z}$, except that the eigenvalue $\rho_0 = 0$ now has multiplicity ℓ-r.

Now the matrices U_j and V_j are defined in terms of the eigenvalues of the matrices $\bar{X}'\bar{Z}\bar{Z}'\bar{X}$ and $\bar{Z}'\bar{X}\bar{X}'\bar{Z}$, respectively.

The columns of the $k \times (k-r)$ matrix U_0 are the eigenvectors of $\bar{X}'\bar{Z}\bar{Z}'\bar{X}$ which correspond to the eigenvalue 0. The columns of the $k \times m_j$ matrix U_j are the eigenvectors of $\bar{X}'\bar{Z}\bar{Z}'\bar{X}$ corresponding to the eigenvalue ρ_j, $j = 1, 2, \ldots, M$.

Similarly, the columns of the $\ell \times (\ell - r)$ matrix V_0 are the eigenvectors of $\bar{Z}'\overline{XX}'\bar{Z}$ corresponding to the eigenvalue 0, whereas the columns of the $\ell \times m_j$ matrix V_j are the eigenvectors which correspond to the eigenvalue ρ_j, $j = 1, 2, \ldots, M$.
We suppose that: $U_0'U_0 = I_{(k-r)}$, $V_0'V_0 = I_{(\ell-r)}$ and $U_j'U_j = V_j'V_j = I_{(m_j)}$, $j = 1, 2, \ldots, M$.
Finally, we note with respect to (8.31) and (8.33) that: $v_{1j} = v_{2j} = 0$ for $j = 0, 1, \ldots, M$ if and only if $\mu \in M(X) \cap M(Z)$ and also

$$d_0 = \sum_{j=0}^{M} v_{2j} = \frac{\mu' M_Z \mu}{\sigma^2}, \quad d_1 = \sum_{j=0}^{M} v_{1j} = \frac{\mu' M_X \mu}{\sigma^2},$$

for $\mu \in M(X) \cup M(Z)$ and $\sigma > 0$.

9. Interpretation of the tests in terms of the original hypotheses

In this section we shall derive the tests ϕ_1, ϕ_2 and ϕ_3 on heuristic grounds and interprete the test statistics S_1, S_2 and S_3 in terms of the original linear hypotheses.
The original problem of testing linear hypotheses can be stated as follows:
On the basis of the vector of observations $y \sim n(\mu, \sigma^2 I)$ we want to test H_0: $\mu = X\beta$, $\beta \in \mathbb{R}^k$ against H_1: $\mu = Z\gamma$, $\gamma \in \mathbb{R}^\ell$, where under H_1, $\gamma \in \mathbb{R}^\ell$ is such that $Z\gamma \neq X\beta$, $\beta \in \mathbb{R}^k$.
Now under H_0 we have the linear model $y = X\beta + u$ and the total sum of squares $y'y$ can be decomposed into two parts

$$(9.1) \qquad y'y = \hat{\beta}'X'X\hat{\beta} + \hat{u}_X'\hat{u}_X,$$

where $\hat{\beta}'X'X\hat{\beta}$ is the explained sum of squares due to the variation in X and $\hat{u}_X'\hat{u}_X$ is the residual sum of squares, which is not explained by the variation in X.
As usual, $\hat{\beta}$ and $\hat{u}_X$ are given by: $\hat{\beta} = (X'X)^{-1}X'y$ and $\hat{u}_X = y - X\hat{\beta}$.
Analogously, under H_1 we have the linear model $y = Z\gamma + u$ and the decomposition

$$(9.2) \qquad y'y = \hat{\gamma}'Z'Z\hat{\gamma} + \hat{u}_Z'\hat{u}_Z,$$

where $\hat{\gamma} = (Z'Z)^{-1}Z'y$ and $\hat{u}_Z = y - Z\hat{\gamma}$.
Here $\hat{\gamma}'Z'Z\hat{\gamma}$ is the explained sum of squares due to the variation in Z and $\hat{u}_Z'\hat{u}_Z$ is the residual sum of squares.
On heuristic grounds it seems natural to reject H_0 if $\hat{u}_X'\hat{u}_X$ is large and $\hat{u}_Z'\hat{u}_Z$ is small, that is, if the ratio $\hat{u}_Z'\hat{u}_Z/\hat{u}_X'\hat{u}_X$ is small.
In other words, as a test statistic we take

$$(9.3) \qquad S_1 = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X},$$

and reject H_0 (the critical region) if

$$(9.4) \qquad S_1 \leq c_1.$$

This, however, is precisely the test ϕ_1 as proposed in the foregoing sections. As is already noted before, S_1 can also be derived from the GLR criterion.

Let us next consider the probability distributions of the denominator $\hat{u}_X'\hat{u}_X$ and the numerator $\hat{u}_Z'\hat{u}_Z$ of the test statistic S_1.

From $y \sim n(\mu, \sigma^2 I)$ it easily follows that

$$(9.5) \qquad \frac{\hat{u}_X'\hat{u}_X}{\sigma^2} \sim \chi^2(n-k, d_1)$$

$$\frac{\hat{u}_Z'\hat{u}_Z}{\sigma^2} \sim \chi^2(n-\ell, d_0)$$

Under H_0 we have:

$$(9.6) \qquad d_0 = \frac{\beta' X' M_Z X \beta}{\sigma^2} \geq 0$$

$$d_1 = 0,$$

and under H_1:

$$(9.7) \qquad d_0 = 0$$

$$d_1 = \frac{\gamma' Z' M_X Z \gamma}{\sigma^2} > 0.$$

The above results enable us to compute the means and variances of $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$, under H_0 and H_1, respectively.

Under H_0 we get:

$$(9.8) \qquad \begin{aligned} E(\hat{u}_X'\hat{u}_X) &= \sigma^2(n-k) \\ E(\hat{u}_Z'\hat{u}_Z) &= \sigma^2(n-\ell + d_0) \\ \mathrm{Var}(\hat{u}_X'\hat{u}_X) &= 2\sigma^4(n-k) \\ \mathrm{Var}(\hat{u}_Z'\hat{u}_Z) &= 2\sigma^4(n-\ell + 2d_0), \end{aligned}$$

and under H_1:

(9.9)
$$E(\hat{u}_X'\hat{u}_X) = \sigma^2(n-k + d_1)$$
$$E(\hat{u}_Z'\hat{u}_Z) = \sigma^2(n-\ell)$$
$$Var(\hat{u}_X'\hat{u}_X) = 2\sigma^4(n-k + 2d_1)$$
$$Var(\hat{u}_Z'\hat{u}_Z) = 2\sigma^4(n-\ell).$$

We recall that d_0 can be thought of as a measure of distance from a point $\mu = X\beta$ under H_0 to H_1 $(\mathcal{M}(Z))$. Similarly, d_1 can be considered as measuring the distance from a point $\mu = Z\gamma$ under H_1 to H_0 $(\mathcal{M}(X))$.
Note that $E(\hat{u}_X'\hat{u}_X)$ is small under H_0 and large under H_1. It is also seen that under H_1 the expected value $E(\hat{u}_X'\hat{u}_X)$ increases with d_1 and that $E(\hat{u}_X'\hat{u}_X) \to \infty$ if $d_1 \to \infty$.
On the other hand, $E(\hat{u}_Z'\hat{u}_Z)$ is large under H_0 and small under H_1. Further, $E(\hat{u}_Z'\hat{u}_Z)$ increases with d_0 under H_0 and $E(\hat{u}_Z'\hat{u}_Z) \to \infty$ if $d_0 \to \infty$.
At this point it should be observed that the denominator $\hat{u}_X'\hat{u}_X$ and the numerator $\hat{u}_Z'\hat{u}_Z$ of S_1 are not stochastically independent. Consequently, besides the means and variances of $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$, the covariance of $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$ is an important characteristic of the joint distribution of these random variables.
In order to find $Cov(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z)$ we first recall from Section 2 that:

(9.10)
$$\frac{\hat{u}_X'\hat{u}_X}{\sigma^2} = \sum_{j=0}^{M} U_{1j} + U_4$$
$$\frac{\hat{u}_Z'\hat{u}_Z}{\sigma^2} = \sum_{j=0}^{M} U_{2j} + U_4,$$

where U_{10}, U_{20}, (U_{11}, U_{21}), ..., (U_{1M}, U_{2M}) and U_4 are mutually independent random variables.
We know that $U_{10} \sim \chi^2(\ell-r, v_{10})$, $U_{20} \sim \chi^2(k-r, v_{20})$, $U_4 \sim \chi^2(m)$, whereas (U_{1j}, U_{2j}), $j = 1, 2, \ldots, M$ has the characteristic function $\psi_j(t_1, t_2)$ as given in (2.26) of Section 2.

By making use of the above mentioned mutual independence we obtain

$$(9.11) \qquad \mathrm{Cov}[(U_{10} + \sum_{j=1}^{M} U_{1j} + U_4), (U_{20} + \sum_{j=1}^{M} U_{2j} + U_4)]$$

$$= \sum_{j=1}^{M} \mathrm{Cov}(U_{1j}, U_{2j}) + \mathrm{Var}(U_4).$$

As is shown in Appendix A, by using the general result

$$\mathrm{Cov}(U_{1j}, U_{2j}) = - \left.\frac{\partial^2 \ln \psi_j(t_1, t_2)}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0},$$

we get

$$(9.12) \qquad \mathrm{Cov}(U_{1j}, U_{2j}) = 2\rho_j m_j, \; j = 1, 2, \ldots, M,$$

for all $v \in \omega_0 \cup \omega_1$.
Since $\mathrm{Var}(U_4) = 2m$, where $m = n+p-k-\ell$, it follows from (9.10), (9.11) and (9.12) that

$$(9.13) \qquad \mathrm{Cov}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z) = 2\sigma^4(m + \sum_{j=1}^{M} \rho_j m_j),$$

for all (μ, σ) with $\mu \in M(X) \cup M(Z)$ and $\sigma > 0$.
It should be noted that $\mathrm{Cov}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z)$ does not depend on μ, that is, it takes on the same value under H_0 as well as H_1.
The above results enable us to compute the correlation coefficient of $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$. This coefficient is a measure of the dependence between $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$. We have under H_0:

$$(9.14) \qquad \mathrm{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z) = \frac{m + \sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k)(n-\ell+2d_0)}},$$

and under H_1:

$$(9.15) \qquad \mathrm{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z) = \frac{m + \sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k+2d_1)(n-\ell)}}.$$

Since $m > 0$, we always have

$$0 < \text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z) \leq 1,$$

and $\text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z) = 1$ if and only if $M(X) = M(Z)$ (that is, the trivial case of nested models).

In the latter situation we have: $p = r = k = \ell$, $M = 0$, $d_0 = d_1 = 0$ and also $\hat{u}_X'\hat{u}_X = \hat{u}_Z'\hat{u}_Z$ with probability 1.

When $p = r$, i.e., if $M = 0$, the correlation coefficient becomes

$$\frac{m}{\sqrt{(n-k)(n-\ell+2d_0)}}$$

under H_0, and

$$\frac{m}{\sqrt{(n-k+2d_1)(n-\ell)}}$$

under H_1.

It is seen that there always exists a positive dependence between $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$.

Further, if we consider the correlation coefficient as a function of the parameters (μ, σ), for given linear hypotheses, we see that $\text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z)$ depends on (μ, σ) through d_0 and d_1.

It easily follows that $\text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z)$ has the following maximum value

$$\frac{m + \sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k)(n-\ell)}}$$

when $d_0 = d_1 = 0$, i.e., for all points (μ, σ) with $\mu \in M(X) \cap M(Z)$, $\sigma > 0$. These points satisfy $\mu = X\beta = Z\gamma$ and form the common boundary between H_0 and H_1.

It is also seen, that under H_0 the correlation coefficient is a decreasing function of d_0 and that $\text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z) \to 0$ if $d_0 \to \infty$.

Similarly, $\text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z)$ is a decreasing function of d_1 under H_1 and $\text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z) \to 0$ if $d_1 \to \infty$.

The following table summarizes the moments of $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$.

Table 3: Moments of the numerator and denominator of S_1

	H_0: $\mu = X\beta$	H_1: $\mu = Z\gamma$
$E(\hat{u}_X'\hat{u}_X)$	$\sigma^2(n-k)$	$\sigma^2(n-k+d_1)$
$E(\hat{u}_Z'\hat{u}_Z)$	$\sigma^2(n-\ell+d_0)$	$\sigma^2(n-\ell)$
$Var(\hat{u}_X'\hat{u}_X)$	$2\sigma^4(n-k)$	$2\sigma^4(n-k+2d_1)$
$Var(\hat{u}_Z'\hat{u}_Z)$	$2\sigma^4(n-\ell+2d_0)$	$2\sigma^4(n-\ell)$
$Corr.(\hat{u}_X'\hat{u}_X,\ \hat{u}_Z'\hat{u}_Z)$	$\dfrac{m + \sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k)(n-\ell+2d_0)}}$	$\dfrac{m + \sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k+2d_1)(n-\ell)}}$

In the above discussion we ignored the fact that a part of the variation in X can be explained by Z and vice versa. That is, we did not take into account that X and Z can be correlated.
For instance, suppose that $\hat{u}_X'\hat{u}_X$, is small, or equivalently, that the explained sum of squares $\hat{\beta}'X'X\hat{\beta}$ is large.
Then the statistic

$$\frac{\hat{\beta}'X'X\hat{\beta}}{\hat{u}_X'\hat{u}_X}$$

has a large value and we could decide to accept H_0.
In doing so, however, we totally ignore the matrix Z, while at the same time it is possible that the statistic

$$\frac{\hat{\gamma}'Z'Z\hat{\gamma}}{\hat{u}_Z'\hat{u}_Z}$$

takes on a large value too, which points into the direction of accepting H_1.

The large value of $\hat{\beta}'X'X\hat{\beta}/\hat{u}_X'\hat{u}_X$ can be possibly due to the fact that the variation in X is partly explained by the variation of Z.
What we need is that part of X which does not depend on Z. To this extend we consider the following decomposition of X

(9.16) $\quad X = Z(Z'Z)^{-1}Z'X + M_Z X,$

where $M_Z = I - Z(Z'Z)^{-1}Z'$.
Here $Z(Z'Z)^{-1}Z'X$ is that part of X which is explained by Z and $M_Z X$ is the residual part of X not explained by Z. Note that the columnvectors of $M_Z X$ are the residual vectors after least-squares regression of X on Z.
These columnvectors span a (k-p)-dimensional subspace. If X_2 is the $n \times (k-p)$ submatrix of X as defined in the foregoing section, it follows that the columns of the $n \times (k-p)$ submatrix $M_Z X_2$ of $M_Z X$ form a basis for this subspace. That is, the matrix X adjusted for the influence of Z can be represented by $X_{2*} = M_Z X_2$.
Next we consider the explained sum of squares due to the adjusted X-matrix. In other words, we consider the linear model $y = X_{2*}\beta_{2*} + u$ and the decomposition

(9.17) $\quad y'y = \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*} + \hat{u}_{X_{2*}}'\hat{u}_{X_{2*}},$

where $\hat{\beta}_{2*} = (X_{2*}'X_{2*})^{-1}X_{2*}'y$ and $\hat{u}_{X_{2*}} = y - X_{2*}\hat{\beta}_{2*}$.
The explained sum of squares due to X adjusted for Z (i.e., due to X_{2*}) is equal to $\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}$ and a reasonable test statistic, instead of $\hat{\beta}'X'X\hat{\beta}/\hat{u}_X'\hat{u}_X$, seems to be

(9.18) $\quad S_2 = \dfrac{\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}}{\hat{u}_X'\hat{u}_X}.$

Since we expect S_2 to be small under H_1, we reject H_0 if $S_2 \leq c_2$.
This is precisely the test ϕ_2 as proposed in the foregoing sections (see (8.24) of Section 8).
Ofcourse, we can also adjust Z for the influence of X and, by using similar arguments as before, we obtain the test which rejects H_0 when

$$\frac{\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}}{\hat{u}_Z'\hat{u}_Z}$$

is large, or equivalently, when

$$(9.19) \qquad S_3 = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{\gamma}_{2*}'Z_{2*}'X_{2*}\hat{\gamma}_{2*}} \leq c_3.$$

Now this is precisely the test ϕ_3 as derived in the foregoing sections (see (8.30) of Section 8).

Here $\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}$ is the explained sum of squares due to Z adjusted for X, i.e., due to $Z_{2*} = M_X Z_2$, where $\hat{\gamma}_{2*} = (Z_{2*}'Z_{2*})^{-1}Z_{2*}'y$.

Again it is informative to compute the means, variances and correlation coefficients of the numerator and denominator of the test statistics under H_0 and H_1, respectively.

We start with S_2. From

$$(9.20) \qquad \begin{aligned} &\frac{\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}}{\sigma^2} = \frac{y'X_{2*}(X_{2*}'X_{2*})^{-1}X_{2*}'y}{\sigma^2} \sim \chi^2(k-p,\ d_0) \\ &\frac{\hat{u}_X'\hat{u}_X}{\sigma^2} \sim \chi^2(n-k,\ d_1), \end{aligned}$$

with d_0 and d_1 as given in (9.6) and (9.7), we can easily compute the means and variances of $\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}$ and $\hat{u}_X'\hat{u}_X$.

Further it can be seen from

$$(9.21) \qquad \begin{aligned} &\frac{\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}}{\sigma^2} = \sum_{j=0}^{M} U_{2j} \\ &\frac{u_X'u_X}{\sigma^2} = \sum_{j=0}^{M} U_{1j} + U_4 \end{aligned}$$

that

$$(9.22) \qquad \begin{aligned} \mathrm{Cov}(\hat{u}_X'\hat{u}_X,\ \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}) &= 2\sigma^4 \sum_{j=1}^{M} \rho_j m_j, && \text{if } r > p \\ \text{"} \qquad\qquad &= 0, && \text{if } r = p \end{aligned}$$

Again the covariance does not depend on $\mu \in \mathcal{M}(X) \cup \mathcal{M}(Z)$.

With the aid of (9.20) and (9.22) we get:

Table 4: Moments of the numerator and denominator of S_2

	H_0: $\mu = X\beta$	H_1: $\mu = Z\gamma$
$E(\hat{u}_X'\hat{u}_X)$	$\sigma^2(n-k)$	$\sigma^2(n-k+d_1)$
$E(\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*})$	$\sigma^2(k-p+d_0)$	$\sigma^2(k-p)$
$Var(\hat{u}_X'\hat{u}_X)$	$2\sigma^4(n-k)$	$2\sigma^4(n-k+2d_1)$
$Var(\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*})$	$2\sigma^4(k-p+2d_0)$	$2\sigma^4(k-p)$
$Corr.(\hat{u}_X'\hat{u}_X, \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*})$	$\frac{\sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k)(k-p+2d_0)}}$	$\frac{\sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k+2d_1)(k-p)}}$

It follows that we always have

$$0 \leq Corr.(\hat{u}_X'\hat{u}_X, \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}) < 1$$

and

$$Corr.(\hat{u}_X'\hat{u}_X, \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}) = 0$$

if and only if $r = p$, i.e., if $M = 0$.
If we consider the correlation coefficient as a function of the parameters (μ, σ), for given linear hypotheses, it is easily seen that it has the maximum value

$$\frac{\sum_{j=1}^{M} \rho_j m_j}{\sqrt{(n-k)(k-p)}}$$

for all points (μ, σ) on the boundary of H_0 and H_1, that is, points with $\mu \in M(X) \cap M(Z)$, $\sigma > 0$ $(d_0 = d_1 = 0)$.

The correlation coefficient decreases with d_0 under H_0 and with d_1 under H_1. Also, $\text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}) \to 0$ if $d_0 \to \infty$ or if $d_1 \to \infty$.
In the second place we consider S_3. The means, variances and correlation coefficient of the denominator $\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}$ and the numerator $\hat{u}_Z'\hat{u}_Z$ can be derived in a similar way as before.
We get:

Table 5: Moments of the numerator and denominator of S_3

	H_0: $\mu = X\beta$	H_1: $\mu = Z\gamma$
$E(\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*})$	$\sigma^2(\ell-p)$	$\sigma^2(\ell-p+d_1)$
$E(\hat{u}_Z'\hat{u}_Z)$	$\sigma^2(n-\ell+d_0)$	$\sigma^2(n-\ell)$
$\text{Var}(\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*})$	$2\sigma^4(\ell-p)$	$2\sigma^4(\ell-p+2d_1)$
$\text{Var}(\hat{u}_Z'\hat{u}_Z)$	$2\sigma^4(n-\ell+2d_0)$	$2\sigma^4(n-\ell)$
$\text{Corr.}(\hat{u}_Z'\hat{u}_Z, \hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*})$	$\dfrac{\sum_{j=1}^{M} \rho_j m_j}{\sqrt{(\ell-p)(n-\ell+2d_0)}}$	$\dfrac{\sum_{j=1}^{M} \rho_j m_j}{\sqrt{(\ell-p+2d_1)(n-\ell)}}$

In this case we also have

$$0 \leq \text{Corr.}(\hat{u}_Z'\hat{u}_Z, \hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}) < 1$$

and

$$\text{Corr.}(\hat{u}_Z'\hat{u}_Z, \hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}) = 0$$

if and only if $r = p$, i.e., if $M = 0$.
Considered as a function of (μ, σ), the correlation coefficient has the maximum value

$$\frac{\sum_{j=1}^{M} \rho_j m_j}{\sqrt{(\ell-p)(n-\ell)}}$$

at all points (μ, σ) with $\mu \in M(Z) \cap M(Z)$, $\sigma > 0$ $(d_0 = d_1 = 0)$. Again the correlation coefficient decreases with d_0 under H_0 and with d_1 under H_1, whereas $\text{Corr.}(\hat{u}_Z'\hat{u}_Z, \hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}) \to 0$ if $d_0 \to \infty$ or if $d_1 \to \infty$.

Finally, we make the following remarks.

1. The test statistics S_1, S_2 and S_3 are ratios of quadratic forms in normally distributed random variables. That is, the numerator and denominator of these statistics are χ^2-distributed random variables. However, due to the fact that, in general, the numerator and denominator are not stochastically independent, the ratio does not have a F distribution. In particular this holds true for S_1.

Now we have seen above that in the special case $r = p$ (which was called case (A) in Section 6), the correlation coefficient of the numerator and denominator of S_2 and S_3 is equal to zero. In general this does not imply that the numerator and demoninator are stochastically independent. However, as we saw in Section 6, when $r = p$ the numerator and denominator of S_2 and S_3 are independent. As a consequence, S_2 and S_3 (multiplied by a suitable constant) have a F distribution (see (6.16), (6.17) and (6.18) of Section 6) when $r = p$.

2. The correlation between the numerator and denominator of S_2 and S_3 is smaller than the correlation between the numerator and denominator of S_1. That is,

$$(9.23) \qquad \text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}) < \text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z)$$

and also

$$(9.24) \qquad \text{Corr.}(\hat{u}_Z'\hat{u}_Z, \hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}) < \text{Corr.}(\hat{u}_X'\hat{u}_X, \hat{u}_Z'\hat{u}_Z).$$

We only prove (9.23). The proof of (9.24) is quite analogous.

First consider the situation under H_0. Then (9.23) becomes

$$(9.25) \qquad \frac{\Sigma\, \rho_j m_j}{\sqrt{(n-k)(k-p+2d_0)}} < \frac{m + \Sigma\, \rho_j m_j}{\sqrt{(n-k)(n-\ell+2d_0)}},$$

where $d_0 \geq 0$.

By making use of $m = n+p-k-\ell = n-\ell-(k-p) = n-\ell+2d_0 - (k-p+2d_0)$, the inequality (9.25) can be written as:

$$(9.26) \qquad \frac{\Sigma\, \rho_j m_j}{k-p+2d_0} < \sqrt{\frac{n-\ell+2d_0}{k-p+2d_0}} - \sqrt{\frac{k-p+2d_0}{n-\ell+2d_0}} + \frac{\Sigma\, \rho_j m_j}{k-p+2d_0} \sqrt{\frac{k-p+2d_0}{n-\ell+2d_0}}$$

Let

$$x = \frac{\Sigma\, \rho_j m_j}{k-p+2d_0}$$

$$y = \sqrt{\frac{k-p+2d_0}{n-\ell+2d_0}}$$

then it follows that $0 \leq x < 1$ and $0 < y < 1$, since $0 \leq \Sigma\, \rho_j m_j < r-p \leq k-p$ and therefore $0 \leq \Sigma\, \rho_j m_j < k-p+2d_0$, whereas $m = n+p-k-\ell = n-\ell+2d_0 - (k-p+2d_0) > 0$ implies that

$$0 < \frac{k-p+2d_0}{n-\ell+2d_0} < 1.$$

Hence (9.26) can be rewritten as

$$(9.27) \qquad x < \frac{1}{y} - y + xy,$$

for $0 \leq x < 1$ and $0 < y < 1$.
Now (9.25) is equivalent to

$$(1-y)x < \frac{1-y^2}{y},$$

which can be rewritten as

$$(1-y)x < \frac{(1-y)(1+y)}{y}.$$

Multiplying the terms at both sides of the inequality sign by $y/(1-y)$ yields

$$(9.28) \qquad xy < 1+y.$$

Obviously, the inequality (9.28) holds true for $0 \leq x < 1$ and $0 < y < 1$, which proves (9.25).
In the second place consider H_1, then (9.23) becomes

$$(9.29) \qquad \frac{\Sigma\ \rho_j m_j}{\sqrt{(n-k+2d_1)(k-p)}} < \frac{m + \Sigma\ \rho_j m_j}{\sqrt{(n-k+2d_1)(n-\ell)}},$$

where $d_1 > 0$.
This is equivalent to

$$(9.30) \qquad \frac{\Sigma\ \rho_j m_j}{\sqrt{k-p}} < \frac{m + \Sigma\ \rho_j m_j}{\sqrt{n-\ell}}.$$

Since (9.25) holds true for $d_0 = 0$, the inequality (9.30) follows at once from (9.25). This shows the truth of (9.29) and completes the proof of (9.23).

10. A large sample approximation to the critical values and p-values of the tests

As we saw in Section 7, the computation of the critical values and p-values of the test ϕ_1, ϕ_2, ϕ_3 requires numerical integration. However, when the sample size n is large, the critical values and p-values can be approximated in a very simple way, as we shall see below.

We know that, in order to find the critical values and p-values, we need the distribution functions of the test statistics S_1, S_2 and S_3 on the boundary of H_0 and H_1. In other words, if $F_{i,v}(s) = P_v(S_i \leq s)$, i = 1, 2, 3, the critical values and p-values are computed from $F_{i,0}(s)$, that is from the distribution functions at the parameter point v = 0. As we know, the point v = 0 uniquely corresponds to the points (μ, σ) satisfying $\mu \in M(X) \cap M(Z)$, $\sigma > 0$.

We recall that the test statistics can be written as the following ratios of quadratic forms (sums of squares)

$$S_1 = \frac{\hat{u}_Z'\hat{u}_Z/\sigma^2}{\hat{u}_X'\hat{u}_X/\sigma^2}$$

$$(10.1) \qquad S_2 = \frac{\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}/\sigma^2}{\hat{u}_X'\hat{u}_X/\sigma^2}$$

$$S_3 = \frac{\hat{u}_Z'\hat{u}_Z/\sigma^2}{\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}/\sigma^2}$$

Note that, under the hypothesis v = 0 (i.e., $\mu \in M(X) \cap M(Z)$, the numerator and denominator of S_1, S_2 and S_3 are χ^2-distributed random variables. We have

$$\hat{u}_X'\hat{u}_X/\sigma^2 \sim \chi^2(n-k)$$

$$\hat{u}_Z'\hat{u}_Z/\sigma^2 \sim \chi^2(n-\ell)$$

$$(10.2) \qquad \hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}/\sigma^2 \sim \chi^2(k-p)$$

$$\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}/\sigma^2 \sim \chi^2(\ell-p)$$

Now in the applications it is often convenient to divide these χ^2 variables by their mean values (number of degrees of freedom) and to work with the mean sums of squares, instead of the sums of squares. In doing this, we obtain the following modified test statistics:

$$S_1^* = \frac{\hat{u}_Z'\hat{u}_Z/[\sigma^2(n-\ell)]}{\hat{u}_X'\hat{u}_X/[\sigma^2(n-k)]} = S_1\,\frac{n-k}{n-\ell}$$

$$(10.3)\qquad S_2^* = \frac{\hat{\beta}_{2*}'X_{2*}'X_{2*}\hat{\beta}_{2*}/[\sigma^2(k-p)]}{\hat{u}_X'\hat{u}_X/[\sigma^2(n-k)]} = S_2\,\frac{n-k}{k-p}$$

$$S_3^* = \frac{\hat{u}_Z'\hat{u}_Z/[\sigma^2(n-\ell)]}{\hat{\gamma}_{2*}'Z_{2*}'Z_{2*}\hat{\gamma}_{2*}/[\sigma^2(\ell-p)]} = S_3\,\frac{\ell-p}{n-\ell}\,.$$

If $F_{i,v}^*(s)$ is the distribution function of S_i^*, i.e.,

$$(10.4)\qquad F_{i,v}^*(s) = P_v(S_i^* \leq s),\ i = 1, 2, 3,$$

we can express the test ϕ_i in terms of the test statistic S_i^*. We get for $i = 1, 2, 3$:

The test ϕ_i rejects H_0 if $S_i^* \leq c_i^*$, where the critical value c_i^* satisfies $F_{i,0}^*(c_i^*) = \alpha$.

Equivalently, if we work with p-values the test ϕ_i rejects H_0 if $F_{i,0}^*(S_i^*) \leq \alpha$.

Now it easily follows from

$$F_{1,v}^*(s) = F_{1,v}\left(\frac{n-\ell}{n-k}\,s\right)$$

$$(10.5)\qquad F_{2,v}^*(s) = F_{2,v}\left(\frac{k-p}{n-k}\,s\right)$$

$$F_{3,v}^*(s) = F_{3,v}\left(\frac{n-\ell}{\ell-p}\,s\right),$$

for all $v \in \omega_0 \cup \omega_1$, that

$$c_1^* = \frac{n-k}{n-\ell}\,c_1$$

$$(10.6)\qquad c_2^* = \frac{n-k}{k-p}\,c_2$$

$$c_3^* = \frac{\ell-p}{n-\ell}\,c_3,$$

and also

(10.7) $\quad F^*_{i,0}(S^*_i) = F_{i,0}(S_i)$, $i = 1, 2, 3$.

That is, the critical value c^*_i can easily be found from the original critical value c_i, whereas the p-value $F^*_{i,0}(S^*_i)$ is equal to the original p-value $F_{i,0}(S_i)$.
In fact, the reason for introducing the test statistic S^*_i is that the distribution functions at $v = 0$ of these statistics can be approximated in a simple way when n is large. This enables us to approximate the critical values c^*_i and the p-values $F^*_{i,0}(S^*_i)$ for large n.
As is shown in Appendix D the distribution functions $F^*_{i,0}(s)$ of the test statistics S^*_i, $i = 1, 2, 3$, can be approximated with the aid of the standard-normal distribution and the χ^2 distribution, respectively, when n is large.
We start with the approximation of $F^*_{1,0}(s)$.

(10.8) $$F^*_{1,0}(s) \approx \Phi\left[\frac{\sqrt{n-\ell}\,(s-1)}{\sqrt{2\beta_n(s)}}\right],\ s > 0 \text{ and } s \neq 1,$$

where

(10.9) $$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}}\,dt$$

and

(10.10) $$\beta_n(s) = \left(\frac{n-\ell}{n-k}\right)s^2 - 2\left(\frac{n-k-\ell+tr}{n-k}\right)s + 1.$$

The quantity tr in (10.10) is defined by

(10.11) $$tr = tr[(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X] = tr[(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z] = p + \sum_{j=1}^{M} \rho_j m_j.$$

It is also shown in Appendix D that from (10.8) we can derive the following approximation to the point s_α which satisfies $F^*_{1,0}(s_\alpha) = \alpha$.

$$s_\alpha \approx \frac{b_n - \sqrt{b_n^2 - a_n c_n}}{a_n} \text{ if } \Phi[-\sqrt{\frac{n - \max(k, \ell)}{2}}] < \alpha \le \tfrac{1}{2}$$

(10.12)

$$s_\alpha \approx \frac{b_n + \sqrt{b_n^2 - a_n c_n}}{a_n} \text{ if } \tfrac{1}{2} \le \alpha < \Phi[\sqrt{\frac{n - \max(k, \ell)}{2}}],$$

where the coefficients a_n, b_n and c_n are given by

$$a_n = 1 - \frac{2t_\alpha^2}{n-k}$$

(10.13) $$b_n = 1 - \frac{2t_\alpha^2(n-k-\ell+tr)}{(n-k)(n-\ell)}$$

$$c_n = 1 - \frac{2t_\alpha^2}{n-\ell}\ .$$

Here t_α is the point which satisfies $\Phi(t_\alpha) = \alpha$, i.e.,

(10.14) $$t_\alpha = \Phi^{-1}(\alpha).$$

The restriction

$$\Phi[-\sqrt{\frac{n - \max(k, \ell)}{2}}] < \alpha < \Phi[\sqrt{\frac{n - \max(k, \ell)}{2}}]$$

in (10.12) is not serious in practice, since

$$\Phi[-\sqrt{\frac{n - \max(k, \ell)}{2}}] \to \Phi(-\infty) = 0$$

$$\Phi[\sqrt{\frac{n - \max(k, \ell)}{2}}] \to \Phi(\infty) = 1$$

if $n \to \infty$ which shows that for large n we have:

$$\Phi[-\sqrt{\frac{n - \max(k, \ell)}{2}}] \approx 0, \quad \Phi[\sqrt{\frac{n - \max(k, \ell)}{2}}] \approx 1.$$

With the aid of (10.8) and (10.12) we obtain large sample approximations to the p-value $F^*_{1,0}(S^*_1)$ and the critical value c^*_1 of the test ϕ_1.

$$(10.15) \quad F^*_{1,0}(S^*_1) \approx \Phi\left[\frac{\sqrt{n-\ell}\,(S^*_1 - 1)}{\sqrt{2\beta_n(S^*_1)}}\right]$$

Since in the applications we only consider significance levels α smaller than $\frac{1}{2}$, the approximation of c^*_1 becomes:

$$(10.16) \quad c^*_1 \approx \frac{b_n - \sqrt{b_n^2 - a_n c_n}}{a_n}$$

These results show that the critical value and the p-value of the test ϕ_1 can easily be approximated from the standard-normal distribution. Next we consider the large sample approximation to $F^*_{2,0}(s)$. From Appendix D we have:

$$(10.17) \quad F^*_{2,0}(s) \approx G_{k-p}[(k-p)(1 + \frac{s-1}{\sqrt{\beta_n(s)}})], \; s > 0,$$

where

$$(10.18) \quad G_i(x) = \int_0^x \frac{t^{\frac{i}{2} - 1} e^{-\frac{1}{2}t}}{\Gamma(\frac{i}{2})\, 2^{\frac{i}{2}}}\, dt$$

and

$$(10.19) \quad \beta_n(s) = (\frac{k-p}{n-k})s^2 - 2(\frac{tr-p}{n-k})s + 1.$$

With the aid of (10.17) we obtain the following approximations to the point s_α which satisfies $F^*_{2,0}(s_\alpha) = \alpha$.

$$(10.20) \quad \begin{aligned} s_\alpha &\approx \frac{b_n - \sqrt{b_n^2 - a_n c}}{a_n} \text{ if } 0 \leq \alpha \leq G_{k-p}(k-p) \\ s_\alpha &\approx \frac{b_n + \sqrt{b_n^2 - a_n c}}{a_n} \text{ if } G_{k-p}(k-p) \leq a < G_{k-p}[k-p + \sqrt{(k-p)(n-k)}], \end{aligned}$$

where the coefficients a_n, b_n and c are given by

$$a_n = 1 - \frac{q_\alpha^2(k-p)}{n-k}$$

(10.21) $$b_n = 1 - \frac{q_\alpha^2(tr-p)}{n-k}$$

$$c = 1 - q_\alpha^2$$

The point q_α in (10.21) is defined as follows

(10.22) $$q_\alpha = \frac{f\alpha}{k-p} - 1,$$

where f_α satisfies $G_{k-p}(f_\alpha) = \alpha$, i.e.,

(10.23) $$f_\alpha = G_{k-p}^{-1}(\alpha).$$

The restriction

$$\alpha < G_{k-p}[k-p + \sqrt{(k-p)(n-k)}]$$

in (10.20) is not serious in practice, since

$$G_{k-p}[k-p + \sqrt{(k-p)(n-k)}] \to G_{k-p}(\infty) = 1$$

if $n\to\infty$, which shows that for large n we have

$$G_{k-p}[k-p + \sqrt{(k-p)(n-k)}] \approx 1.$$

With the aid of (10.17) we get the following large sample approximation to the p-value $F_{2,0}^*(S_2^*)$:

(10.24) $$F_{2,0}^*(S_2^*) = G_{k-p}[(k-p)(1 + \frac{S_2^* - 1}{\sqrt{\beta_n(S_2^*)}})].$$

Since $G_i(i) > \frac{1}{2}$ for all i and since in practice we only consider significance levels α smaller than $\frac{1}{2}$, it follows from (10.20) that the approximation of c_2^* becomes equal to:

$$(10.25) \qquad c_2^* \approx \frac{b_n - \sqrt{b_n^2 - a_n c}}{a_n} .$$

The approximations to the critical value and p-value of the test ϕ_2 can easily be computed with the aid of the $\chi^2(k-p)$ distribution.
Finally we consider the approximation to $F_{3,0}^*(s)$.

From Appendix D we have:

$$(10.26) \qquad F_{3,0}^*(s) \approx 1 - G_{\ell-p}[(\ell-p)(1 + \frac{1-s}{\sqrt{\beta_n(s)}})], \; s > 0$$

where now $\beta_n(s)$ is defined by

$$(10.27) \qquad \beta_n(s) = s^2 - 2(\frac{tr-p}{n-\ell})s + (\frac{\ell-p}{n-\ell}).$$

The result (10.26) enables us to find an approximation to the points s_α which satisfies $F_{3,0}^*(s_\alpha) = \alpha$. We have:

$$s_\alpha \approx \frac{b_n - \sqrt{b_n^2 - ac_n}}{a} \text{ if } 1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}] < \alpha \leq$$

$$1 - G_{\ell-p}(\ell-p), \text{ provided that } \alpha \neq 1 - G_{\ell-p}[2(\ell-p)]$$

$$(10.28) \qquad s_\alpha \approx \frac{c_n}{2b_n} \text{ if } \alpha = 1 - G_{\ell-p}[2(\ell-p)]$$

$$s_\alpha \approx \frac{b_n + \sqrt{b_n^2 - ac_n}}{a} \text{ if } 1 - G_{\ell-p}(\ell-p) \leq \alpha \leq 1,$$

where the coefficients a, b_n and c_n are defined as follows

$$a = 1 - q_\alpha^2$$

$$(10.29) \qquad b_n = 1 - \frac{q_\alpha^2(tr-p)}{n-\ell}$$

$$c_n = 1 - \frac{q_\alpha^2(\ell-p)}{n-\ell} .$$

The point q_α in (10.29) is given by

$$(10.30) \quad q_\alpha = \frac{f_{1-\alpha}}{\ell-p} - 1,$$

where $f_{1-\alpha}$ satisfies $G_{\ell-p}(f_{1-\alpha}) = 1-\alpha$, i.e.,

$$(10.31) \quad f_{1-\alpha} = G^{-1}_{\ell-p}(1-\alpha).$$

Note that

$$1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}] \to 1 - G_{\ell-p}(\infty) = 0,$$

if $n\to\infty$, which shows that for large n

$$1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}] \approx 0.$$

Therefore, the restriction

$$1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}] < \alpha$$

in (10.28) is not serious in practice.

From (10.26) we obtain a large sample approximation to the p-value $F^*_{3,0}(S^*_3)$.

$$(10.32) \quad F^*_{3,0}(S^*_3) \approx 1 - G_{\ell-p}[(\ell-p)(1 + \frac{1 - S^*_3}{\sqrt{\beta_n(S^*_3)}})].$$

In practice we mostly consider significance levels α smaller than 0.3 and since $1 - G_i(i) > 0.3$ for all i, it follows from (10.28) that the approximation of c^*_3 becomes

$$(10.33) \quad \begin{aligned} c^*_3 &\approx \frac{b_n - \sqrt{b_n^2 - ac_n}}{a}, \quad \alpha \neq 1 - G_{\ell-p}[2(\ell-p)] \\ c^*_3 &\approx \frac{c_n}{2b_n}, \quad \alpha = 1 - G_{\ell-p}[2(\ell-p)]. \end{aligned}$$

In this case, the approximations to the critical value and the p-value of the test ϕ_3 can easily be computed from the $\chi^2(\ell-p)$ distribution. In order to facilitate the use of the approximations to s_α as given in (10.20) and (10.28), the following table contains the values of $G_i(i)$ and $1 - G_i(i)$ for $i = 1, 2, \ldots, 10$.

Table 6: Values of $G_i(i)$ and $1 - G_i(i)$

i	$G_i(i)$	$1 - G_i(i)$
1	0.683	0.317
2	0.632	0.368
3	0.608	0.392
4	0.594	0.406
5	0.584	0.416
6	0.577	0.423
7	0.571	0.429
8	0.567	0.433
9	0.563	0.437
10	0.560	0.440

11. Required computations for the applications

In this section we shall give a description of the computations required in order to apply the tests in practice.
The data set in a problem of testing linear hypotheses consists of (y, X, Z). On the basis of these data we want to test

$$H_0: y \sim n(X\beta, \sigma^2 I) \quad \text{vs.} \quad H_1: y \sim n(Z\gamma, \sigma^2 I),$$

where β, γ and σ are unknown.
From the data set (y, X, Z), where y is a n-vector, X is a $n \times k$ matrix with rank k and Z is a $n \times \ell$ matrix with rank ℓ, we make the following computations.

1. If $k < \ell$, compute the matrix $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$ and the eigenvalues of this matrix. When $\ell < k$, we compute $(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z$ and the eigenvalues of the latter matrix. In the case $k = \ell$ it does not matter which of the above two matrices is computed.
The above computations yield the multiplicity p of the eigenvalue 1, the number M of different eigenvalues which are strictly between 0 and 1, the values $\rho_1, \rho_2, \ldots, \rho_M$ of these eigenvalues and the corresponding multiplicities $m_1, m_2, \ldots, m_M$. We note that p as well as M can be equal to 0.

2. Compute $r = p + \sum_{j=1}^{M} m_j$, consider the classification as given in Section 6 and according to Table 2 of Section 6 decide which of the tests ϕ_1, ϕ_2 or ϕ_3 shall be used.

3. Compute $\hat{\beta} = (X'X)^{-1}X'y$, $\hat{\gamma} = (Z'Z)^{-1}Z'y$, $\hat{u}_X = y - X\hat{\beta}$, $\hat{u}_Z = y - Z\hat{\gamma}$, $\hat{u}_X'\hat{u}_X$ and $\hat{u}_Z'\hat{u}_Z$.
If the test ϕ_1 is taken we compute the test statistic

$$S_1 = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X}.$$

When ϕ_2 or ϕ_3 are used we first construct the matrix $G = [X_2 \vdots Z]$ or

$G = [X \vdots Z_2]$ as described in Section 8 and then compute
$\hat{\delta} = (G'G)^{-1}G'y$, $\hat{u}_G = y - G\hat{\delta}$ and $\hat{u}_G'\hat{u}_G$.
If ϕ_2 is used we compute

$$S_2 = \frac{\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G}{\hat{u}_X'\hat{u}_X}.$$

When the test ϕ_3 is taken we compute

$$S_3 = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X - \hat{u}_G'\hat{u}_G}.$$

4. When the test ϕ_i is used we compute the p-value

$$F_{i,0}(S_i) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin \varepsilon_{i,0}(u, S_i)}{u\, \gamma_{i,0}(u, S_i)}du$$

as described in Section 7 and reject H_0 if

$$F_{i,0}(S_i) \leq \alpha,$$

where α is a preassigned significance level.
The above integration requires the computation of coefficients $\lambda_{1j}(S_i)$, $\lambda_{2j}(S_i)$, $j = 0, 1, \ldots, M$, $\lambda(S_i)$ and the degrees of freedom m_{1j}, m_{2j}, $j = 0, 1, \ldots, M$ and m. For all three tests we have

$$\lambda_{1j}(S_i) = \frac{1}{2}(1 - S_i) - \frac{1}{2}\sqrt{(1 - S_i)^2 + 4S_i(1 - \rho_j)}$$

$$\lambda_{2j}(S_i) = \frac{1}{2}(1 - S_i) + \frac{1}{2}\sqrt{(1 - S_i)^2 + 4S_i(1 - \rho_j)},$$

$j = 0, 1, \ldots, M$, where $\rho_0 = 0$, and also

$$m_{1j} = m_{2j} = m_j \text{ for } j = 1, 2, \ldots, M.$$

When the test ϕ_1 is used we have:

$$\lambda(S_1) = 1 - S_1$$

$$m_{10} = \ell - r$$

$$m_{20} = k - r$$

$$m = n+p-k-\ell.$$

If the test ϕ_2 is taken this becomes:

$$\lambda(S_2) = 0$$

$$m_{10} = n+p-k-r$$

$$m_{20} = k-r$$

$$m = 0.$$

For the test ϕ_3 we take:

$$\lambda(S_3) = 0$$

$$m_{10} = \ell - r$$

$$m_{20} = n+p-\ell-r$$

$$m = 0.$$

5. When the sample size n is large we can use the approximations as describes in Section 10.
If the test ϕ_1 is used we compute

$$S_1^* = S_1 \frac{n-k}{n-\ell}$$

and

$$F_{1,0}^*(S_1^*) \approx \Phi\left[\frac{\sqrt{n-\ell}\,(S_1^* - 1)}{\sqrt{2}\ \beta_n(S_1^*)}\right].$$

We reject H_0 if this approximated p-value is smaller than or equal to α.

Equivalently, we can approximate the critical value c_1^* by

$$c_1^* \approx \frac{b_n - \sqrt{b_n^2 - a_n c_n}}{a_n}$$

and reject H_0 if S_1^* is smaller than or equal to this approximated critical value.

For the test ϕ_2 we compute

$$S_2^* = S_2 \frac{n-k}{k-p}$$

and reject H_0 if

$$F_{2,0}^*(S_2^*) \approx G_{k-p}[(k-p)(1 + \frac{S_2^* - 1}{\sqrt{\beta_n(S_2^*)}})] \leq \alpha,$$

or equivalently, if

$$S_2^* \leq c_2^* \approx \frac{b_n - \sqrt{b_n^2 - a_n c}}{a_n}.$$

Similarly, if the test ϕ_3 is taken, we compute

$$S_3^* = S_3 \frac{\ell-p}{n-\ell}$$

and reject H_0 if

$$F_{3,0}^*(S_3^*) \approx 1 - G_{\ell-p}[(\ell-p)(1 + \frac{1 - S_3^*}{\sqrt{\beta_n(S_3^*)}})] \leq \alpha,$$

or equivalently, if

$$S_3^* \leq c_3^* \approx \frac{b_n - \sqrt{b_n^2 - ac_n}}{a}.$$

6. Sometimes one is interested in the value of the power function of the test ϕ_i at the parameter point (β, σ) under H_0 or (γ, σ) under H_1.

Now we know from Section 7 that the power function $\pi(\phi_i, v)$ of the test ϕ_i is given by

$$\pi(\phi_i, v) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin \varepsilon_{i,v}(u, c_i)}{u\, \gamma_{i,v}(u, c_i)} du.$$

Thus, once the critical value c_i and the parameter vector v are given, we can compute the power through numerical integration.
The critical value c_i can be found by solving the equation

$$\alpha = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin \varepsilon_{i,0}(u, c_i)}{u\, \gamma_{i,0}(u, c_i)} du.$$

A large sample approximation of c_i can be derived from the large sample approximation of c_i^* as given under 5. and the relations (see (10.6) of Section 10)

$$c_1 = \frac{n-\ell}{n-k} c_1^*$$

$$c_2 = \frac{k-p}{n-k} c_2^*$$

$$c_3 = \frac{n-\ell}{\ell-p} c_3^*.$$

For the computation of the vector $v = (v_{10}, v_{11}, \ldots v_{1M}, v_{20}, v_{21}, \ldots, v_{2M})$ in terms of (β, σ) under H_0 or (γ, σ) under H_1, we refer to Section 8, see formulae (8.31) - (8.34).

12. Summary and conclusion

In this study the principle of invariance is applied to the problem of testing linear hypotheses.
Before we apply invariance considerations, the original problem is transformed into an equivalent problem with a more simple structure, by means of a suitable linear transformation.
The transformed problem is invariant under a group of transformations which map the sample space onto itself. These transformatins consist of changes of scale, certain translations and certain rotations.
They can be interpreted as changes of the coordinate system in which the observations are expressed.
Since the problem remains invariant under certain changes of the coordinate system, a natural procedure is to require that the tests exhibit the same property. That is, we only consider tests which are invariant with respect to the above group of transformations.
Now the class of invariant tests is the totality of tests depending only on the maximal invariant statistic. Within this restricted class of tests we try to find the UMP level α test.
In general, for the problem of testing linear hypotheses we obtain a two-dimensional maximal invariant statistic, which can easily be computed from the original observations. In the special case of (nontrivial) nested linear hypotheses the maximal invariant statistic turns out to be one-dimensional and there exists a UMP invariant level α test which coincides with the classical F test.
On the other hand, however, for the problem of testing nonnested linear hypotheses te existence of a UMP invariant level α test is an unsolved problem.
In order to derive reasonable tests for the nonnested case we do not only require a test to be invariant and to have level α, but also to be unbiased, computable, usable and strictly discriminating.
Since we are looking for unbiased level α tests we can restrict attention to the class of α-similar tests, i.e., the tests which have rejection probability α at parameter points on the boundary between H_0 and H_1.
Now a class of invariant tests is constructed which are α-similar, computable and usable.

The power function of these tests is derived and it turns out to depend on the value of $r = \text{rank}(X'Z)$ whether these tests have level α, are unbiased and strictly discriminating.
In particular we consider the tests ϕ_1, ϕ_2 and ϕ_3, which are based on the test statistics S_1, S_2 and S_3, respectively.
The test ϕ_1 is the α-similar generalized likelihood-ratio test, ϕ_2 is a test which under all circumstances is exact, i.e., ϕ_2 always has level α. The test ϕ_3 has guaranteed power, that is, the power function of ϕ_3 always exceeds the level α under H_1.
As was said before, whether the tests ϕ_1, ϕ_2 and ϕ_3 possess the above mentioned desirable properties depends on the value of $r = \text{rank}(X'Z)$.
We investigate every possible situation and select the appropriate test.
Further we derive the distribution functions of the test statistics S_1, S_2 and S_3 and show that the critical values, the p-values and the power functions of the tests can be computed through numerical integration.
The values of the test statistics S_1, S_2 and S_3 can easily be computed from the original data (y, X, Z). The test statistics turn out to be ratios of sums of squares which have a natural interpretation in terms of the original linear hypotheses (linear models).
When the sample size n is large the critical values and p-values of the tests can very easily be approximated from the standard-normal and χ^2 distribution.
In the special case of nested linear hypotheses our general tests turn out to be equivalent to the well-known F test.
Finally we give a review of the computations which are required in order to apply the tests ϕ_1, ϕ_2 and ϕ_3 in practice.

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Appendix A

The characteristic function $\psi(t_1, t_2)$

In this appendix we shall derive the joint characteristic function $\psi(t_1, t_2)$ of the random variables (U_1, U_2), which are defined by

$$\text{(A.1)} \qquad U_1 = u_1'u_1 \qquad U_2 = u_2'u_2,$$

where the m-dimensional random vectors u_1 and u_2 have the following probability distribution

$$\text{(A.2)} \qquad \binom{u_1}{u_2} \sim n\left(\binom{\delta_1}{\delta_2}, \begin{bmatrix} I & \sqrt{\rho}I \\ \sqrt{\rho}I & I \end{bmatrix}\right).$$

The coefficient ρ in (A.2) satisfies $0 < \rho < 1$ and the m-dimensional vectors δ_1 and δ_2 have the property that $\delta_1 = 0$ or $\delta_2 = 0$.
Moreover, we shall compute the covariance of U_1 and U_2.
In order to find $\psi(t_1, t_2)$ we need the following general result:
Let the n-dimensional random vector x have a $n(\mu, V)$ distribution, where V is nonsingular, and let T be a symmetric n×n matrix, then

$$\text{(A.3)} \qquad \phi(T) = E(e^{ix'Tx}) = |I - 2iTV|^{-\frac{1}{2}} \exp\{i\mu'(I - 2iTV)^{-1}T\mu\},$$

where i denotes the imaginary unit.

Proof:
Since V is positive definite we have $V = SS'$, where S is a nonsingular n×n matrix. If we take $y = S^{-1}x$, it follows that $x = Sy$ and we get

$$\text{(A.4)} \qquad \phi(T) = E(e^{iy'S'TSy}) = E(e^{iy'By}),$$

where $B = S'TS$ in a symmetric n×n matrix and $y \sim n(S^{-1}\mu, I)$.
Let Λ be the diagonal matrix of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix B and H the corresponding orthogonal matrix of eigenvectors, then B can be written as $B = H\Lambda H'$ and it is seen from (A.4) that

(A.5) $\phi(T) = E(e^{iy'H\Lambda H'y}) = E(e^{iz'\Lambda z})$,

where $z = H'y \sim n(\theta, I)$ with $\theta = H'S^{-1}\mu$.
If Z_j denotes the jth element of the vector z and if $W_j = Z_j^2$, j = 1, 2, ..., n, it follows from (A.5) that

(A.6) $$\phi(T) = E(\exp\{i \sum_{j=1}^{n} \lambda_j Z_j^2\}) = E(\exp\{i \sum_{j=1}^{n} \lambda_j W_j\}).$$

Now it is easily verified that the random variables $W_1, W_2, \ldots, W_n$ are mutually stochastically independent and that $W_j \sim \chi^2(1, \theta_j^2)$, j = 1, 2, ..., n, where θ_j denotes the jth element of the vector θ.
This gives

(A.7) $$\phi(T) = E(\prod_{j=1}^{n} \exp\{i\lambda_j W_j\}) = \prod_{j=1}^{n} E(\exp\{i\lambda_j W_j\}) = \prod_{j=1}^{n} \phi_j(\lambda_j),$$

where $\phi_j(t)$ is the characteristic function of a random variable possessing a $\chi^2(1, \theta_j^2)$ distribution, that is,

$$\phi_j(t) = (1 - 2it)^{-\frac{1}{2}}\exp\{\frac{it\ \theta_j^2}{1-2it}\}.$$

Substitution of the latter result into (A.7) yields:

(A.8) $$\phi(T) = [\prod_{j=1}^{n} (1-2i\lambda_j)^{-\frac{1}{2}}]\exp\{\sum_{j=1}^{n} \frac{i\lambda_j\theta_j^2}{1-2i\lambda_j}\}$$

$$= |I-2i\Lambda|^{-\frac{1}{2}}\exp\{i\theta'(I-2i\Lambda)^{-1}\Lambda\theta\}.$$

With the aid of the relations $\Lambda = H'BH$, $\theta = H'S^{-1}\mu$, $B = S'TS$ and $SS' = V$ we can express the determinant $|I-2i\Lambda|$ and the quadratic form $\theta'(I-2i\Lambda)^{-1}\Lambda\theta$ in terms of the original parameters μ, V and T.
We have

(A.9) $$|I-2i\Lambda| = |I-2iH'BH| = |H'(I-2iB)H|$$

$$= |H'|\ |I-2iB|\ |H| = |I-2iB| = |I-2iS'TS|$$

$$= |S'(I-2iTSS')(S')^{-1}| = |S'|\ |I-2iTSS'|\ |(S')^{-1}|$$

$$= |I-2iTSS'| = |I-2iTV|,$$

and

$$\text{(A.10)}\qquad \theta'(I-2i\Lambda)^{-1}\Lambda\theta = \mu'(S')^{-1}H(I-2iH'BH)^{-1}H'BS^{-1}\mu$$

$$= \mu'(S')^{-1}H[H'(I-2iB)H]^{-1}H'BS^{-1}\mu$$

$$= \mu'(S')^{-1}(I-2iB)^{-1}BS^{-1}\mu = \mu'(S')^{-1}(I-2iS'TS)^{-1}S'T\mu$$

$$= \mu'(S')^{-1}[S'(I-2iTSS')(S')^{-1}]^{-1}S'T\mu$$

$$= \mu'(I-2iTSS')^{-1}T\mu = \mu'(I-2iTV)^{-1}T\mu.$$

Finally, substitution of (A.9) and (A.10) into (A.8) gives

$$\phi(T) = |I-2iTV|^{-\frac{1}{2}}\exp\{i\mu'(I-2iTV)^{-1}T\mu\},$$

which completes the proof of (A.3).
Now let the random variables (U_1, U_2) and the random vectors u_1 and u_2 be as defined in (A.1) and (A.2), then it is easily seen that the joint characteristic function $\psi(t_1, t_2)$ of (U_1, U_2) can be written as:

$$\text{(A.11)}\qquad \psi(t_1, t_2) = E(e^{it_1U_1+it_2U_2}) =$$

$$E(\exp\{it_1u_1'u_1 + it_2u_2'u_2\}) = E(e^{ix'Tx}),$$

where

$$x = \binom{u_1}{u_2} \text{ and } T = \begin{bmatrix} t_1I & 0 \\ 0 & t_2I \end{bmatrix}.$$

It follows from (A.2) that $x \sim n(\mu, V)$ with

$$\mu = \binom{\delta_1}{\delta_2} \text{ and } V = \begin{bmatrix} I & \sqrt{\rho}I \\ \sqrt{\rho}I & I \end{bmatrix}.$$

Hence, (A.11) together with (A.3) imply that

$$\text{(A.12)}\qquad \psi(t_1, t_2) = \phi(T) = |I-2iTV|^{-\frac{1}{2}}\exp\{i\mu'(I-2iTV)^{-1}T\mu\},$$

where T, μ and V are as defined above.

Substitution of the expressions for T, μ and V into I-2iTV and Tμ yields:

$$I-2iTV = \begin{bmatrix} (1-2it_1)I & -2it_1\sqrt{\rho}I \\ -2it_2\sqrt{\rho}I & (1-2it_2)I \end{bmatrix},$$

and

$$T\mu = \binom{t_1\delta_1}{t_2\delta_2}.$$

Next consider a matrix of the form

$$\begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix},$$

where $a \neq 0$ and $d \neq 0$, then it follows from the well-known result

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{22}|\ |A_{11} - A_{12}A_{22}^{-1}A_{21}|$$

that

$$\text{(A.13)} \qquad \begin{vmatrix} aI & bI \\ cI & cI \end{vmatrix} = |dI|\ |aI - (bI)(dI)^{-1}(cI)|$$

$$= d^m|(a - \frac{bc}{d})I| = d^m(a - \frac{bc}{d})^m = (ad - bc)^m.$$

Moreover, it is easily seen that:

$$\text{(A.14)} \qquad \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} dI & -bI \\ -cI & aI \end{bmatrix}.$$

Therefore, if we take $a = 1-2it_1$, $b = -2it_1\sqrt{\rho}$, $c = -2it_2\sqrt{\rho}$ and $d = 1-2it_2$, it follows from the above expression for the matrix I-2iTV that

$$\text{(A.15)} \qquad |I-2iTV| = [1-2it_1-2it_2-4(1-\rho)t_1t_2]^m$$

and

$$\text{(A.16)} \qquad (I-2iTV)^{-1} = [1-2it_1-2it_2-4(1-\rho)t_1t_2]^{-1}\begin{bmatrix} (1-2it_2)I & 2it_1\sqrt{\rho}I \\ 2it_2\sqrt{\rho}I & (1-2it_1)I \end{bmatrix}.$$

Upon substituting (A.15), (A.16) and the expression for μ and $T\mu$ into (A.12), we obtain

$$(A.17) \qquad \psi(t_1,t_2) = [1-2it_1-2it_2-4(1-\rho)t_1t_2]^{-\frac{m}{2}}$$

$$\exp\{\frac{(it_1+2t_1t_2)\delta_1'\delta_1+(it_2+2t_1t_2)\delta_2'\delta_2-4\sqrt{\rho}t_1t_2\delta_1'\delta_2}{1-2it_1-2it_2-4(1-\rho)t_1t_2}\}.$$

Finally, since $\delta_1 = 0$ or $\delta_2 = 0$ we always have $\delta_1'\delta_2 = 0$ and it is seen that

$$(A.18) \qquad \psi(t_1, t_2) = [1-2it_1-2it_2-4(1-\rho)t_1t_2]^{-\frac{m}{2}}$$

$$\exp\{\frac{(it_1+2t_1t_2)v_1 + (it_2+2t_1t_2)v_2}{1-2it_1-2it_2-4(1-\rho)t_1t_2}\},$$

where $v_1 = \delta_1'\delta_1$ and $v_2 = \delta_2'\delta_2$.
Note that

$$\psi(t_1, 0) = (1-2it_1)^{-\frac{m}{2}} \exp\{\frac{it_1v_1}{1-2it_1}\}$$

and

$$\psi(0, t_2) = (1-2it_2)^{-\frac{m}{2}}\exp\{\frac{it_2v_2}{1-2it_2}\},$$

which shows that $U_1 = u_1'u_1 \sim \chi^2(m, v_1)$ and $U_2 = u_2'u_2 \sim \chi^2(m, v_2)$, a fact which, of course, can be directly concluded from (A.2).
Next we shall compute $Cov(U_1, U_2)$.
To this extend we use the general result

$$(A.19) \qquad Cov(U_1, U_2) = -\frac{\partial^2 \ln \psi(t_1, t_2)}{\partial t_1 \partial t_2}\Bigg|_{t_1=t_2=0}.$$

From (A.18) we obtain

$$(A.20) \qquad \ln \psi(t_1, t_2) = -\frac{m}{2} \ln g(t_1, t_2) + \frac{h(t_1, t_2)}{g(t_1, t_2)},$$

with

$$g(t_1, t_2) = 1-2it_1-2it_2-4(1-\rho)t_1t_2$$

(A.21)

$$h(t_1, t_2) = (it_1+2t_1t_2)v_1 + (it_2+2t_1t_2)v_2.$$

Differentiation of (A.20) to t_1 and t_2 yields:

(A.22) $$\frac{\partial^2 \ln\psi}{\partial t_1 \partial t_2} = -\frac{m}{2}\left[\frac{\partial^2 g}{\partial t_1 \partial t_2} g - \frac{\partial g}{\partial t_1}\frac{\partial g}{\partial t_2}\right]/g^2$$

$$+ \left[\frac{\partial^2 h}{\partial t_1 \partial t_2} g - \frac{\partial h}{\partial t_1}\frac{\partial g}{\partial t_2}\right]/g^2$$

$$- \left[\frac{\partial g}{\partial t_1}\frac{\partial h}{\partial t_2} + h\frac{\partial^2 g}{\partial t_1 \partial t_2}\right]/g^2 + 2h\frac{\partial g}{\partial t_1}\frac{\partial g}{\partial t_2}/g^3.$$

With the aid of the definitions of $g(t_1, t_2)$ and $h(t_1, t_2)$ in (A.21) we get

$$\frac{\partial g}{\partial t_1} = -2i-4(1-\rho)t_2$$

$$\frac{\partial g}{\partial t_2} = -2i-4(1-\rho)t_1$$

$$\frac{\partial h}{\partial t_1} = (i+2t_2)v_1+2t_2v_2$$

(A.23)

$$\frac{\partial h}{\partial t_2} = 2t_1v_1+(i+2t_1)v_2$$

$$\frac{\partial^2 g}{\partial t_1 \partial t_2} = -4(1-\rho)$$

$$\frac{\partial^2 h}{\partial t_1 \partial t_2} = 2(v_1+v_2).$$

Upon substituting $t_1 = t_2 = 0$ into (A.23) we have

$$\left.\frac{\partial g}{\partial t_1}\right|_{t_1=t_2=0} = \left.\frac{\partial g}{\partial t_2}\right|_{t_1=t_2=0} = -2i$$

$$\left.\frac{\partial h}{\partial t_1}\right|_{t_1=t_2=0} = iv_1$$

(A.24) $$\left.\frac{\partial h}{\partial t_2}\right|_{t_1=t_2=0} = iv_2$$

$$\left.\frac{\partial^2 g}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0} = -4(1-\rho)$$

$$\left.\frac{\partial^2 h}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0} = 2(v_1+v_2)$$

By making use of $g(0,0) = 1$, $h(0,0) = 0$ (see (A.21)) and (A.24) we obtain from (A.22):

$$\left.\frac{\partial^2 \ln\psi}{\partial t_1 \partial t_2}\right|_{t_1=t_2=0} = -\frac{m}{2}[-4(1-\rho) - (-2i)^2]$$

$$+ 2(v_1+v_2) - (iv_1)(-2i) - (-2i)(iv_2)$$

$$= -\frac{m}{2}(-4+4\rho+4) + 2v_1+2v_2-2v_1-2v_2$$

$$= -2m\rho.$$

Hence, it follows from (A.19) that

(A.25) $$\mathrm{Cov}(U_1, U_2) = 2m\rho.$$

Note that the covariance of U_1 and U_2 does not depend on the parameters v_1 and v_2.

Appendix B

The characteristic function and distribution function of a linear combination of mutually independent χ^2 random variables

The purpose of this appendix is to derive some properties of the characteristic function and the distribution function of a linear combination of mutually stochastically independent chi-square random variables.
Before doing this, we need some general results. Most of these results will be stated without proof. For the proofs we refer to Chung [2] and Kawata [5].
Let X be a random variable with distribution function F(x). Then the characteristic function $\phi(t)$ of X is defined by

$$\text{(B.1)} \qquad \phi(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} dF(x),$$

$-\infty < t < \infty$, where i denotes the imaginary unit.
Some well-known properties of $\phi(t)$ are:

(i) $\phi(t)$ always exists and $|\phi(t)| \leq 1$, where $|\phi(t)|$ denotes the modulus of the complex-valued function $\phi(t)$.
(ii) $\phi(0) = 1$.
(iii) $\overline{\phi(t)} = \phi(-t)$, where $\overline{\phi(t)}$ denotes the complex conjugate of $\phi(t)$.
(iv) There exists a 1:1 correspondence between F(x) and $\phi(t)$.

When the distribution function F(x) of the random variable X is known, we can find the corresponding characteristic function $\phi(t)$ from (B.1). Conversely, if we know the characteristic function $\phi(t)$ of X, the corresponding distribution function F(x) is given by the so called inversion formula of Lévy:

$$\text{(B.2)} \qquad \frac{F(x_2) + F(x_2^-)}{2} - \frac{F(x_1) + F(x_1^-)}{2} = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} \phi(t)dt,$$

for $x_1 < x_2$, where F(x-) denotes the left-hand limit of F(x) at the point x and where

$$\frac{e^{-itx_1} - e^{-itx_2}}{it}\,\phi(t)\Big|_{t=0} \overset{\text{def.}}{=} \lim_{t\to 0} \frac{e^{-itx_1} - e^{-itx_2}}{it}\,\phi(t) = x_2 - x_1.$$

If F(x) is continuous at x_1 and x_2 the inversion formula becomes:

$$\text{(B.3)} \qquad F(x_2) - F(x_1) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it}\,\phi(t)dt,\ x_1 < x_2.$$

Under the condition $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$ the inversion formula can be simplified. In the first place, if $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$, it can be shown that F(x) is everywhere continuous and that

$$\text{(B.4)} \qquad F(x_2) - F(x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itx_1} - e^{-itx_2}}{it}\,\phi(t)dt,\ x_1 < x_2.$$

In the second place, when $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$, we can prove the stronger result that F(x) is absolutely continuous with density function

$$\text{(B.5)} \qquad f(x) = F'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\phi(t)dt,\ -\infty < x < \infty.$$

Moreover, under the stronger condition $\int_{-\infty}^{\infty} |t\phi(t)|dt < \infty$ we have

$$\text{(B.6)} \qquad f'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -ite^{-itx}\phi(t)dt,\ -\infty < x < \infty.$$

The inversion formula (B.2) does not give the distribution function itself, but the difference $F(x_2) - F(x_1)$ at the continuity points x_1 and x_2.

The following inversion formula due to Gil-Palaez gives F(x) directly at the continuity point x:

$$\text{(B.7)} \qquad \frac{F(x) + F(x-)}{2} = \frac{1}{2} - \lim_{\substack{\delta\downarrow 0 \\ T\uparrow\infty}} \frac{1}{\pi} \int_{\delta}^{T} \{Im(\phi(t)e^{-itx})/t\}dt,$$

for $-\infty < x < \infty$, where $Im(\phi(t)e^{-itx})$ denotes the imaginary part of the complex-valued function $\phi(t)e^{-itx}$.

For a proof of (B.7) we refer to Kawata [5].

When F(x) is continuous at x we have

$$\text{(B.8)} \qquad F(x) = \frac{1}{2} - \lim_{\substack{\delta \downarrow 0 \\ T \uparrow \infty}} \frac{1}{\pi} \int_{\delta}^{T} \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt, \quad -\infty < x < \infty.$$

Next we shall prove that under a number of special conditions the right-hand side of formula (B.7) can be simplified. We first prove the following result.

If $E(|X|) = \int_{-\infty}^{\infty} |x| dF(x) < \infty$, then

$$\text{(B.9)} \qquad \lim_{t \downarrow 0} \{\mathrm{Im}(\phi(t)e^{-itx})/t\} = E(X) - x,$$

for $-\infty < x < \infty$, where $E(X) = \int_{-\infty}^{\infty} x dF(x)$.

Proof:

With the aid of (B.1) we have

$$\mathrm{Im}(\phi(t)e^{-itx}) = \frac{\phi(t)e^{-itx} - \phi(-t)e^{itx}}{2i} =$$

$$\left[\int_{-\infty}^{\infty} e^{ity}dF(y)e^{-itx} - \int_{-\infty}^{\infty} e^{-ity}dF(y)e^{itx}\right]/(2i) =$$

$$\int_{-\infty}^{\infty} \frac{e^{it(y-x)} - e^{-it(y-x)}}{2i} dF(y) = \int_{-\infty}^{\infty} \sin t(y-x)dF(y).$$

This implies that

$$\text{(B.10)} \qquad \{\mathrm{Im}(\phi(t)e^{-itx})/t\} = \int_{-\infty}^{\infty} \frac{\sin t(y-x)}{t} dF(y).$$

If for any x, $g(y) \overset{\text{def.}}{=} |y| + |x|$, we get

$$\left|\frac{\sin t(y-x)}{t}\right| \leq |y-x| \leq |y| + |x| = g(y).$$

Since $\int_{-\infty}^{\infty} g(y)dF(y) = \int_{-\infty}^{\infty} (|y| + |x|)dF(y)$

$$= \int_{-\infty}^{\infty} |y|dF(y) + |x| \int_{-\infty}^{\infty} dF(y) = E(|X|) + |x| < \infty,$$

and $\lim_{t\downarrow 0} \frac{\sin t(y-x)}{t} = y-x$,

it follows from the dominated convergence theorem of Lebesgue that

$$\lim_{t\downarrow 0} \int_{-\infty}^{\infty} \frac{\sin t(y-x)}{t} dF(y) = \int_{-\infty}^{\infty} (y-x)dF(y)$$

$$= \int_{-\infty}^{\infty} ydF(y) - x \int_{-\infty}^{\infty} dF(y) = E(X) - x.$$

With the aid of the latter result we obtain from (B.10):

$$\lim_{t\downarrow 0}\{\mathrm{Im}(\phi(t)e^{-itx})/t\} = \lim_{t\downarrow 0} \int_{-\infty}^{\infty} \frac{\sin t(y-x)}{t} dF(y) = E(X) - x,$$

which completes the proof of (B.9).

In the second place, if $E(|X|) < \infty$ and $\int_{1}^{\infty} |\frac{\phi(t)}{t}|dt < \infty$, it can be shown that

$$\text{(B.11)} \qquad \frac{F(x) + F(x-)}{2} = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt,$$

$-\infty < x < \infty$, where

$$\mathrm{Im}(\phi(t)e^{-itx})/t \Big|_{t=0} \overset{\text{def.}}{=} \lim_{t\downarrow 0}\{\mathrm{Im}(\phi(t)e^{-itx})/t\} = E(X) - x.$$

<u>Proof:</u>

Since $E(|X|) < \infty$, it follows from (B.9) that $\lim_{t\downarrow 0}\{\mathrm{Im}(\phi(t)e^{-itx})/t\} = E(X) - x$, which proves the second part of (B.11).

Now consider the right-hand side of (B.7).

We get:

$$\text{(B.12)} \qquad \lim_{\substack{\delta\downarrow 0\\ T\uparrow\infty}} \frac{1}{\pi} \int_{\delta}^{T} \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt = \lim_{\substack{\delta\downarrow 0\\ T\uparrow\infty}} \frac{1}{\pi} \int_{-\infty}^{\infty} I_{(\delta,T)}(t)\{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt,$$

where $I_{(\delta,T)}(t)$ denotes the indicator function of the set (δ, T). From (B.10) it is seen that

$$|\text{Im}(\phi(t)e^{-itx})/t| = |\int_{-\infty}^{\infty} \frac{\sin t(y-x)}{t} dF(y)|$$

$$\leq \int_{-\infty}^{\infty} |\frac{\sin t(y-x)}{t}| dF(y) \leq \int_{-\infty}^{\infty} |y-x| dF(y)$$

$$\leq \int_{-\infty}^{\infty} (|y| + |x|) dF(y) = E(|X|) + |x|.$$

We also have

$$|\text{Im}(\phi(t)e^{-itx}/t| = |\frac{\phi(t)e^{-itx} - \phi(-t)e^{itx}}{2it}|$$

$$\leq |\frac{\phi(t)e^{-itx}}{2it}| + |\frac{\phi(-t)e^{itx}}{2it}| = |\frac{\phi(t)}{t}|.$$

Therefore, if the function $g(t)$ is defined by

$$\begin{aligned} g(t) &= 0, & -\infty < t < 0 \\ &= E(|X|) + |x|, & 0 \leq t < 1 \\ &= |\frac{\phi(t)}{t}|, & 1 \leq t < \infty \end{aligned}$$

it follows that

$$|I_{(\delta,T)}(t) \{\text{Im}(\phi(t)e^{-itx})/t\}| \leq g(t)$$

and

$$\int_{-\infty}^{\infty} g(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{1} \{E(|X|) + |x|\}dt + \int_{1}^{\infty} |\frac{\phi(t)}{t}| dt$$

$$= E(|X|) + |x| + \int_{1}^{\infty} |\frac{\phi(t)}{t}| dt < \infty.$$

Since,

$$\lim_{\substack{\delta\downarrow 0\\ T\uparrow\infty}} I_{(\delta,T)}(t)\ \{\mathrm{Im}(\phi(t)e^{-itx})/t\} =$$

$$I_{(0,\infty)}(t)\ \{\mathrm{Im}(\phi(t)e^{-itx})/t\},$$

the dominated convergence theorem implies that

$$\lim_{\substack{\delta\downarrow 0\\ T\uparrow\infty}} \frac{1}{\pi}\int_{-\infty}^{\infty} I_{(\delta,T)}(t)\ \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt =$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} I_{(0,\infty)}(t)\ \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt,$$

where $I_{(0,\infty)}(t)$ is the indicator function of $(0, \infty)$.

Substitution of the latter results into (B.12) yields:

(B.13) $$\lim_{\substack{\delta\downarrow 0\\ T\uparrow\infty}} \frac{1}{\pi}\int_{\delta}^{T} \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt = \frac{1}{\pi}\int_{0}^{\infty} \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt.$$

Finally, substitution of (B.13) into (B.7) completes the proof of (B.11).

Of course, when $F(x)$ is continuous at x we can write (B.11) in the following way

(B.14) $$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_{0}^{\infty} \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt,$$

$-\infty < x < \infty$.

The integrand $\mathrm{Im}(\phi(t)e^{-itx})/t$ in the above expressions can always be written as

(B.15) $$\mathrm{Im}(\phi(t)e^{-itx})/t = \frac{1}{t}\ |\phi(t)|\ \sin\{\arg(\phi(t)) - tx\},$$

where $\arg(\phi(t))$ denotes the principal argument of the complex-valued function $\phi(t)$.

Proof:
Any complex number z can be written as

$$z = |z|e^{i\arg(z)} = |z| \cos\{\arg(z)\} + i|z| \sin\{\arg(z)\}$$

which implies that $\mathrm{Im}(z) = |z| \sin\{\arg(z)\}$.
Hence,

$$\mathrm{Im}(\phi(t)e^{-itx}) = |\phi(t)e^{-itx}| \sin\{\arg(\phi(t)e^{-itx})\}.$$

Now we have:

$$|\phi(t)e^{-itx}| = |\phi(t)| \quad |e^{-itx}| = |\phi(t)|$$

and

$$\arg(\phi(t)e^{-itx}) = \arg(\phi(t)) + \arg(e^{-itx})$$

$$= \arg(\phi(t)) - tx.$$

From these results it is seen that

$$\mathrm{Im}(\phi(t)e^{-itx})/t = \frac{1}{t}|\phi(t)| \sin\{\arg(\phi(t)) - tx\},$$

which proves the desired result.
So far no special assumptions were made about the random variable X. Now we shall study the special case where X is defined as a linear combination of mutually stochastically independent chi-square random variables, i.e.,

(B.16) $$X = \sum_{j=1}^{M} \alpha_j U_j,$$

where $U_1, U_2, \ldots, U_M$ are mutually stochastically independent random variables with $U_j \sim \chi^2(m_j, \theta_j)$, $j = 1, 2, \ldots, M$.

Further, let $\alpha = \min\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_M|\}$ and $m = \sum_{j=1}^{M} m_j$, then $\alpha > 0$ and $m = 1, 2, 3, \ldots$

Also note that $\theta_j \geq 0$ for $j = 1, 2, \ldots, M$.

When X is defined through (B.16), the distribution function F(x) satisfies:

(B.17) F(x) is absolutely continuous.

Proof:

Since the distribution function of $U_j \sim \chi^2(m_j, \theta_j)$ is absolutely continuous, it easily follows that the distribution function of $V_j = \alpha_j U_j$ is absolutely continuous. This implies that $X = \sum_{j=1}^{M} V_j$ where $V_1, V_2, \ldots, V_M$ are mutually independent random variables with absolutely continuous distribution functions. In other words, F(x) is the M-fold convolution of absolutely continuous distributions. Hence, in order to prove that F(x) is absolutely continuous it suffices to show that the convolution of 2 absolutely continuous distribution functions is again absolutely continuous.

Let $F_1(x)$ and $F_2(x)$ be absolutely continuous with densities $f_1(x) = F_1'(x)$ and $f_2(x) = F_2'(x)$.

Let F(x) be the convolution of $F_1(x)$ and $F_2(x)$, i.e.,

$$F(x) = \int_{-\infty}^{\infty} F_1(x-u)dF_2(u),$$

then we have

$$F(x) = \int_{-\infty}^{\infty} F_1(x-u)f_2(u)du = \int_{-\infty}^{\infty} [\int_{-\infty}^{x-u} f_1(w)dw]f_2(u)du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_1(z-u)f_2(u)dzdu = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_1(z-u)f_2(u)dudz$$

$$= \int_{-\infty}^{x} f(z)dz,$$

where $f(z) = \int_{-\infty}^{\infty} f_1(z-u)f_2(u)du$.

From the latter result it follows that $F(x)$ is absolutely continuous with density function $f(x) = F'(x) = \int_{-\infty}^{\infty} f_1(x-u)f_2(u)du$ and this completes the proof of (B.17).

The fact that $U_j \sim \chi^2(m_j, \theta_j)$, implies $E(U_j) = m_j + \theta_j$ and this enables us to compute $E(X)$, we have

(B.18) $$E(|X|) < \infty \text{ and } E(X) = \sum_{j=1}^{M} \alpha_j(m_j + \theta_j).$$

Proof:

$$|X| = |\sum_{j=1}^{M} \alpha_j U_j| \leq \sum_{j=1}^{M} |\alpha_j| \; |U_j|.$$

Hence,

$$E(|X|) \leq E(\sum_{j=1}^{M} |\alpha_j| \; |U_j|) = \sum_{j=1}^{M} |\alpha_j| \; E(|U_j|)$$

$$= \sum_{j=1}^{M} |\alpha_j| \; E(U_j) = \sum_{j=1}^{M} |\alpha_j| \; (m_j + \theta_j) < \infty.$$

Also,

$$E(X) = E(\sum_{j=1}^{M} \alpha_j U_j) = \sum_{j=1}^{M} \alpha_j \; E(U_j) = \sum_{j=1}^{M} \alpha_j(m_j + \theta_j).$$

Next we shall show that X has the following characteristic function

(B.19) $$\phi(t) = [\prod_{j=1}^{M} (1-2i\alpha_j t)^{-\frac{m_j}{2}}] \exp\{it \sum_{j=1}^{M} \frac{\theta_j \alpha_j}{1-2i\alpha_j t}\}.$$

Proof:

Since $U_j \sim \chi^2(m_j, \theta_j)$ it follows that

$$\phi_j(t) = E(e^{itU_j}) = (1-2it)^{-\frac{m_j}{2}} \exp\{\frac{it\theta_j}{1-2it}\}.$$

Now we get

$$\phi(t) = E(e^{itX}) = E(e^{it\sum_{j=1}^{M}\alpha_j U_j}) = E(\prod_{j=1}^{M} e^{i\alpha_j t U_j})$$

$$= \prod_{j=1}^{M} E(e^{i(\alpha_j t)U_j}) = \prod_{j=1}^{M} \phi_j(\alpha_j t),$$

which proves the desired result.

The modulus and argument of $\phi(t)$ are given by

(B.20)

$$|\phi(t)| = [\prod_{j=1}^{M} (1+4\alpha_j^2 t^2)^{-\frac{m_j}{4}}] \exp\{-2 \sum_{j=1}^{M} \frac{\theta_j \alpha_j^2 t^2}{1+4\alpha_j^2 t^2}\}$$

$$\arg(\phi(t)) = \sum_{j=1}^{M} [(\frac{m_j}{2}) \operatorname{arctg}(2\alpha_j t) + \frac{\theta_j \alpha_j t}{1+4\alpha_j^2 t^2}]$$

<u>Proof</u>:
see Imhof [4] and Koerts and Abrahamse [6].

The function $|\phi(t)|$ can be bounded above as follows:

(B.21) $\quad |\phi(t)| \leq (1+4\alpha^2 t^2)^{-\frac{m}{4}}, \; -\infty < t < \infty,$

where $\alpha = \min\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_M|\}$ and $m = \sum_{j=1}^{M} m_j$.

<u>Proof</u>:

From (B.20) it is seen that $|\phi(t)| \leq \prod_{j=1}^{M} (1+4\alpha_j^2 t^2)^{-\frac{m_j}{4}}$.
Now we have $1+4\alpha_j^2 t^2 \geq 1+4\alpha^2 t^2$, which implies that

$$(1+4\alpha_j^2 t^2)^{-\frac{m_j}{4}} \leq (1+4\alpha^2 t^2)^{-\frac{m_j}{4}}.$$

This gives:

$$|\phi(t)| \leq \prod_{j=1}^{M} (1+4\alpha_j^2 t^2)^{-\frac{m_j}{4}} \leq \prod_{j=1}^{M} (1+4\alpha^2 t^2)^{-\frac{m_j}{4}}$$

$$= (1+4\alpha^2 t^2)^{-\frac{1}{4}\sum_{j=1}^{M} m_j} = (1+4\alpha^2 t^2)^{-\frac{m}{4}},$$

which completes the proof.

The characteristic fucntion $\phi(t)$ of X has the property that

(B.22) $$\int_1^\infty \left|\frac{\phi(t)}{t}\right| dt < \infty.$$

Proof:
From (B.21) it follows that

$$\int_1^\infty \left|\frac{\phi(t)}{t}\right| dt \leq \int_1^\infty t^{-1}(1+4\alpha^2 t^2)^{-\frac{m}{4}} dt.$$

After the transformation $z = (1+4\alpha^2 t^2)^{-1}$ with inverse transformation $t = (2\alpha)^{-1} z^{-\frac{1}{2}}(1-z)^{\frac{1}{2}}$ and jacobian

$$\left|\frac{dt}{dz}\right| = (4\alpha)^{-1} z^{-\frac{3}{2}}(1-z)^{-\frac{1}{2}},$$

we get

$$\int_1^\infty t^{-1}(1+4\alpha^2 t^2)^{-\frac{m}{4}} dt = \frac{1}{2}\int_0^A z^{\frac{m}{4}-1}(1-z)^{-1} dz,$$

where $A = (1+4\alpha^2)^{-1} < 1$.
Since $z < A < 1$ it follows that $(1-z)^{-1} < (1-A)^{-1} < \infty$ and this yields

$$\frac{1}{2}\int_0^A z^{\frac{m}{4}-1}(1-z)^{-1} dz \leq \frac{1}{2}(1-A)^{-1}\int_0^A z^{\frac{m}{4}-1} dz =$$

$$\frac{1}{2}(1-A)^{-1}\left[\frac{4}{m} z^{\frac{m}{4}}\Big|_0^A\right] = \frac{1}{2}(1-A)^{-1}\frac{4}{m} A^{\frac{m}{4}}$$

$$= (2m\alpha^2)^{-1}(1+4\alpha^2)^{-\frac{m}{4}+1}.$$

The above results imply that

$$\int_1^\infty \left|\frac{\phi(t)}{t}\right| dt \leq \int_1^\infty t^{-1}(1+4\alpha^2 t^2)^{-\frac{m}{4}} dt$$

$$= \frac{1}{2}\int_0^A z^{\frac{m}{4}-1}(1-z)^{-1}dz \leq (2m\alpha^2)^{-1}(1+4\alpha^2)^{-\frac{m}{4}+1} < \infty,$$

which proves the desired result.

Next we shall derive sufficient conditions for the integrability of the functions $|\phi(t)|$ and $|t\,\phi(t)|$, respectively.

(B.23)

$$\text{If } m \geq 3 \text{ then } \int_{-\infty}^{\infty} |\phi(t)| dt < \infty$$

$$\text{If } m \geq 5 \text{ then } \int_{-\infty}^{\infty} |t\,\phi(t)| dt < \infty .$$

Proof:

In the first place we have

$$\int_{-\infty}^{\infty} (1+4\alpha^2 t^2)^{-\frac{m}{4}} dt = 2\int_0^{\infty} (1+4\alpha^2 t^2)^{-\frac{m}{4}} dt =$$

$$(2\alpha)^{-1}\int_0^1 z^{\frac{m-2}{4}-1}(1-z)^{\frac{1}{2}-1} dz = \frac{\sqrt{\pi}\,\Gamma(\frac{m-2}{4})}{2\alpha\,\Gamma(\frac{m}{4})} < \infty,$$

if $m \geq 3$, where use has been made of the above transformation $z = (1+\alpha^2 t^2)^{-1}$.

Hence, if $m \geq 3$ it follows from (B.21) that

$$\int_{-\infty}^{\infty} |\phi(t)| dt \leq \int_{-\infty}^{\infty} (1+4\alpha^2 t^2)^{-\frac{m}{4}} dt = \frac{\sqrt{\pi}\,\Gamma(\frac{m-2}{4})}{2\alpha\,\Gamma(\frac{m}{4})} < \infty,$$

which proves the first line of (B.23).

In the second place, if $m \geq 5$ we get in a similar way

$$\int_{-\infty}^{\infty} |t|\ (1+4\alpha^2t^2)^{-\frac{m}{4}}dt = 2\int_0^{\infty} t(1+4\alpha^2t^2)^{-\frac{m}{4}}dt =$$

$$(4\alpha^2)^{-1}\int_0^1 z^{\frac{m-4}{4}-1}dz = \frac{1}{\alpha^2(m-4)} < \infty.$$

From (B.21) we therefore have,

$$\int_{-\infty}^{\infty} |t\ \phi(t)|dt \leq \int_{-\infty}^{\infty} |t|\ (1+4\alpha^2t^2)^{-\frac{m}{4}}dt = \frac{1}{\alpha^2(m-4)} < \infty,$$

if $m \geq 5$ and this shows the second statement of (B.23).
With the aid of (B.23) we can express the density function $f(x) = F'(x)$ of X and the derivative $f'(x)$ of $f(x)$ in terms of $\phi(t)$.

(B.24)

$$\text{If } m \geq 3 \text{ then } f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt,\ -\infty < x < \infty$$

$$\text{If } m \geq 5 \text{ then } f'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -ite^{-itx}\phi(t)dt,\ -\infty < x < \infty.$$

Proof:

The results follow at once from (B.23) and the general results (B.5) and (B.6), respectively.
Now we shall prove that the inversion formula for $\phi(t)$ as given in (B.19) takes the form

(B.25) $$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \{\text{Im}(\phi(t)e^{-itx})t\}dt,$$

$-\infty < x < \infty$, where

$$\text{Im}(\phi(t)e^{-itx})/t\Big|_{t=0} \overset{\text{def.}}{=} \lim_{t\downarrow 0}\{\text{Im}(\phi(t)e^{-itx})/t\}$$

$$= E(X) - x = \sum_{j=1}^{M} \alpha_j(m_j + \theta_j) - x.$$

Proof:

We know that $E(|X|) < \infty$ and $\int_1^\infty |\frac{\phi(t)}{t}|dt < \infty$, see (B.18) and (B.22), respectively.

Hence, it follows from the general result (B.11) that

$$\frac{F(x) + F(x-)}{2} = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \{\mathrm{Im}(\phi(t)e^{-itx})/t\}dt,$$

$-\infty < x < \infty$ and

$$\mathrm{Im}(\phi(t)e^{-itx})/t\Big|_{t=0} \overset{\text{def.}}{=} \lim_{t\downarrow 0}\{\mathrm{Im}(\phi(t)e^{-itx})/t\} = E(X) - x.$$

The proof is completed by observing that $F(x-) = F(x)$ for all x, which is implied by (B.17), and by making use of $E(X) = \sum_{j=1}^{M} \alpha_j(m_j + \theta_j)$, see (B.18).

In order to compute F(x) from the given function $\phi(t)$ it is more convenient to rewrite (B.25) as follows:

(B.26) $$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\{\varepsilon(u) - \frac{1}{2}ux\}}{u\,\gamma(u)}du,$$

$-\infty < x < \infty$, where

$$\gamma(u) = \left[\prod_{j=1}^{M}(1+\alpha_j^2u^2)^{\frac{m_j}{4}}\right]\exp\left\{\frac{1}{2}\sum_{j=1}^{M}\frac{\theta_j\alpha_j^2u^2}{1+\alpha_j^2u^2}\right\}$$

$$\varepsilon(u) = \frac{1}{2}\sum_{j=1}^{M}\left[m_j\,\mathrm{arctg}(\alpha_j u) + \frac{\theta_j\alpha_j u}{1+\alpha_j^2u^2}\right]$$

and where

$$\frac{\sin\{\varepsilon(u) - \frac{1}{2}ux\}}{u\,\gamma(u)}\Bigg|_{u=0} \overset{\text{def.}}{=} \lim_{u\downarrow 0}\frac{\sin\{\varepsilon(u) - \frac{1}{2}ux\}}{u\,\gamma(u)}$$

$$= \frac{1}{2}E(X) - \frac{1}{2}x = \frac{1}{2}\sum_{j=1}^{M}\alpha_j(m_j + \theta_j) - \frac{1}{2}x.$$

Proof:

With the aid of (B.15) it follows from (B.25) that

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{t}\,|\phi(t)|\,\sin\{\arg(\phi(t)) - tx\}dt,\ -\infty < x < \infty.$$

After the transformation u = 2t with inverse transformation $t = \frac{1}{2}u$ and jacobian $|\frac{dt}{du}| = \frac{1}{2}$ we get

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{u}\,|\phi(\tfrac{1}{2}u)|\,\sin\{\arg(\phi(\tfrac{1}{2}u)) - \tfrac{1}{2}ux\}du.$$

Now the first part of (B.26) follows at once from (B.20) by setting $\gamma(u) = |\phi(\frac{1}{2}u)|^{-1}$ and $\varepsilon(u) = \arg(\phi(\frac{1}{2}u))$.

In order to prove the second part of (B.26) we observe that

$$\frac{\sin\{\varepsilon(u) - \frac{1}{2}ux\}}{u\,\gamma(u)} = \frac{1}{u}\,|\phi(\tfrac{1}{2}u)|\,\sin\{\arg(\phi(\tfrac{1}{2}u)) - \tfrac{1}{2}ux\}$$

$$= \mathrm{Im}(\phi(\tfrac{1}{2}u)e^{-iu(\frac{1}{2}x)})/u = \mathrm{Im}(\phi^*(u)e^{-iux^*})/u,$$

where $x^* = \frac{1}{2}x$ and

$$\phi^*(u) = [\prod_{j=1}^{M}(1-2i\alpha_j^*u)^{-\frac{m_j}{4}}]\exp\{iu\sum_{j=1}^{M}\frac{\theta_j\alpha_j^*}{1-2i\alpha_j^*u}\}$$

with $\alpha_j^* = \frac{1}{2}\alpha_j$, j = 1, 2, ..., M.

That is $\phi^*(u)$ is the characteristic function of the random variable $X^* = \sum_{j=1}^{M}\alpha_j^*U_j$, where the U_j's are defined as before.

Hence, it follows from (B.9) and (B.18) that

$$\lim_{u\downarrow 0}\{\mathrm{Im}(\phi^*(u)e^{-iux^*})/u\} = E(X^*) - x^*$$

$$= \sum_{j=1}^{M}\alpha_j^*(m_j + \theta_j) - x^* = \frac{1}{2}\sum_{j=1}^{M}\alpha_j(m_j + \theta_j) - \frac{1}{2}x$$

and this implies that

$$\lim_{u \downarrow 0} \frac{\sin\{\varepsilon(u) - \frac{1}{2} ux\}}{u\,\gamma(u)} = \frac{1}{2} E(X) - \frac{1}{2} x = \frac{1}{2} \sum_{j=1}^{M} \alpha_j (m_j + \theta_j) - \frac{1}{2} x$$

as was to be proved.
For the numerical computation of $F(x)$ at a fixed point x, the integral in (B.26) is evaluated in two steps.

a) $\int_0^\infty \frac{\sin\{\varepsilon(u) - \frac{1}{2} ux\}}{u\,\gamma(u)} du$ is approximated by

$\int_0^U \frac{\sin\{\varepsilon(u) - \frac{1}{2} ux\}}{u\,\gamma(u)} du$ for sufficiently large U.

b) $\int_0^U \frac{\sin\{\varepsilon(u) - \frac{1}{2} ux\}}{u\,\gamma(u)} du$ is approximated by using the compound rule of Simpson.

The truncation error in a) and the approximation error in b) both can be made arbitrarily small.
For a computer program which computes $F(x)$ in the way indicated above, we refer to Koerts and Abrahamse [6].
As far as the truncation error in a) is concerned, we have the following result:

$$\text{(B.27)} \qquad \left| \int_U^\infty \frac{\sin\{\varepsilon(u) - \frac{1}{2} ux\}}{u\,\gamma(u)} du \right| \leq T(U), \quad U > 0,$$

where

$$T(U) = 2m^{-1} c_1^{-1} U^{-\frac{m}{2}} \exp\{-\frac{1}{2} \sum_{j=1}^{M} \frac{\theta_j \alpha_j^2 U^2}{1+\alpha_j^2 U^2}\}$$

and

$$c_1 = \prod_{j=1}^{M} |\alpha_j|^{\frac{m_j}{2}}.$$

Proof:

$$\left| \int_U^\infty \frac{\sin\{\varepsilon(u) - \frac{1}{2} ux\}}{u\ \gamma(u)} du \right| \le \int_U^\infty \left| \frac{\sin\{\varepsilon(u) - \frac{1}{2} ux\}}{u\ \gamma(u)} \right| du \le \int_U^\infty \frac{1}{u\ \gamma(u)} du.$$

Since $1+\alpha_j^2 u^2 \ge \alpha_j^2 u^2$ and $\exp\{\frac{1}{2} \sum_{j=1}^{M} \frac{\theta_j \alpha_j^2 u^2}{1+\alpha_j^2 u^2}\}$ is nondecreasing in $u > 0$, it follows from the definition of $\gamma(u)$ that

$$u\ \gamma(u) > u \prod_{j=1}^{M} (\alpha_j^2 u^2)^{\frac{m_j}{4}} \exp\{\frac{1}{2} \sum_{j=1}^{M} \frac{\theta_j \alpha_j^2 U^2}{1+\alpha_j^2 U^2}\}$$

$$= u^{\frac{m}{2}+1} c_1 \exp\{\frac{1}{2} \sum_{j=1}^{M} \frac{\theta_j \alpha_j^2 U^2}{1+\alpha_j^2 U^2}\},$$

for $u \ge U$.

This implies that

$$\int_U^\infty \frac{1}{u\ \gamma(u)} du \le c_1^{-1} \exp\{-\frac{1}{2} \sum_{j=1}^{M} \frac{\theta_j \alpha_j^2 U^2}{1+\alpha_j^2 U^2}\} \int_U^\infty u^{-\frac{m}{2}-1} du$$

$$= 2m^{-1} c_1^{-1} U^{-\frac{m}{2}} \exp\{-\frac{1}{2} \sum_{j=1}^{M} \frac{\theta_j \alpha_j^2 U^2}{1+\alpha_j^2 U^2}\},$$

which completes the proof of (B.27).

Note that in the case $\theta_j = 0$ for all j we get

$$T(U) = 2m^{-1} c_1^{-1} U^{-\frac{m}{2}}.$$

Further it is easily seen that $T(U)$ is a strictly decreasing function of $U > 0$ with $T(U) \to 0$ as $U \to \infty$.

When $\theta_j > 0$ for at least one j we have:

$$2m^{-1} c_1^{-1} c_2^{-1} U^{-\frac{m}{2}} < T(U) < 2m^{-1} c_1^{-1} U^{-\frac{m}{2}},\ U > 0,$$

where $c_2 = \exp\{\frac{1}{2} \sum_{j=1}^{M} \theta_j\}$.

Appendix C

The distribution function of Q_s considered as a function of the parameters

In this appendix we shall investigate how the distribution function of a particular linear combination of mutually stochastically independent chi-square random variables depends on the noncentrality parameters. To be more specific, consider the random variable Q_s, which for each fixed $s > 0$ is defined by

$$\text{(C.1)} \qquad Q_s = \sum_{j=0}^{M} [\lambda_{1j}(s)V_{1j}(s) + \lambda_{2j}(s)V_{2j}(s)] + \lambda(s)V(s).$$

The random variables $V_{10}(s)$, $V_{11}(s)$, ..., $V_{1M}(s)$, $V_{20}(s)$, $V_{21}(s)$, ..., $V_{2M}(s)$, $V(s)$ are mutually stochastically independent and the coefficients $\lambda_{1j}(s)$ and $\lambda_{2j}(s)$ are given by

$$\text{(C.2)} \qquad \begin{aligned} \lambda_{1j}(s) &= \tfrac{1}{2}(1-s) - \tfrac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)} \\ \lambda_{2j}(s) &= \tfrac{1}{2}(1-s) + \tfrac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)}, \end{aligned}$$

$j = 0, 1, ..., M$, where $\rho_0 = 0 < \rho_1 < \rho_2 < ... < \rho_M < 1$.
The coefficient $\lambda(s)$ is arbitrary and the random variables $V(s)$, $V_{1j}(s)$ and $V_{2j}(s)$ possess the following probability distributions:

$$\text{(C.3)} \qquad \begin{aligned} &V(s) \sim \chi^2(m) \\ &V_{1j}(s) \sim \chi^2(m_{1j}, \tau_{1j}(s)) \\ &V_{2j}(s) \sim \chi^2(m_{2j}, \tau_{2j}(s)) \end{aligned}$$

where $m \geq 0$, $m_{1j} \geq 0$, $m_{2j} \geq 0$, $j = 0, 1, ..., M$ and $m_{1j} = m_{2j} = m_j$ for $j = 1, 2, ..., M$.
The noncentrality parameters $\tau_{1j}(s)$ and $\tau_{1j}(s)$ in (C.3) are of the form

$$\text{(C.4)} \qquad \begin{aligned} \tau_{1j}(s) &= c_{1j}(s)v_{2j} + d_{1j}(s)v_{1j} \\ \tau_{1j}(s) &= c_{2j}(s)v_{2j} + d_{2j}(s)v_{1j}, \end{aligned}$$

for $j = 0, 1, \ldots, M$, where the parameters v_{1j} and v_{2j} satisfy $v_{1j} \geq 0$, $v_{2j} \geq 0$ and where the coefficients $c_{1j}(s)$, $c_{2j}(s)$, $d_{1j}(s)$ and $d_{2j}(s)$ are given by

$$c_{1j}(s) = \frac{\lambda_{1j}(s) + s}{\lambda_{1j}(s)[\lambda_{1j}(s) - \lambda_{2j}(s)]}$$

$$c_{2j}(s) = \frac{\lambda_{2j}(s) + s}{\lambda_{2j}(s)[\lambda_{2j}(s) - \lambda_{1j}(s)]}$$

(C.5)

$$d_{1j}(s) = \frac{-s[\lambda_{1j}(s) - 1]}{\lambda_{1j}(s)[\lambda_{1j}(s) - \lambda_{2j}(s)]}$$

$$d_{2j}(s) = \frac{-s[\lambda_{2j}(s) - 1]}{\lambda_{2j}(s)[\lambda_{2j}(s) - \lambda_{1j}(s)]}.$$

With respect to the above coefficients we make the following remarks. In the first place we note that

$$\lambda_{10}(s) = -s, \ \lambda_{20}(s) = 1$$

(C.6)

$$-s < \lambda_{1j}(s) < 0, \ 0 < \lambda_{2j}(s) < 1, \ j = 1, 2, \ldots, M.$$

Secondly, it is not difficult to verify that

$$\lambda_{1j}(s) + \lambda_{2j}(s) = 1-s$$

(C.7)

$$\lambda_{1j}(s) \, \lambda_{2j}(s) = -s(1 - \rho_j),$$

for $j = 0, 1, \ldots, M$.
From (C.5) and (C.6) it easily follows that

$$c_{10}(s) = 0, \ c_{20}(s) = 1$$

(C.8)

$$d_{10}(s) = 1, \ d_{20}(s) = 0$$

and also that $c_{1j}(s) > 0$, $c_{2j}(s) > 0$, $d_{1j}(s) > 0$ and $d_{2j}(s) > 0$ for $j = 1, 2, \ldots, M$.

Since the noncentrality parameters $\tau_{1j}(s)$ and $\tau_{2j}(s)$ of the chi-square random variables $V_{1j}(s)$ and $V_{2j}(s)$ depend on the parameters v_{1j} and v_{2j}, it will be clear that the distribution function of the random variable Q_s depends on the parameters v_{1j} and v_{2j}.
We shall investigate this dependence in the following two situations:

(I) $\quad v_{1j} = 0,\ v_{2j} \geq 0,\ j = 0, 1, \ldots, M.$

(II) $\quad v_{1j} \geq 0,\ v_{2j} = 0,\ j = 0, 1, \ldots, M.$

That is, if $v = (v_1, v_2)$ with $v_1 = (v_{10}, v_{11}, \ldots, v_{1M})$ and $v_2 = (v_{20}, v_{21}, \ldots, v_{2M})$, we shall derive the partial derivatives

$$\frac{\partial\, G_v(x, s)}{\partial v_{2j}},\ j = 0, 1, \ldots, M$$

in situation (I), and

$$\frac{\partial\, G_v(x, s)}{\partial v_{1j}},\ j = 0, 1, \ldots, M$$

in situation (II).
Here $G_v(x, s)$ denotes the distribution function of Q_s, i.e.,

$$G_v(x, s) = P(Q_s \leq x).$$

From (C.1) and the result (B.19) of Appendix B it follows that the characteristic function $\psi_v(t, s) = E(e^{itQ_s})$ of Q_s takes the form

$$\text{(C.9)} \qquad \psi_v(t, s) = [\prod_{j=0}^{M} (1-2i\lambda_{1j}(s)t)^{-\frac{m_{1j}}{2}} (1-2i\lambda_{2j}(s)t)^{-\frac{m_{2j}}{2}}]$$

$$(1-2i\lambda(s)t)^{-\frac{m}{2}} \exp\{it \sum_{j=0}^{M} [\frac{\lambda_{1j}(s)\tau_{1j}(s)}{1-2i\lambda_{1j}(s)t} + \frac{\lambda_{2j}(s)\tau_{2j}(s)}{1-2i\lambda_{2j}(s)t}]\},$$

where i denotes the imaginary unit.
By making use of the result (B.25) of Appendix B, we can express $G_v(x, s)$ in terms of $\psi_v(t, s)$ as follows:

$$\text{(C.10)} \qquad G_v(x, s) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \{\text{Im}(\psi_v(t, s)e^{-itx})/t\}dt,$$

$-\infty < x < \infty$, where $\text{Im}(\psi_v(t, s)e^{-itx})$ denotes the imaginary part of the complex-valued function $\psi_v(t, s)e^{-itx}$ and where

$$\text{Im}(\psi_v(t, s)e^{-itx})/t\Big|_{t=0} \overset{\text{def.}}{=} \lim_{t \downarrow 0}\{\text{Im}(\psi_v(t, s)e^{-itx})/t\}$$

$$= E(Q_s) - x = \sum_{j=0}^{M} [\lambda_{1j}(s)(m_{1j} + \tau_{1j}(s))$$

$$+ \lambda_{2j}(s)(m_{2j} + \tau_{2j}(s))] + \lambda(s)m - x.$$

Now we shall first derive $\partial G_v(x, s)/\partial v_{2j}$ for $j = 0, 1, \ldots, M$ in case (I).

Since we have $v_{1j} = 0$, $j = 0, 1, \ldots, M$ it follows that $\tau_{1j}(s) = c_{1j}(s)v_{2j}$ and $\tau_{1j}(s) = c_{2j}(s)v_{2j}$ (see (C.4)) and this shows that $\psi_v(t, s)$ takes the form:

$$\text{(C.11)} \qquad \psi_v(t, s) = [\prod_{j=0}^{M} (1-2i\lambda_{1j}(s)t)^{-\frac{m_{1j}}{2}} (1-2i\lambda_{2j}(s)t)^{-\frac{m_{2j}}{2}}]$$

$$(1-2i\lambda(s)t)^{-\frac{m}{2}} \exp\{it \sum_{j=0}^{M} [\frac{\lambda_{1j}(s)c_{1j}(s)v_{2j}}{1-2i\lambda_{1j}(s)t} + \frac{\lambda_{2j}(s)c_{2j}(s)v_{2j}}{1-2i\lambda_{2j}(s)t}]\}.$$

If $\tilde{v} = (0, \tilde{v}_2)$ with $\tilde{v}_2 = (v_{20}, v_{21}, \ldots, v_{2j}+\Delta, \ldots, v_{2M})$, it is seen from (C.10) that

$$\text{(C.12)} \qquad \frac{\partial G_v(x, s)}{\partial v_{2j}} = \lim_{\Delta \to \infty} \frac{G_{\tilde{v}}(x, s) - G_v(x, s)}{\Delta} =$$

$$\lim_{\Delta \to 0} -\frac{1}{\pi} \int_0^\infty \frac{\text{Im}(\psi_{\tilde{v}}(t, s)e^{-itx}) - \text{Im}(\psi_v(t, s)e^{-itx})}{t\Delta} dt,$$

for $j = 0, 1, \ldots, M$.

We start with the case $j = 0$. Since $c_{10}(s) = 0$, $c_{20}(s) = 1$ and $\lambda_{20}(s) = 1$, we can write:

$$\psi_v(t, s) = f(t) \exp\{\frac{itv_{20}}{1-2it}\},$$

where

$$f(t) \overset{\text{def.}}{=} \psi_v(t, s) \exp\{\frac{-itv_{20}}{1-2it}\}.$$

This yields

$$\psi_{\tilde{v}}(t, s) = f(t) \exp\{\frac{it(v_{20} + \Delta)}{1-2it}\}$$

$$= f(t) \exp\{\frac{itv_{20}}{1-2it}\} \exp\{\frac{it\Delta}{1-2it}\}$$

$$= \psi_v(t, s) \exp\{\frac{it\Delta}{1-2it}\}.$$

By making use of $Im(z) = (z - \bar{z})/(2i)$ for a complex number z and

$$\overline{\psi_v(t, s)e^{-itx}} = \psi_v(-t, s)e^{itx},$$

it follows that

$$\text{(C.13)} \quad \frac{Im(\psi_{\tilde{v}}(t, s)e^{-itx}) - Im(\psi_v(t, s)e^{-itx})}{t\Delta} = \tfrac{1}{2}[h(t, \Delta) + h(-t, \Delta)],$$

where

$$h(t, \Delta) = \psi_v(t, s)e^{-itx}\left(\frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta}\right).$$

Substitution of (C.13) into (C.12) gives:

$$\text{(C.14)} \quad \frac{\partial G_v(x, s)}{\partial v_{20}} = \lim_{\Delta \to 0} -\frac{1}{\pi}\int_0^\infty \frac{1}{2}[h(t, \Delta) + h(-t, \Delta)]dt.$$

If the modulus of the integrand $\frac{1}{2}[h(t, \Delta) + h(-t, \Delta)]$ can be bounded by a positive, integrable function of t which does not depend on Δ, we can apply Lebesgue's dominated convergence theorem and interchange the operations lim and $\int$.

We therefore first consider $|\frac{1}{2}[h(t, \Delta) + h(-t, \Delta)]|$.
Since $\overline{h(t, \Delta)} = h(-t, \Delta)$ and $|\overline{h(t, \Delta)}| = |h(t, \Delta)|$, it follows that

(C.15) $$|\frac{1}{2}[h(t, \Delta) + h(-t, \Delta)]| \leq \frac{1}{2}|h(t, \Delta)| + \frac{1}{2}|h(-t, \Delta)|$$

$$= |h(t, \Delta)| = |\psi_v(t, s)| \; |\frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta}|,$$

where use has been made of the definition of $h(t, \Delta)$.
In order to obtain the desired result we shall prove that for any $A < 0$

(C.16) $$|\frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta}| \leq e^{-\frac{1}{2}A} |(1-2it)^{-1}|$$

for $A < \Delta < \infty$ and all t.

<u>Proof</u>:
Let $\Delta > 0$, then

$$\frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta} = \frac{\exp\{\frac{ity}{1-2it}\}}{it\Delta}\Bigg|_{y=0}^{y=\Delta} = (1-2it)^{-1} \frac{1}{\Delta} \int_0^\Delta \exp\{\frac{ity}{1-2it}\}dy,$$

which shows that

$$\left| \frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta} \right| \leq |(1-2it)^{-1}| \frac{1}{\Delta} \int_0^\Delta |\exp\{\frac{ity}{1-2it}\}|dy.$$

Since

$$|\exp\{\frac{ity}{1-2it}\}| = \exp\{\frac{-2yt^2}{1+4t^2}\} \leq 1$$

for $y > 0$ and all t, we get

$$\left| \frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta} \right| \leq |(1-2it)^{-1}| \frac{1}{\Delta} \int_0^\Delta dy$$

$$= |(1-2it)^{-1}| \leq e^{-\frac{1}{2}A} |(1-2it)^{-1}|,$$

for any $A < 0$, $\Delta > 0$ and all t.

Next suppose that $\Delta < 0$, then we have

$$\frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta} = - \frac{\exp\{\frac{ity}{1-2it}\}}{it\Delta}\Bigg|_{y=\Delta}^{y=0}$$

$$= (1-2it)^{-1} \frac{1}{(-\Delta)} \int_{\Delta}^{0} \exp\{\frac{ity}{1-2it}\}dy$$

Now we get

$$|\exp\{\frac{ity}{1-2it}\}| = \exp\{\frac{-2yt^2}{1+4t^2}\} \leq \exp\{\frac{-2\Delta t^2}{1+4t^2}\}$$

for $\Delta < y < 0$ and all t.

This yields:

$$\left| \frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta} \right| \leq |(1-2it)^{-1}| \frac{1}{(-\Delta)} \int_{\Delta}^{0} |\exp\{\frac{ity}{1-2it}\}|dy$$

$$\leq |(1-2it)^{-1}| \exp\{\frac{-2\Delta t^2}{1+4t^2}\} \frac{1}{(-\Delta)} \int_{\Delta}^{0} dy$$

$$= |(1-2it)^{-1}| \exp\{\frac{-2\Delta t^2}{1+4t^2}\} \leq |(1-2it)^{-1}| \exp\{\frac{-2At^2}{1+4t^2}\}$$

$$\leq |(1-2it)^{-1}| e^{-\frac{1}{2}A}$$

for any $A < 0$, $A < \Delta < 0$ and all t, where use has been made of

$$\exp\{\frac{-2At^2}{1+4t^2}\} \leq \lim_{t\to\pm\infty} \exp\{\frac{-2At^2}{1+4t^2}\} = e^{-\frac{1}{2}A}, \; A < 0.$$

This completes the proof of (C.16).

The substitution of (C.16) into (C.15) yields:

(i) $$| \tfrac{1}{2}[h(t, \Delta) + h(-t, \Delta)]| \leq e^{-\frac{1}{2}A} | \psi_{0,v}(t, s)|$$

for any $A < 0$, $A < \Delta < \infty$ and all t, where

$$\text{(C.17)} \qquad \psi_{0,v}(t, s) = (1-2it)^{-1} \psi_v(t, s).$$

Since $(1-2it)^{-1}$ is the characteristic function of a random variable with a $\chi^2(2)$ distribution, it follows that the function $\psi_{0,v}(t, s)$ is the characteristic function of the random variable

$$Q_{0,s} = Q_s + W,$$

where W is independent of $V_{1j}(s)$, $V_{2j}(s)$ and $V(s)$ and $W \sim \chi^2(2)$. With the aid of $\lambda_{20}(s) = 1$ and (C.1) it is seen that $Q_{0,s}$ can be written as:

$$\text{(C.18)} \qquad Q_{0,s} = \sum_{j=0}^{M} [\lambda_{1j}(s)\tilde{V}_{1j}(s) + \lambda_{2j}(s)\tilde{V}_{2j}(s)] + \lambda(s)\tilde{V}(s),$$

where $\tilde{V}_{10}(s)$, $\tilde{V}_{11}(s)$, ..., $\tilde{V}_{1M}(s)$, $\tilde{V}_{20}(s)$, ..., $\tilde{V}_{2M}(s)$, $\tilde{V}(s)$ are mutually stochastically independent and have the following distributions

$$\text{(C.19)} \qquad \begin{aligned} &\tilde{V}(s) \sim \chi^2(\tilde{m}) \\ &\tilde{V}_{1j}(s) \sim \chi^2(\tilde{m}_{1j}, \tau_{1j}(s)) \\ &\tilde{V}_{2j}(s) \sim \chi^2(\tilde{m}_{2j}, \tau_{2j}(s)), \end{aligned}$$

$j = 0, 1, \ldots, M$.
The degrees of freedom are given by

$$\text{(C.20)} \qquad \begin{aligned} &\tilde{m} = m \\ &\tilde{m}_{1j} = m_{1j}, \; j = 0, 1, \ldots, M \\ &\tilde{m}_{20} = m_{20} + 2 \\ &\tilde{m}_{2j} = m_{2j}, \; j = 1, 2, \ldots, M \end{aligned}$$

Note that $Q_{0,s}$ is again a random variable of the type considered in (C.1) and that $Q_{0,s}$ can be obtained from Q_s through replacing m_{20} by $m_{20} + 2 = \tilde{m}_{20}$.

As is shown in Appendix B, see (B.23), if the sum of the degrees of freedom $\sum_{j=0}^{M} (\tilde{m}_{1j} + \tilde{m}_{2j}) + \tilde{m} \geq 3$, it follows that

$$\text{(C.21)} \qquad \int_{-\infty}^{\infty} |\psi_{0,v}(t, s)| dt < \infty.$$

Now we have

$$\sum_{j=0}^{M} (\tilde{m}_{1j} + \tilde{m}_{2j}) + \tilde{m} = \sum_{j=0}^{M} (m_{1j} + m_{2j}) + m + 2 \geq m_{20} + 2 \geq 3,$$

since $m_{20} \geq 1$ and this shows (C.21).

Returning to the positive function $e^{-\frac{1}{2}A} |\psi_{0,v}(t, s)|$ in the right-hand side of (i), it is seen from (C.21) that

$$\text{(ii)} \qquad \int_{0}^{\infty} e^{-\frac{1}{2}A} |\psi_{0,v}(t, s)| dt = e^{-\frac{1}{2}A} \int_{-\infty}^{\infty} I_{(0,\infty)}(t) |\psi_{0,v}(t, s)| dt$$

$$\leq e^{-\frac{1}{2}A} \int_{-\infty}^{\infty} |\psi_{0,v}(t, s)| dt < \infty,$$

where $I_{(0,\infty)}(t)$ denotes the indicator function of $(0, \infty)$.

From

$$\lim_{\Delta \to 0} \frac{\exp\{\frac{it\Delta}{1-2it}\} - 1}{it\Delta} = \lim_{\Delta \to 0} \left(\frac{it(1-2it)^{-1} \exp\{\frac{it\Delta}{1-2it}\}}{it} \right)$$

$$= (1-2it)^{-1} \lim_{\Delta \to 0} \exp\{\frac{it\Delta}{1-2it}\} = (1-2it)^{-1},$$

it follows that

$$\text{(iii)} \qquad \lim_{\Delta \to 0} \frac{1}{2} [h(t, \Delta) + h(-t, \Delta)] =$$

$$\frac{1}{2} [(1-2it)^{-1} \psi_v(t, s) e^{-itx} + (1+2it)^{-1} \psi_v(-t, s) e^{itx}]$$

$$= \frac{1}{2} [\psi_{0,v}(t, s) e^{-itx} + \psi_{0,v}(-t, s) e^{itx}].$$

The results (i), (ii) and (iii) enable us to apply the dominated convergence theorem to (C.14) and this yields

$$\text{(C.22)}\qquad \frac{\partial G_v(x,\ s)}{\partial v_{20}} = -\frac{1}{\pi}\int_0^\infty \frac{1}{2}\,[e^{-itx}\psi_{0,v}(t,\ s) + e^{itx}\psi_{0,v}(-t,\ s)]dt.$$

Since (C.21) implies that both integrals

$$\int_0^\infty |e^{-itx}\psi_{0,v}(t,\ s)|dt \text{ and } \int_0^\infty |e^{itx}\psi_{0,v}(-t,\ s)|dt$$

are finite, we get

$$\frac{1}{\pi}\int_0^\infty \frac{1}{2}\,[e^{-itx}\psi_{0,v}(t,\ s) + e^{itx}\psi_{0,v}(-t,\ s)]dt =$$

$$\frac{1}{2\pi}\,[\int_0^\infty e^{-itx}\psi_{0,v}(t,\ s)dt + \int_0^\infty e^{itx}\psi_{0,v}(-t,\ s)dt] =$$

$$\frac{1}{2\pi}\,[\int_0^\infty e^{-itx}\psi_{0,v}(t,\ s)dt + \int_{-\infty}^0 e^{-itx}\psi_{0,v}(t,\ s)dt] =$$

$$\frac{1}{2\pi}\int_{-\infty}^\infty e^{-itx}\psi_{0,v}(t,\ s)dt.$$

With the aid of the latter result, (C.22) can be rewritten as

$$\text{(C.23)}\qquad \frac{\partial G_v(x,\ s)}{\partial v_{20}} = -\frac{1}{2\pi}\int_{-\infty}^\infty e^{-itx}\psi_{0,v}(t,\ s)dt.$$

Now let $G_{0,v}(x,\ s)$ be the distribution function of $Q_{0,s}$, i.e., $G_{0,v}(x,\ s) = P(Q_{0,s} \leq x)$ and let the probability density function of $Q_{0,s}$ be denoted by $g_{0,v}(x,\ s)$, that is,

$$g_{0,v}(x,\ s) = G'_{0,v}(x,\ s) = \frac{\partial G_{0,v}(x,\ s)}{\partial x}\ .$$

Since $\sum_{j=0}^{M} (\tilde{m}_{1j} + \tilde{m}_{2j}) + \tilde{m} \geq 3$, it follows from the result (B.24) of Appendix B that

(C.24) $$g_{0,v}(x, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi_{0,v}(t, s)dt,$$

for $-\infty < x < \infty$.

Finally, the substitution of (C.24) into (C.23) yields

(C.25) $$\frac{\partial G_v(x, s)}{\partial v_{20}} = - g_{0,v}(x, s)$$

for $-\infty < x < \infty$, all $s > 0$, all $v = (0, v_2)$ and any $\lambda(s)$. This completes the case $j = 0$.

Next we consider the case $j = k$, where $k = 1, 2, \ldots, M$. We start with formula (C.12) and in this case the function $\psi_{\tilde{v}}(t, s)$ can be written as

$$\psi_{\tilde{v}}(t, s) = \psi_v(t, s) \exp\{it\Delta[\frac{c_{1k}(s)\lambda_{1k}(s)}{1-2i\lambda_{1k}(s)t} + \frac{c_{2k}(s)\lambda_{2k}(s)}{1-2i\lambda_{2k}(s)t}]\}.$$

Substitution of this result into (C.12) gives

(C.26) $$\frac{\partial G_v(x, s)}{\partial v_{2k}} = \lim_{\Delta\to 0} -\frac{1}{\pi}\int_0^{\infty} \frac{1}{2}[h(t, \Delta) + h(-t, \Delta)]dt,$$

where now $h(t, \Delta)$ is defined by

$$h(t, \Delta) = \psi_v(t, s)e^{-itx}(\frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta}),$$

and where

$$\ell(t) = \frac{c_{1k}(s)\lambda_{1k}(s)}{1-2i\lambda_{1k}(s)t} + \frac{c_{2k}(s)\lambda_{2k}(s)}{1-2i\lambda_{2k}(s)t}\ .$$

Now we have

(C.27) $$\left|\frac{1}{2}[h(t, \Delta) + h(-t, \Delta)]\right| \leq |h(t, \Delta)|$$
$$= |\psi_v(t, s)| \left|\frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta}\right|,$$

and we therefore first consider the modulus of $[\exp\{it\Delta\ \ell(t)\} - 1]/(it\Delta)$. We shall prove that for any $A < 0$:

(C.28) $$\left|\frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta}\right| \leq e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} |\ell(t)|$$

for $A < \Delta < \infty$ and all t.

Proof:

Let $\Delta > 0$, then

$$\frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta} = \frac{\exp\{ity\ \ell(t)\}}{it\Delta}\Bigg|_{y=0}^{y=\Delta} = \ell(t)\ \frac{1}{\Delta}\int_0^\Delta \exp\{ity\ \ell(t)\}dy,$$

which shows that

$$\left|\ \frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta}\ \right| \leq |\ell(t)|\ \frac{1}{\Delta}\int_0^\Delta |\exp\{ity\ \ell(t)\}|dy.$$

Since

$$|\exp\{ity\ \ell(t)\}| = \exp\{-2y[\frac{c_{1k}(s)\lambda_{1k}^2(s)t^2}{1+4\lambda_{1k}^2(s)t^2} + \frac{c_{2k}(s)\lambda_{2k}^2(s)t^2}{1+4\lambda_{2k}^2(s)t^2}]\}$$

$$= \exp\{-2y\ w(t)\} \leq 1$$

for $y > 0$ and all t, we get

$$\left|\ \frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta}\ \right| \leq |\ell(t)|\cdot\frac{1}{\Delta}\int_0^\Delta dy$$

$$= |\ell(t)| \leq e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}}\ |\ell(t)|,$$

for any $A < 0$ and all t, where

$$w(t) = \frac{c_{1k}(s)\lambda_{1k}^2(s)t^2}{1+4\lambda_{1k}^2(s)t^2} + \frac{c_{2k}(s)\lambda_{2k}^2(s)t^2}{1+4\lambda_{2k}^2(s)t^2}\ .$$

If $\Delta < 0$ we have

$$\frac{\exp\{it\Delta\ \ell(t)\} -1}{it\Delta} = -\ \frac{\exp\{ity\ \ell(t)\}}{it\Delta}\Bigg|_{y=\Delta}^{y=0}$$

$$= \ell(t)\ \frac{1}{(-\Delta)}\int_\Delta^0 \exp\{ity\ \ell(t)\}dy.$$

Since

$$|\exp\{ity\ \ell(t)\}| = \exp\{-2y\ w(t)\} \leq \exp\{-2\Delta\ w(t)\}$$

for $\Delta < y < 0$ and all t, it follows that

$$\left| \frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta} \right| \leq |\ell(t)| \frac{1}{(-\Delta)} \int_{\Delta}^{0} |\exp\{ity\ \ell(t)\}| dy$$

$$\leq |\ell(t)|\ \exp\{-2\Delta\ w(t)\} \frac{1}{(-\Delta)} \int_{\Delta}^{0} dy = |\ell(t)|\ \exp\{-2\Delta\ w(t)\}$$

$$\leq |\ell(t)|\ \exp\{-2A\ w(t)\} \leq |\ell(t)|\ \exp\{-\tfrac{1}{2}A(c_{1k}(s) + c_{2k}(s))\}$$

for any $A < 0$, $A < \Delta < 0$ and all t, where use has been made of

$$\exp\{-2A\ w(t)\} \leq \lim_{t\to\pm\infty} \exp\{-2A\ w(t)\} = \exp\{-\tfrac{1}{2}A(c_{1k}(s) + c_{2k}(s))\}.$$

From (C.5) and (C.7) it can be seen that

(C.29) $$c_{1k}(s) + c_{2k}(s) = \frac{1}{1-\rho_k},$$

which shows that

$$\left| \frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta} \right| \leq |\ell(t)|\ e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}}$$

for any $A < 0$, $A < \Delta < 0$ and all t.
This completes the proof of (C.28).
Now substitution of (C.28) into (C.27) yields

(C.30) $$\left| \frac{1}{2} [h(t, \Delta) + h(-t, \Delta)] \right| \leq e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} |\ell(t)\psi_v(t, s)|,$$

for any $A < 0$, $A < \Delta < \infty$ and all t.
With the aid of (C.5), (C.7) and (C.29) it is not difficult to verify that

$$\text{(C.31)} \qquad \ell(t) = \frac{\lambda_{1k}(s)c_{1k}(s)}{1-2i\lambda_{1k}(s)t} + \frac{\lambda_{2k}(s)c_{2k}(s)}{1-2i\lambda_{2k}(s)t} = \frac{(1+2ist)}{(1-2i\lambda_{1k}(s)t)(1-2i\lambda_{2k}(s)t)},$$

and the latter result shows that (C.30) can be rewritten as

$$\text{(iv)} \qquad \left|\frac{1}{2}[h(t, \Delta) + h(-t, \Delta)]\right| \leq e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} \left|(1+2ist)\psi_{k,v}(t, s)\right|,$$

for any $A < 0$, $A < \Delta < \infty$ and all t, where

$$\text{(C.32)} \qquad \psi_{k,v}(t, s) = (1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1}\psi_v(t, s).$$

In order to show that the positive function in the right-hand side of (iv) is integrable, we first investigate the function $\psi_{k,v}(t, s)$ as defined in (C.32).
Since $(1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1}$ is the characteristic function of

$$\lambda_{1k}(s)W_1 + \lambda_{2k}(s)W_2,$$

where $W_1 \sim \chi^2(2)$, $W_2 \sim \chi^2(2)$ and W_1 and W_2 are independent, it follows that $\psi_{k,v}(t, s)$ is the characteristic function of the random variable

$$Q_{k,s} = Q_s + \lambda_{1k}(s)W_1 + \lambda_{2k}(s)W_2$$

where W_1 and W_2 are independent of $V_{1j}(s)$, $V_{2j}(s)$ and $V(s)$.
From these results it can be seen that

$$\text{(C.33)} \qquad Q_{k,s} = \sum_{j=0}^{M} [\lambda_{1j}(s)\tilde{V}_{1j}(s) + \lambda_{2j}(s)\tilde{V}_{2j}(s)] + \lambda(s)\tilde{V}(s),$$

where $\tilde{V}_{10}(s)$, $\tilde{V}_{11}(s)$, ..., $\tilde{V}_{1M}(s)$, $\tilde{V}_{20}(s)$, ..., $\tilde{V}_{2M}(s)$, $\tilde{V}(s)$ are mutually stochastically independent with

$$\tilde{V}(s) \sim \chi^2(\tilde{m})$$

$$\text{(C.34)} \qquad \tilde{V}_{1j}(s) \sim \chi^2(\tilde{m}_{1j}, \tau_{1j}(s))$$

$$\tilde{V}_{2j}(s) \sim \chi^2(\tilde{m}_{2j}, \tau_{2j}(s)),$$

$j = 0, 1, \ldots, M$.

The degrees of freedom are given by

$$\tilde{m} = m$$

$$(C.35)\quad \left.\begin{array}{l} \tilde{m}_{1j} = m_{1j} \\ \tilde{m}_{2j} = m_{2j} \end{array}\right\}, \; j = 0, 1, \ldots, M, \; j \neq k$$

$$\tilde{m}_{1k} = m_{1k} + 2 = m_k + 2$$

$$\tilde{m}_{2k} = m_{2k} + 2 = m_k + 2.$$

The random variable $Q_{k,s}$ is again of the type considered in (C.1) and it can be obtained from Q_s through raplacing m_{1k} and m_{2k} by $m_{1k} + 2 = \tilde{m}_{1k}$ and $m_{2k} + 2 = \tilde{m}_{2k}$, respectively.

If the sum of the degrees of freedom

$$\sum_{j=0}^{M} (\tilde{m}_{1j} + \tilde{m}_{2j}) + \tilde{m} \geq 5,$$

it follows from (B.23) of Appendix B that

$$(C.36)\quad \begin{array}{l} \int_{-\infty}^{\infty} |\psi_{k,v}(t, s)|dt < \infty \\ \int_{-\infty}^{\infty} |t\psi_{k,v}(t, s)|dt < \infty \end{array}$$

Now we have

$$\sum_{j=0}^{M} (\tilde{m}_{1j} + \tilde{m}_{2j}) + \tilde{m} = \sum_{j=0}^{M} (m_{1j} + m_{2j}) + m + 4$$

$$\geq m_{1k} + m_{2k} + 4 = 2m_k + 4 \geq 6,$$

since $m_k \geq 1$, and this shows (C.36).

With the aid of (C.36) we get

(v)
$$\int_0^\infty e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} \left|(1+2ist)\psi_{k,v}(t,\ s)\right|dt =$$

$$e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} \int_{-\infty}^{\infty} I_{(0,\infty)}(t)\ \left|(1+2ist)\psi_{k,v}(t,\ s)\right|dt \leq$$

$$e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} \int_{-\infty}^{\infty} \left|(1+2ist)\psi_{k,v}(t,\ s)\right|dt \leq$$

$$e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} \int_{-\infty}^{\infty} \{|\psi_{k,v}(t,\ s)| + 2s|t\ \psi_{k,v}(t,\ s)|\}dt$$

$$= e^{-\frac{1}{2}\frac{A}{(1-\rho_k)}} \{\int_{-\infty}^{\infty} |\psi_{k,v}(t,\ s)|dt + 2s \int_{-\infty}^{\infty} |t\ \psi_{k,v}(t,\ s)|dt\} < \infty.$$

Finally it is seen from

$$\lim_{\Delta\to 0} \frac{\exp\{it\Delta\ \ell(t)\} - 1}{it\Delta} = \lim_{\Delta\to 0}\left(\frac{it\ \ell(t)\ \exp\{it\Delta\ \ell(t)\}}{it}\right)$$

$$= \ell(t) \lim_{\Delta\to 0} \exp\{it\Delta\ \ell(t)\} = \ell(t) =$$

$$(1+2ist)(1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1},$$

that

(vi)
$$\lim_{\Delta\to 0} \frac{1}{2}\ [h(t,\ \Delta) + h(-t,\ \Delta)] =$$

$$\frac{1}{2}\ [(1+2ist)\psi_{k,v}(t,\ s)e^{-itx} + (1-2ist)\psi_{k,v}(-t,\ s)e^{itx}].$$

The results (iv), (v) and (vi) show that we can apply the dominated convergence theorem of Lebesgue to (C.26) and this yields:

(C.37)
$$\frac{\partial G_v(x,\ s)}{\partial v_{2k}} = -\frac{1}{\pi}\int_0^\infty \frac{1}{2}\ [(1+2ist)e^{-itx}\psi_{k,v}(t,\ s)$$

$$+ (1-2ist)e^{itx}\psi_{k,v}(-t,\ s)]dt.$$

Since (C.36) implies that both integrals
$\int_0^\infty |(1+2ist)e^{-itx}\psi_{k,v}(t, s)|dt$ and $\int_0^\infty |(1-2ist)e^{itx}\psi_{k,v}(-t, s)|dt$ are finite, it is not difficult to verify that (C.37) can be rewritten as

$$\text{(C.38)} \qquad \frac{\partial G_v(x, s)}{\partial v_{2k}} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}(1+2ist)\psi_{k,v}(t, s)dt.$$

Let $G_{k,v}(x, s)$ be the distribution function of the random variable $Q_{k,s}$ and let $g_{k,v}(x, s)$ denote the corresponding probability density function, i.e.,

$$g_{k,v}(x, s) = G'_{k,v}(x, s) = \frac{\partial G_{k,v}(x, s)}{\partial x}.$$

Since $\sum_{j=0}^{M} (\tilde{m}_{1j} + \tilde{m}_{2j}) + \tilde{m} \geq 6$, it follows from (B.24) of Appendix B that

$$g_{k,v}(x, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\psi_{k,v}(t, s)dt$$

$$g'_{k,v}(x, s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -ite^{-itx}\psi_{k,v}(t, s)dt,$$

where

$$g'_{k,v}(x, s) = \frac{\partial g_{k,v}(x, s)}{\partial x}.$$

Substitution of the latter results into (C.38) yields:

$$\text{(C.39)} \qquad \frac{\partial G_v(x, s)}{\partial v_{2k}} = -[g_{k,v}(x, s) - 2sg'_{k,v}(x, s)],$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (0, v_2)$ and any $\lambda(s)$.

If we compare this result to the case $j = 0$, see (C.25), it is seen that $\partial G_v(x, s)/\partial v_{2j}$, $j = 1, 2, \ldots, M$ can no longer be expressed in terms of a probability density function only.

However, under certain conditions on the degrees of freedom parameters m_{10}, m and the coefficient $\lambda(s)$, it can be shown that

$$\frac{\partial G_v(x, s)}{\partial v_{2k}} = -g^*_{k,v}(x, s),$$

where $g^*_{k,v}(x, s)$ is a probability density function. In order to see this, we reconsider (C.38) and define

$$\text{(C.40)} \qquad \psi^*_{k,v}(t, s) = (1+2ist)\psi_{k,v}(t, s)$$

Now suppose that $m_{10} \geq 2$. Since $\lambda_{10}(s) = -s$ and $\psi_{k,v}(t, s) = (1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1}\psi_v(t, s)$, it follows from (C.9) that $\psi_{k,v}(t, s)$ contains the term $(1+2ist)^{-\frac{m_{10}}{2}}$ with $m_{10}/2 \geq 1$. Hence, $\psi^*_{k,v}(t, s)$ is again a characteristic function and it is not difficult to verify that the corresponding random variable, say $Q^*_{k,s}$, can be written as:

$$\text{(C.41)} \qquad Q^*_{k,s} = \sum_{j=0}^{M} [\lambda_{1j}(s)\tilde{V}^*_{1j}(s) + \lambda_{2j}(s)\tilde{V}^*_{2j}(s)] + \lambda(s)\tilde{V}^*(s),$$

where the random variables at the right-hand side are mutually stochastically independent and where

$$\text{(C.42)} \qquad \begin{aligned} &\tilde{V}^*(s) \sim \chi^2(\tilde{m}^*) \\ &\left.\begin{aligned} &\tilde{V}^*_{1j}(s) \sim \chi^2(\tilde{m}^*_{1j}, \tau_{1j}(s)) \\ &\tilde{V}^*_{2j}(\theta) \sim \chi^2(\tilde{m}^*_{2j}, \tau_{2j}(s)) \end{aligned}\right\}, \ j = 0, 1, \ldots, M. \end{aligned}$$

The degrees of freedom are given by

$$\text{(C.43)} \qquad \begin{aligned} &\tilde{m}^* = \tilde{m} \\ &\tilde{m}^*_{10} = \tilde{m}_{10} - 2 \\ &\tilde{m}^*_{2j} = \tilde{m}_{1j}, \ j = 1, 2, \ldots, M \\ &\tilde{m}^*_{2j} = \tilde{m}_{2j}, \ j = 0, 1, \ldots, M, \end{aligned}$$

where $\tilde{m}$, $\tilde{m}_{1j}$ and $\tilde{m}_{2j}$ are as defined in (C.35).

Note that $\tilde{m}^*_{10} = \tilde{m}_{10} - 2 = m_{10} - 2 \geq 0$, since $m_{10} \geq 2$.

With the aid of (C.35) and $m_{10} \geq 2$ it is also seen that

(C.44) $$\sum_{j=0}^{M} (\tilde{m}^*_{1j} + \tilde{m}^*_{2j}) + \tilde{m}^* \geq 6.$$

Since $Q^*_{k,s}$ is a random variable of the type considered in (C.1), it follows from (C.44) and (B.24) of Appendix B that

(C.45) $$g^*_{k,v}(x, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \psi^*_{k,v}(t, s)dt,$$

where $g^*_{k,v}(x, s)$ is the probability density function of $Q^*_{k,s}$.
The substitution of (C.40) into (C.38) gives

$$\frac{\partial G_v(x, s)}{\partial v_{2k}} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \psi^*_{k,v}(t, s)dt$$

and it follows from (C.45) that, under the condition $m_{10} \geq 2$, we have

(C.46) $$\frac{\partial G_v(x, s)}{\partial v_{2k}} = -g^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (0, v_2)$ and any $\lambda(s)$.
Next we consider the situation where $m \geq 2$ and $\lambda(s) = -s$ for all $s > 0$.
Since $\lambda_{10}(s) = -s = \lambda(s)$, it follows that (C.9) becomes

(C.47) $$\psi_v(t, s) = [\prod_{j=0}^{M} (1-2i\lambda_{1j}(s)t)^{-\frac{\hat{m}_{1j}}{2}} (1-2i\lambda_{2j}(s)t)^{-\frac{\hat{m}_{2j}}{2}}]$$

$$\exp\{it \sum_{j=0}^{M} [\frac{\lambda_{1j}(s)\tau_{1j}(s)}{1-2i\lambda_{1j}(s)t} + \frac{\lambda_{2j}(s)\tau_{2j}(s)}{1-2i\lambda_{2j}(s)t}]\},$$

where

$$\hat{m}_{10} = m_{10} + m \geq 2$$

(C.48) $$\hat{m}_{1j} = m_{1j},\ j = 1, 2, \ldots, M$$

$$\hat{m}_{2j} = m_{2j},\ j = 0, 1, \ldots, M.$$

That is, Q_s can be written as

$$(C.49)\qquad Q_s = \sum_{j=0}^{M} [\lambda_{1j}(s)V_{1j}(s) + \lambda_{2j}(s)V_{2j}(s)],$$

where the random variables at the right-hand side are mutually independent and where

$$(C.50)\qquad \begin{aligned} V_{1j}(s) &\sim \chi^2(\hat{m}_{1j}, \tau_{1j}(s)) \\ V_{2j}(s) &\sim \chi^2(\hat{m}_{2j}, \tau_{2j}(s)), \end{aligned}$$

$j = 0, 1, \ldots, M$.

It is seen that Q_s in (C.49) can be obtained from (C.1) through replacing $\lambda(s)$ by 0 and m_{10} by $m_{10} + m = \hat{m}_{10} \geq 2$. Since the result (C.46) holds true for any $\lambda(s)$, it follows that

$$(C.51)\qquad \frac{\partial G_v(x, s)}{\partial v_{2k}} = -g^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$ and all $v = (0, v_2)$, where $g^*_{k,v}(x, s)$ is the probability density function corresponding to the characteristic function (see (C.40)):

$$\psi^*_{k,v}(t, s) = (1+2ist)\ \psi_{k,v}(t, s) =$$

$$(1+2ist)(1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1}\psi_v(t, s).$$

This completes the situation (I).

We proceed with case (II), i.e., we suppose that $v_{1j} \geq 0$, $v_{2j} = 0$, $j = 0, 1, \ldots, M$ and we shall derive $\partial G_v(x, s)/\partial v_{1j}$.

It is seen from (C.4) that $\tau_{1j}(s) = d_{1j}(s)v_j$ and $\tau_{1j}(s) = d_{2j}(s)v_{1j}$, which shows that $\psi_v(t, s)$ takes the form

$$(C.52)\qquad \psi_v(t, s) = [\sum_{j=0}^{M} (1-2i\lambda_{1j}(s)t)^{-\frac{m_{1j}}{2}}(1-2i\lambda_{2j}(s)t)^{-\frac{m_{2j}}{2}}](1-2i\lambda(s)t)^{-\frac{m}{2}}$$

$$\exp\{it \sum_{j=0}^{M} [\frac{\lambda_{1j}(s)d_{1j}(s)v_{1j}}{1-2i\lambda_{1j}(s)t} + \frac{\lambda_{2j}(s)d_{2j}(s)v_{1j}}{1-2i\lambda_{2j}(s)t}]\}$$

Now we take $\tilde{v} = (\tilde{v}_1, 0)$ with $\tilde{v}_1 = (v_{10}, v_{11}, \ldots, v_{1j} + \Delta, \ldots, v_{1M})$ and it follows from (C.10) that

$$\text{(C.53)} \quad \frac{\partial G_v(x,\ s)}{\partial v_{1j}} = \lim_{\Delta \to 0} \frac{G_{\tilde{v}}(x,\ s) - G_v(x,\ s)}{\Delta} =$$

$$\lim_{\Delta \to 0} -\frac{1}{\pi} \int_0^\infty \frac{\mathrm{Im}(\psi_{\tilde{v}}(t,\ s)e^{-itx}) - \mathrm{Im}(\psi_v(t,\ s)e^{-itx})}{t\Delta} dt,$$

for $j = 0, 1, \ldots, M$.

By using similar arguments as before it can be shown that

$$\text{(C.54)} \quad \frac{\partial G_v(x,\ s)}{\partial v_{10}} = \frac{s}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \psi_{0,v}(t,\ s)dt,$$

where

$$\text{(C.55)} \quad \psi_{0,v}(t,\ s) = (1+2ist)^{-1}\ \psi_v(t,\ s)$$

is a characteristic function which satisfies

$$\int_{-\infty}^{\infty} |\psi_{0,v}(t,\ s)|dt < \infty.$$

Hence, if $h_{0,v}(x,\ s)$ is the corresponding probability density function, we have

$$\text{(C.56)} \quad h_{0,v}(x,\ s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \psi_{0,v}(t,\ s)dt.$$

Substitution of (C.56) into (C.54) yields

$$\text{(C.57)} \quad \frac{\partial G_v(x,\ s)}{\partial v_{10}} = s\ h_{0,v}(x,\ s),$$

for $-\infty < x < \infty$, all $s > 0$, all $v = (v_1,\ 0)$ and any $\lambda(s)$.

When $j = k = 1, 2, \ldots, M$ we get

$$\text{(C.58)} \quad \frac{\partial G_v(x,\ s)}{\partial v_{1k}} = \frac{s}{2\pi} \int_{-\infty}^{\infty} e^{-itx}(1-2it)\psi_{k,v}(t,\ s)dt,$$

where

$$\text{(C.59)} \quad \psi_{k,v}(t,\ s) = (1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1}\ \psi_v(t,\ s)$$

is a characteristic function satisfying

(C.60)
$$\int_{-\infty}^{\infty} |\psi_{k,v}(t, s)|dt < \infty$$
$$\int_{-\infty}^{\infty} |t\ \psi_{k,v}(t, s)|dt < \infty.$$

If $h_{k,v}(x, s)$ denotes the probability density function corresponding to $\psi_{k,v}(t, s)$ and if

$$h'_{k,v}(x, s) = \frac{\partial h_{k,v}(x, s)}{\partial x},$$

it follows from (C.60) that

(C.61)
$$h_{k,v}(x, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\psi_{k,v}(t, s)dt$$
$$h'_{k,v}(x, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -ite^{-itx}\psi_{k,v}(t, s)dt.$$

Substitution of (C.61) into (C.58) gives

$$\text{(C.62)} \qquad \frac{\partial G_v(x, s)}{\partial v_{1k}} = s[h_{k,v}(x, s) + 2h'_{k,v}(x, s)],$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (v_1, 0)$ and any $\lambda(s)$.

When the function $\psi^*_{k,v}(t, s)$ is defined by

$$\text{(C.63)} \qquad \psi^*_{k,v}(t, s) = (1-2it)\psi_{k,v}(t, s),$$

it can be shown in a similar manner as before that, under the condition $m_{20} \geq 2$, $\psi^*_{k,v}(t, s)$ is again a characteristic function with the property that

$$\text{(C.64)} \qquad h^*_{k,v}(x, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\psi^*_{k,v}(t, s)dt,$$

where $h^*_{k,v}(x, s)$ is the probability density function corresponding to $\psi^*_{k,v}(t, s)$.

Therefore, if $m_{20} \geq 2$, it is seen from (C.63) and (C.64) that (C.58) becomes

$$\text{(C.65)} \qquad \frac{\partial G_v(x, s)}{\partial v_{1k}} = sh^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (v_1, 0)$ and any $\lambda(s)$.

Finally we consider the case where $m \geq 2$ and $\lambda(s) = 1$ for all $s > 0$. Since $\lambda_{20}(s) = 1 = \lambda(s)$, it follows that $\psi_v(t, s)$ in (C.9) can be written as:

$$\text{(C.66)} \qquad \psi_v(t, s) = [\prod_{j=0}^{M} (1-2i\lambda_{1j}(s)t)^{-\frac{\hat{m}_{1j}}{2}} (1-2i\lambda_{2j}(s)t)^{-\frac{\hat{m}_{2j}}{2}}]$$

$$\exp\{it \sum_{j=0}^{M} [\frac{\lambda_{1j}(s)\tau_{1j}(s)}{1-2i\lambda_{1j}(s)t} + \frac{\lambda_{2j}(s)\tau_{2j}(s)}{1-2i\lambda_{2j}(s)t}]\},$$

where

$$\hat{m}_{1j} = m_{1j}, \; j = 0, 1, \ldots, M$$

$$\text{(C.67)} \qquad \hat{m}_{20} = m_{20} + m \geq 2$$

$$\hat{m}_{2j} = m_{2j}, \; j = 1, 2, \ldots, M$$

This means that Q_s takes the form

$$\text{(C.68)} \qquad Q_s = \sum_{j=0}^{M} [\lambda_{1j}(s)V_{1j}(s) + \lambda_{2j}(s)V_{2j}(s)],$$

where the random variables at the right-hand side are mutually independent and where

$$\text{(C.69)} \qquad \begin{aligned} V_{1j}(s) &\sim \chi^2(\hat{m}_{1j}, \tau_{1j}(s)) \\ V_{2j}(s) &\sim \chi^2(\hat{m}_{2j}, \tau_{2j}(s)), \end{aligned}$$

$j = 0, 1, \ldots, M$.

That is, Q_s in (C.68) can be obtained from (C.1) through replacing $\lambda(s)$ by 0 and m_{20} by $m_{20} + m = \hat{m}_{20} \geq 2$. Since (C.65) holds true for any $\lambda(s)$ it follows that, under the condition $m \geq 2$ and $\lambda(s) = 1$ for all $s > 0$, we have

$$\text{(C.70)} \qquad \frac{\partial G_v(x, s)}{\partial v_{1k}} = s\, h^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$ and all $v = (v_1, 0)$, where $h^*_{k,v}(x, s)$ is the probability density function corresponding to the characteristic function $\psi^*_{k,v}(t, s)$ as defined in (C.63). This completes the case (II).

Summarizing the above results, if Q_s is defined by (C.1) - (C.5), then the characteristic function $\psi_v(t, s)$ of Q_s is given by (C.9) and the distribution function $G_v(x, s)$ can be found through (C.10), where $v = (v_1, v_2)$ with $v_1 = (v_{10}, v_{11}, \ldots, v_{1M})$ and $v_2 = (v_{20}, v_{21}, \ldots, v_{2M})$. The partial derivatives of $G_v(x, s)$ with respect to the parameters v_{1j} and v_{2j} under the hypotheses (I) and (II), respectively, are given by:

(I) $\qquad v_{1j} = 0,\ v_{2j} \geq 0,\ j = 0, 1, \ldots, M.$

$$\text{(a)} \qquad \frac{\partial G_v(x, s)}{\partial v_{20}} = -\, g_{0,v}(x, s),$$

for $-\infty < x < \infty$, all $s > 0$, all $v = (0, v_2)$ and any $\lambda(s)$, where $g_{0,v}(x, s)$ is the probability density function corresponding to the characteristic function

$$\psi_{0,v}(t, s) = (1-2it)^{-1}\, \psi_v(t, s).$$

$$\text{(b)} \qquad \frac{\partial G_v(x, s)}{\partial v_{2k}} = -\, [g_{k,v}(x, s) - 2s g'_{k,v}(x, s)],$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (0, v_2)$ and any $\lambda(s)$, where $g_{k,v}(x, s)$ is the probability density function corresponding to the characteristic function

$$\psi_{k,v}(t, s) = (1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1}\ \psi_v(t, s),$$

and where

$$g'_{k,v}(x, s) = \frac{\partial g_{k,v}(x, s)}{\partial x}.$$

(c) If $m_{10} \geq 2$, the function

$$\psi^*_{k,v}(t, s) = (1+2ist)\psi_{k,v}(t, s)$$

is again a characteristic function and

$$\frac{\partial G_v(x, s)}{\partial v_{2k}} = -\ g^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (0, v_2)$ and any $\lambda(s)$, where $g^*_{k,v}(x, s)$ is the probability density function corresponding to $\psi^*_{k,v}(t, s)$.

(d) If $m \geq 2$ and $\lambda(s) = -s$ for all $s > 0$,

$$\psi^*_{k,v}(t, s) = (1+2ist)\psi_{k,v}(t, s)$$

is a characteristic function and

$$\frac{\partial G_v(x, s)}{\partial v_{2k}} = -\ g^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$ and all $v = (0, v_2)$, where again $g^*_{k,v}(x, s)$ is the probability density function corresponding to $\psi^*_{k,v}(t, s)$.

(II) $\quad v_{1j} \geq 0,\ v_{2j} = 0,\ j = 0, 1, \ldots, M.$

(a) $$\frac{\partial G_v(x, s)}{\partial v_{10}} = s\ h_{0,v}(x, s),$$

for $-\infty < x < \infty$, all $s > 0$, all $v = (v_1, 0)$ and any $\lambda(s)$, where $h_{0,v}(x, s)$ is the probability density function corresponding to the characteristic function

$$\psi_{0,v}(t, s) = (1+2ist)^{-1} \psi_v(t, s)$$

(b) $$\frac{\partial G_v(x, s)}{\partial v_{1k}} = s[h_{k,v}(x, s) + 2h'_{k,v}(x, s)].$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (v_1, 0)$ and any $\lambda(s)$, where $h_{k,v}(x, s)$ is the probability density function corresponding to the characteristic function

$$\psi_{k,v}(t, s) = (1-2i\lambda_{1k}(s)t)^{-1}(1-2i\lambda_{2k}(s)t)^{-1} \psi_v(t, s),$$

and where

$$h'_{k,v}(x, s) = \frac{\partial h_{k,v}(x,s)}{\partial x}.$$

(c) If $m_{20} \geq 2$, the function

$$\psi^*_{k,v}(t, s) = (1-2it)\psi_{k,v}(t, s)$$

is again a characteristic function and

$$\frac{\partial G_v(x, s)}{\partial v_{1k}} = s\, h^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (v_1, 0)$ and any $\lambda(s)$, where $h^*_{k,v}(x, s)$ is the probability density function corresponding to $\psi^*_{k,v}(t, s)$.

(d) If $m \geq 2$ and $\lambda(s) = 1$ for all $s > 0$,

$$\psi^*_{k,v}(t, s) = (1-2it)\psi_{k,v}(t, s)$$

is a characteristic function and

$$\frac{\partial G_v(x, s)}{\partial v_{1k}} = s\, h^*_{k,v}(x, s),$$

for $k = 1, 2, \ldots, M$, $-\infty < x < \infty$, all $s > 0$, all $v = (v_1, 0)$, where again $h^*_{k,v}(x, s)$ is the probability density function corresponding to $\psi^*_{k,v}(t, s)$.

Appendix D

A large sample approximation to the distribution function of the test statistics

In this appendix we shall derive a large sample approximation to the distribution function of the test statistics

$$S_1^* = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X}\,\frac{n-k}{n-\ell}$$

(D.1) $$S_2^* = \frac{\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G}{\hat{u}_X'\hat{u}_X}\,\frac{n-k}{k-p}$$

$$S_3^* = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X - \hat{u}_G'\hat{u}_G}\,\frac{\ell-p}{n-\ell},$$

where

$$\hat{u}_X = y - X\hat{\beta}$$

(D.2) $$\hat{u}_Z = y - Z\hat{\gamma}$$

$$\hat{u}_G = y - G\hat{\delta},$$

with

$$\hat{\beta} = (X'X)^{-1}X'y$$

(D.3) $$\hat{\gamma} = (Z'Z)^{-1}Z'y$$

$$\hat{\delta} = (G'G)^{-1}G'y.$$

Here y is a n-dimensional random vector having a $n(\mu, \sigma^2 I)$ distribution. X and Z are given (nonstochastic) matrices, X is of the order $n\times k$ with rank k and Z is of the order $n\times\ell$ with rank ℓ. Further, p is defined by $p = \dim(M(X) \cap M(Z))$, where in general, $M(A)$ denotes the linear (vector-) subspace (of $\mathbb{R}^n$) spanned by the columnvectors of the $n\times m$ matrix A. The matrix G is nonstochastic and of the order $n\times(k+\ell-p)$ with rank $k+\ell-p$.

The columnvectors of G form a basis for the (k+ℓ-p)-dimensional linear subspace $M(X) + M(Z)$. We shall derive approximations to the distributions of the statistics in (D.1) for large n (the sample size) and under the assumption that the parameters (μ, σ) satisfy

(D.4) $$(\mu, \sigma) \in \{(\mu, \sigma) | \mu \in M(X) \cap M(Z), \sigma > 0\}.$$

Throughout this appendix we also assume that $p = \dim(M(X) \cap M(Z))$ is independent of n. It should be noted that $P(S_i^* \leq 0) = 0$, i = 1, 2, 3. We start with the variable S_1^*. Now S_1^* can be written as:

(D.5) $$S_1^* = S_1 \frac{n-k}{n-\ell} ,$$

where

$$S_1 = \frac{\hat{u}_Z'\hat{u}_Z}{\hat{u}_X'\hat{u}_X} .$$

It is easily seen that the event $S_1 \leq s$ is equivalent to $Q_1(s) = \hat{u}_Z'\hat{u}_Z/\sigma^2 - s\hat{u}_X'\hat{u}_X/\sigma^2 \leq 0$.
As is shown in Section 4 and Section 8 the random variable $Q_1(s)$ can be written as a linear combination of mutually stochastically independent χ^2 random variables. Since the coefficients of this linear combination and the degrees of freedom of the χ^2 variables depend on the eigenvalues (and corresponding multiplicities of these eigenvalues) of the k×k matrix $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$, we first consider the latter matrix. It can be shown that this matrix has an eigenvalue 1 with multiplicity p. Now suppose that $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$ has M different eigenvalues ρ_1, ρ_2, ..., ρ_M with $0 < \rho_j < 1$, j = 1, 2, ..., M and multiplicities m_1, m_2, ..., m_M. If $r = p + \sum_{j=1}^{M} m_j$, it is not difficult to see that $r = \text{rank}(X'Z)$ and that $0 \leq p \leq r \leq \min(k, \ell)$. We also have

$$tr = tr[(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X] = p + \sum_{j=1}^{M} \rho_j m_j .$$

It should be observed that the matrices $(X'X)^{-1}X'Z(Z'Z)^{-1}Z'X$ and $(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z$ have the same nonzero eigenvalues.

By using the results of Section 4 and Section 8 and assuming that μ and σ satisfy (D.4), we get for $s > 0$:

$$Q_1(s) = \sum_{j=0}^{M} [\lambda_{1j}(s)V_{1j} + \lambda_{2j}(s)V_{2j}] + (1-s)V, \tag{D.6}$$

where the random variables $V_{10}, V_{11}, \ldots, V_{1M}, V_{20}, \ldots, V_{2M}, V$ are mutually independent with

$$\begin{aligned} &V \sim \chi^2(m) \\ &V_{1j} \sim \chi^2(m_{1j}) \\ &V_{2j} \sim \chi^2(m_{2j}), \end{aligned} \tag{D.7}$$

$j = 0, 1, \ldots, M$ and where

$$\begin{aligned} \lambda_{1j}(s) &= \frac{1}{2}(1-s) - \frac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)} \\ \lambda_{2j}(s) &= \frac{1}{2}(1-s) + \frac{1}{2}\sqrt{(1-s)^2 + 4s(1-\rho_j)}, \end{aligned} \tag{D.8}$$

$j = 0, 1, \ldots, M$.
The coefficient ρ_0 in (D.8) is defined by $\rho_0 = 0$.
The degrees of freedom in (D.7) are given by

$$\begin{aligned} &m = n+p-k-\ell \\ &m_{10} = \ell-r \\ &m_{20} = k-r \\ &m_{1j} = m_{2j} = m_j, \; j = 1, 2, \ldots, M \end{aligned} \tag{D.9}$$

Note that $m + \sum_{j=0}^{M} (m_{1j} + m_{2j}) = (n+p-k-\ell) + (\ell-r) + (k-r) + 2\sum_{j=1}^{M} m_j = n+p-2r + 2(r-p) = n-p$, since $r = p + \sum_{j=1}^{M} m_j$.
It is not difficult to verify that the coefficients in (D.8) satisfy:

$$\begin{aligned} &\lambda_{1j}(s) + \lambda_{2j}(s) = 1-s \\ &\lambda_{1j}(s)\lambda_{2j}(s) = -s(1-\rho_j), \end{aligned} \tag{D.10}$$

for $j = 0, 1, \ldots, M$, and that

$$\begin{aligned} &\lambda_{10}(s) = -s, \ \lambda_{20}(s) = 1 \\ &-s < \lambda_{1j}(s) < 0, \quad 1 < s \leq 1 \\ \text{(D.11)} \quad &-s < \lambda_{1j}(s) < 1-s, \ 1 \leq s < \infty \\ &1-s < \lambda_{2j}(s) < 1, \ 0 < s \leq 1 \\ &0 < \lambda_{2j}(s) < 1, \quad 1 \leq s < \infty, \end{aligned}$$

for $j = 1, 2, \ldots, M$.

Now we define the coefficients $\tau_1(s), \tau_2(s), \ldots, \tau_{n-p}(s)$ as follows: the first m_{10} coefficients $\tau_i(s)$ are equal to $\lambda_{10}(s)$, the second m_{11} coefficients $\tau_i(s)$ are equal to $\lambda_{11}(s), \ldots$, and the last m coefficients $\tau_i(s)$ are equal to 1-s. Then the random variable $Q_1(s)$ in (D.6) can be rewritten as

$$\text{(D.12)} \quad Q_1(s) = \sum_{i=1}^{n-p} \tau_i(s) U_i,$$

where $U_1, U_2, \ldots, U_{n-p}$ are mutually independent random variables with $U_i \sim \chi^2(1)$ for $j = 1, 2, \ldots, n-p$.

Next we return to the statistic S_1^* and consider the event $S_1^* \leq s$. Since $S_1 \leq s$ is equivalent to $Q_1(s) \leq 0$, it follows from (D.5) that the event $S_1^* \leq s$ is equivalent to

$$Q_1(\frac{n-\ell}{n-k} s) = \hat{u}_Z'\hat{u}_Z/\sigma^2 - (\frac{n-\ell}{n-k}) s\hat{u}_X'\hat{u}_X/\sigma^2 \leq 0.$$

If we define $W_1(s)$ by

$$W_1(s) = Q_1(\frac{n-\ell}{n-k} s),$$

it is seen from (D.12) that

$$\text{(D.13)} \quad W_1(s) = \sum_{i=1}^{n-p} \alpha_i(s) U_i,$$

where

$$\alpha_i(s) = \tau_i\left(\frac{n-\ell}{n-k}\, s\right), \tag{D.14}$$

$i = 1, 2, \ldots, n-p$.

This shows that the event $S_1^* \leq s$ is equivalent to $W_1(s) \leq 0$ with $W_1(s)$ as given in (D.13) and that

$$P(S_1^* \leq s) = P(W_1(s) \leq 0). \tag{D.15}$$

Hence, if we can find an approximation to the distribution function of $W_1(s)$ for large n, we can use this result to approximate the probability $P(S_1^* \leq s)$.

Now we shall show that when $s \neq 1$, $\mu_1(s) = E(W_1(s))$ and $\sigma_1^2(s) = Var(W_1(s))$, we have

$$\frac{W_1(s) - \mu_1(s)}{\sigma_1(s)} \overset{F}{\rightarrow} n(0,\ 1) \text{ if } n\rightarrow\infty,$$

where $\overset{F}{\rightarrow}$ denotes convergence in distribution.

We first compute $\mu_1(s)$, $\sigma_1^2(s)$ and the coefficient of skewness $\gamma_1(s)$, which is defined by

$$\gamma_1(s) = \frac{E[(W_1(s) - \mu_1(s))^3]}{[\sigma_1(s)]^3}.$$

The coefficient $\gamma_1(s)$ is a measure of asymmetry of the distribution of $W_1(s)$ or $(W_1(s) - \mu_1(s))/\sigma_1(s)$.

Since $U_1, U_2, \ldots, U_{n-p}$ are mutually independent and

$$E(U_i) = 1$$

$$Var(U_i) = 2$$

$$E[(U_i - 1)^3] = 8,$$

it follows from (D.13) that

$$\mu_1(s) = \sum_{i=1}^{n-p} \alpha_i(s)$$

(D.16) $$\sigma_1^2(s) = 2 \sum_{i=1}^{n-p} [\alpha_i(s)]^2$$

$$E[(W_1(s) - \mu_1(s))^3] = 8 \sum_{i=1}^{n-p} [\alpha_i(s)]^3.$$

By making use of (D.14), the definition of $\tau_i(s)$, (D.10) and (D.9) we obtain:

$$\mu_1(s) = - (n-\ell)(s-1)$$

(D.17) $$\sigma_1^2(s) = 2(n-\ell)\, \beta_n(s)$$

$$\gamma_1(s) = \sqrt{\frac{8}{n-\ell}}\, \frac{\delta_n(s)}{[\beta_n(s)]^{3/2}},$$

where

(D.18) $$\beta_n(s) = (\frac{n-\ell}{n-k})s^2 - 2(\frac{n-k-\ell+tr}{n-k})s + 1$$

$$\delta_n(s) = - (\frac{n-\ell}{n-k})^2 s^3 + 3\, \frac{(n-\ell)(n-k-\ell+tr)}{(n-k)^2} s^2 - 3(\frac{n-k-\ell+tr}{n-k})s + 1.$$

Note that, since $0 \leq tr \leq \min(k, \ell)$, we have

(D.19) $$\lim_{n\to\infty} \beta_n(s) = (1-s)^2$$

$$\lim_{n\to\infty} \delta_n(s) = (1-s)^3,$$

which shows that for $s \neq 1$ we have

$$\lim_{n\to\infty} \frac{\delta_n(s)}{[\beta_n(s)]^{3/2}} = (\frac{1-s}{|1-s|})^3 = \begin{cases} +1 \text{ if } 0 < s < 1 \\ -1 \text{ if } 1 < s < \infty. \end{cases}$$

Hence, it follows from (D.17) that for $s \neq 1$

(D.20) $$\lim_{n\to\infty} \gamma_1(s) = 0.$$

In order to derive the limiting distribution of $(W_1(s) - \mu_1(s))/\sigma_1(s)$ we use the following theorem, due to Liapounov, see Cramér [3].

Let $X_1, X_2, \ldots, X_n$ be mutually stochastically independent random variables with $\mu_i = E(X_i)$ and $\sigma_i^2 = Var(X_i)$.
If $E(|X_i - \mu_i|^3) < \infty$ for all i and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} E(|X_i - \mu_i|^3)}{(\sum_{i=1}^{n} \sigma_i^2)^{3/2}} = 0,$$

then

$$\frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \xrightarrow{F} n(0,\ 1) \text{ if } n\to\infty.$$

Now suppose that $s > 0$ is fixed and $s \neq 1$.
Let X_i be defined by $X_i = \alpha_i(s)U_i$ for $i = 1, 2, \ldots, n-p$.
Then $X_1, X_2, \ldots, X_{n-p}$ are mutually independent and $\mu_i = E(X_i) = \alpha_i(s)$, $\sigma_i^2 = Var(X_i) = 2[\alpha_i(s)]^2$ for $i = 1, 2, \ldots, n-p$.
Further it follows from (D.13) and (D.16) that

$$W_1(s) = \sum_{i=1}^{n-p} X_i$$

(D.21) $$\mu_1(s) = \sum_{i=1}^{n-p} \mu_i$$

$$\sigma_1^2(s) = \sum_{i=1}^{n-p} \sigma_i^2.$$

The condition $E(|X_i - \mu_i|^3) < \infty$ becomes $|\alpha_i(s)|^3\ E(|U_i - 1|^3) < \infty$. Since $U_i \sim \chi^2(1)$, we have $E(|U_i - 1|^3) = E(|U_i^3 - 3U_i^2 + 3U_i - 1|)$ $\leq E(U_i^3 + 3U_i^2 + 3U_i + 1) = 28$, for all i.
From (D.11) it is seen that $|\lambda_{1j}(s)| \leq s$ and $|\lambda_{2j}(s)| \leq 1$ for $s > 0$ and $j = 0, 1, \ldots, M$. Since $|1-s| \leq \max(1, s)$ it follows that $|\tau_i(s)| \leq \max(1, s)$.
According to (D.14) we get

(D.22) $$|\alpha_i(s)| \leq \max(1, \frac{n-\ell}{n-k}s).$$

When $\ell \geq k$ it follows that $\frac{n-\ell}{n-k}s \leq s$ and $|\alpha_i(s)| \leq \max(1, s)$. If $\ell \leq k$, we have

$$\frac{n-\ell}{n-k} = \frac{n-k+k-\ell}{n-k} = 1 + \frac{k-\ell}{n-k} \leq 1 + \frac{k-\ell}{\ell-p+1},$$

since $n-k \geq \ell-p+1$ ($m = n+p-k-\ell \geq 1$).
This shows that for $\ell \leq k$

$$\frac{n-\ell}{n-k}s \leq \frac{k-p+1}{\ell-p+1}s.$$

Hence, when $\ell \leq k$

$$|\alpha_i(s)| \leq \max(1, \frac{k-p+1}{\ell-p+1}s),$$

and we always have

(D.23) $$|\alpha_i(s)| \leq \max(1, s, \frac{k-p+1}{\ell-p+1}s) \overset{\text{def.}}{=} g(s).$$

Note that $g(s)$ does not depend on n.
Together with $E(|U_i - 1|^3) \leq 28$ the result (D.23) yields

(D.24) $$E(|X_i - \mu_i|^3) \leq 28[g(s)]^3 < \infty$$

for all i.
We proceed by considering

$$\frac{\sum_{i=1}^{n-p} E(|X_i - \mu_i|^3)}{(\sum_{i=1}^{n-p} \sigma_i^2)^{3/2}}.$$

By making use of $\sum_{i=1}^{n-p} \sigma_i^2 = \sigma_1^2(s) = 2(n-\ell)\ \beta_n(s)$ (see (D.17) and (D.18)) and (D.24) we get:

$$0 < \frac{\sum_{i=1}^{n-p} E(|X_i - \mu_i|^3)}{(\sum_{i=1}^{n-p} \sigma_i^2)^{3/2}} \leq \frac{(n-p)\ 28[g(s)]^3}{[2(n-\ell)\ \beta_n(s)]^{3/2}}$$

$$= \frac{n-p}{(n-\ell)^{3/2}} \frac{7\sqrt{2}\ [g(s)]^3}{[\beta_n(s)]^{3/2}} = \frac{n-p}{(n-\ell)^{3/2}} B_n(s),$$

where

$$B_n(s) = \frac{7\sqrt{2}\ [g(s)]^3}{[\beta_n(s)]^{3/2}} .$$

If $s \neq 1$, we know that $\lim_{n\to\infty} \beta_n(s) = (1-s)^2 > 0$ and this shows that

$$\lim_{n\to\infty} B_n(s) = 7\sqrt{2}\ [\frac{g(s)}{|1-s|}]^3 < \infty .$$

Since

$$\lim_{n\to\infty} \frac{n-p}{(n-\ell)^{3/2}} = 0,$$

it follows that

$$\lim_{n\to\infty} \frac{n-p}{(n-\ell)^{3/2}} B_n(s) = 0$$

and this implies that

(D.25) $$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-p} E(|X_i - \mu_i|^3)}{(\sum_{i=1}^{n-p} \sigma_i^2)^{3/2}} = 0,$$

when $s \neq 1$.

The conditions of Liapounov's theorem are satisfied and we have shown that for any $s > 0$, $s \neq 1$:

(D.26) $$\frac{W_1(s) - \mu_1(s)}{\sigma_1(s)} = \frac{\sum_{i=1}^{n-p} X_i - \sum_{i=1}^{n-p} \mu_i}{\sqrt{\sum_{i=1}^{n-p} \sigma_i^2}} \xrightarrow{F} n(0,\ 1)$$

if $n \to \infty$.

The result enables us to approximate $P(S_1^* \leq s)$ for large n, s > 0 and $s \neq 1$.

From (D.15) we get:

$$P(S_1^* \leq s) = P(W_1(s) \leq 0) = P[\frac{W_1(s) - \mu_1(s)}{\sigma_1(s)} \leq \frac{-\mu_1(s)}{\sigma_1(s)}]$$

and by making use of (D.26) it follows that

$$\text{(D.27)} \qquad P(S_1^* \leq s) \approx \Phi[\frac{-\mu_1(s)}{\sigma_1(s)}],$$

when n is large, where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

Finally, substitution of $\mu_1(s)$ and $\sigma_1(s)$ from (D.17) into the right-hand side of (D.27) yields for $s > 0$, $s \neq 1$

$$\text{(D.28)} \qquad P(S_1^* \leq s) \approx \Phi[\frac{\sqrt{n-\ell}\,(s-1)}{\sqrt{2\,\beta_n(s)}}],$$

when n is large, where $\beta_n(s)$ is as given in (D.18).

Note that, since $\beta_n \to (1-s)^2$ if $n \to \infty$, we have

$$\Phi[\frac{\sqrt{n-\ell}\,(s-1)}{\sqrt{2\,\beta_n(s)}}] \to \begin{cases} 0 \text{ if } s < 1 \\ 1 \text{ if } s > 1, \end{cases}$$

if $n \to \infty$. This result agrees with the fact that the random variable S_1^* converges in probability (and therefore also in distribution) to the constant 1 if $n \to \infty$.

Now it remains to consider the case where s = 1.

From (D.17) and (D.18) we can compute the mean, variance and coefficient of skewness of $W_1(1)$, we get:

$$\mu_1(1) = 0$$

$$\text{(D.29)} \qquad \sigma_1^2(1) = 2(\frac{n-\ell}{n-k})(k+\ell-2tr)$$

$$\gamma_1(1) = \sqrt{8}\,\frac{(k-\ell)}{(k-\ell-2tr)^{3/2}}\,[\frac{(n-k-\ell+tr) - (k+\ell-2tr)}{\sqrt{(n-k)(n-\ell)}}].$$

Suppose first that $k \neq \ell$, then since $0 \leq tr \leq \min(k, \ell)$, it follows that

$$\lim_{n\to\infty} \left[\frac{(n-k-\ell+tr) - (k+\ell-2tr)}{\sqrt{(n-k)(n-\ell)}}\right] = 1,$$

which shows that the coefficient of skewness $\gamma_1(1)$ does not converge to 0 if $n\to\infty$.

Therefore, if $(W_1(1) - \mu_1(1))/\sigma_1(1) = W_1(1)/\sigma_1(1)$ has a limiting distribution, this limiting distribution is not equal to a $n(0, 1)$ distribution.

In the second place, suppose that $k = \ell$, then we have

$$\mu_1(1) = 0$$

$$\sigma_1^2(1) = 4(k-tr)$$

$$\gamma_1(1) = 0$$

Although now $\gamma_1(1) = 0$ for all n, it can again be shown that $W_1(1)/\sigma_1(1)$ does not have a limiting $n(0, 1)$ distribution.

In order to see this, consider (D.13) for $s = 1$ and $k = \ell$, we get:

$$W_1(1) = \sum_{i=1}^{n-p} \alpha_i(1)U_i = \sum_{i=1}^{n-p} \tau_i(1)U_i$$

$$= \sum_{i=1}^{2(k-p)} \tau_i(1)U_i + \sum_{i=2(k-p)+1}^{n-p} \tau_i(1)U_i,$$

where use has been made of (D.14).

From the fact that $\tau_i(s) = 1-s$ for $i = 2(k-p)+1, \ldots, n-p$ it follows that $\tau_i(1) = 0$ for $i = 2(k-p)+1, \ldots, n-p$ and this shows that

$$W_1(1) = \sum_{i=1}^{2(k-p)} \tau_i(1)U_i,$$

where $U_1, U_2, \ldots, U_{2(k-p)}$ are mutually independent $\chi^2(1)$ variables and where only the $2(k-p)$ coefficients $\tau_i(1)$ depend on n through ρ_j and $\lambda_{1j}(1)$, $\lambda_{2j}(1)$ $j = 0, 1, \ldots, M$, see (D.8). Note that $\lambda_{1j}(1) = -\sqrt{1-\rho_j}$ and $\lambda_{2j}(1) = +\sqrt{1-\rho_j}$ for $j = 0, 1, \ldots, M$.

Hence, when s = 1 and k = ℓ, we have

$$\frac{W_1(1) - \mu_1(1)}{\sigma_1(1)} = \frac{\sum_{i=1}^{2(k-p)} \tau_i(1)U_i}{2\sqrt{k-tr}} ,$$

which shows that if $(W_1(1) - \mu_1(1))/\sigma_1(1)$ has a limiting distribution, this limiting distribution is not a n(0, 1) distribution.
Since the limiting distribution in the case s = 1 (if it exists) is not of a simple form, we shall not consider this case any further.
It should be noted that when k = ℓ there is no need to approximate $P(S_1^* \leq 1)$. This follows from the fact that for k = ℓ, the probability distribution of $W_1(1)$ is symmetric around 0. That is, we have for any n

$$P(S_1^* \leq 1) = P(W_1(1) \leq 0) = \frac{1}{2} .$$

The result (D.28) also enables us to approximate the value of s_α which satisfies

$$P(S_1^* \leq s_\alpha) = \alpha,$$

where α is a preassigned probability.
For large n this value can be approximated by the value of x which satisfies

(D.30) $$\Phi[\frac{\sqrt{n-\ell}\,(x-1)}{\sqrt{2}\,\beta_n(x)}] = \alpha.$$

Let $t_\alpha = \Phi^{-1}(\alpha)$, then (D.30) is equivalent to

(D.31) $$\frac{\sqrt{n-\ell}\,(x-1)}{\sqrt{2}\,\beta_n(x)} = t_\alpha.$$

Since $t_\alpha < 0$ if and only if $\alpha < \frac{1}{2}$, we first consider the case where $\alpha < \frac{1}{2}$ and try to find the solution x of (D.31) which satisfies $0 < x < 1$.
Now (D.31) can be rewritten as

$$f(x) = (n-\ell)(x-1)^2 - 2t_\alpha^2\beta_n(x) = 0,$$

and with the aid of (D.18) this becomes

(D.32) $$f(x) = a_n x^2 - 2b_n x + c_n = 0,$$

where

$$a_n = 1 - \frac{2t_\alpha^2}{n-k}$$

(D.33) $$b_n = 1 - \frac{2t_\alpha^2(n-k-\ell+tr)}{(n-k)(n-\ell)}$$

$$c_n = 1 - \frac{2t_\alpha^2}{n-\ell}.$$

Since n is large we suppose that $n-k > 2t_\alpha^2$ and $n-\ell > 2t_\alpha^2$, which for $t_\alpha < 0$ is equivalent to

$$\alpha > \Phi[-\sqrt{\frac{n-\max(k,\ \ell)}{2}}].$$

For large n this is no serious restriction on α, because

$$\Phi[-\sqrt{\frac{n-\max(k,\ \ell)}{2}}] \to \Phi(-\infty) = 0,$$

if $n\to\infty$.
Further we have from $tr \leq \min(k,\ \ell)$:

$$n-k \geq n-k-\ell+tr \text{ and } n-\ell \geq n-k-\ell+tr.$$

Also, since we exclude the trivial case $p = k = \ell$ (which is equivalent to $M(X) = M(Z) = M(G)$), we never have $n-k = n-k-\ell+tr$ and at the same time $n-\ell = n-k-\ell+tr$.
This shows that

$$0 < a_n < 1$$

$$0 < b_n < 1$$

$$0 < c_n < 1$$

(D.34) $$b_n \geq a_n$$

$$b_n \geq c_n$$

$$b_n^2 > a_n c_n$$

$$a_n - 2b_n + c_n < 0.$$

If we solve (D.32), we get

(D.35)
$$x_1 = \frac{b_n - \sqrt{b_n^2 - a_nc_n}}{a_n}$$

$$x_2 = \frac{b_n + \sqrt{b_n^2 - a_nc_n}}{a_n}.$$

Since $b_n^2 > a_nc_n$ it follows that x_1 and x_2 are real and different. Further it is seen from $a_n > 0$, $f(0) = c_n > 0$ and $f(1) = a_n - 2b_n + c_n < 0$ that $0 < x_1 < 1$ and $x_2 > 1$. Now we were looking for a solution in the interval (0, 1) and the above results show that, for large n, the value of s_α which satisfies $P(S_1^* \leq s_\alpha) = \alpha$ can be approximated by

(D.36)
$$s_\alpha \approx x_1 = \frac{b_n - \sqrt{b_n^2 - a_nc_n}}{a_n},$$

where $\Phi[-\sqrt{\frac{n - \max(k, \ell)}{2}}] < \alpha < \frac{1}{2}$, and where the coefficients a_n, b_n and c_n are as given in (D.33). Next we consider the case where $\alpha > \frac{1}{2}$. Since $t_\alpha > 0$ if and only if $\alpha > \frac{1}{2}$, it follows from (D.31) that we are looking for a solution x which satisfies $x > 1$.
Hence, it is easily seen from the above results that, for large n, the value of s_α can be approximated by

(D.37)
$$s_\alpha \approx x_2 = \frac{b_n + \sqrt{b_n^2 - a_nc_n}}{a_n},$$

when $\frac{1}{2} < \alpha < \Phi[\sqrt{\frac{n-\max(k, \ell)}{2}}]$.

Again the restriction on α is not serious, since

$$\Phi[\sqrt{\frac{n-\max(k, \ell)}{2}}] \rightarrow \Phi(\infty) = 1,$$

if $n\rightarrow\infty$.
Finally we consider $\alpha = \frac{1}{2}$. Since $t_\alpha = 0$ if and only if $\alpha = \frac{1}{2}$, it is seen from (D.32) and (D.33) that $s_\alpha \approx x_1 = x_2 = 1$ in this case.

We also note that the approximations x_1 and x_2 in (D.36) and (D.37), respectively, has the property that $x_1 \to 1$ and $x_2 \to 1$ if $n\to\infty$. This easily follows from the fact that $a_n \to 1$, $b_n \to 1$ and $c_n \to 1$ if $n\to\infty$ (see (D.33)).

We now proceed by deriving a large sample approximation to the distribution function of the statistic S_2^* under the hypothesis (D.4). Here we assume that $p < k$, since otherwise S_2^* is not defined. From (D.1) we see that S_2^* can be witten as

$$\text{(D.38)} \qquad S_2^* = S_2 \frac{n-k}{k-p},$$

where

$$S_2 = \frac{\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G}{\hat{u}_X'\hat{u}_X}.$$

The event $S_2 \leq s$ is equivalent to

$$Q_2(s) = (\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - s\ \hat{u}_X'\hat{u}_X/\sigma^2 \leq 0.$$

In a similar way as before, by using the results of Section 4 and Section 8, it can be shown that for $s > 0$ the random variable $Q_2(s)$ can be written as

$$\text{(D.39)} \qquad Q_2(s) = \sum_{j=0}^{M} [\lambda_{1j}(s)V_{1j} + \lambda_{2j}(s)V_{2j}].$$

Here the coefficients $\lambda_{1j}(s)$ and $\lambda_{2j}(s)$ are as given in (D.8), the random variables $V_{10}, V_{11}, \ldots, V_{1M}, V_{20}, \ldots, V_{2M}$ are mutually independent with

$$\text{(D.40)} \qquad \begin{aligned} V_{1j} &\sim \chi^2(m_{1j}) \\ V_{2j} &\sim \chi^2(m_{2j}), \end{aligned}$$

$j = 0, 1, \ldots, M$, and the degrees of freedom ar now defined by

$$\text{(D.41)} \qquad \begin{aligned} m_{10} &= \ell-r+m = \ell-r+n+p-k-\ell = n+p-k-r \\ m_{20} &= k-r \\ m_{1j} &= m_{2j} = m_j,\ j = 1, 2, \ldots, M \end{aligned}$$

We define the coefficients $\tau_1(s)$, $\tau_2(s)$, ..., $\tau_{n-p}(s)$ as follows: the first m_{10} coefficients $\tau_i(s)$ are equal to $\lambda_{10}(s)$, the second m_{11} coefficients $\tau_i(s)$ are equal to $\lambda_{11}(s)$, ..., the last m_{2M} coefficients $\tau_i(s)$ are equal to $\lambda_{2M}(s)$. Then $Q_2(s)$ in (D.39) can be rewritten as

(D.42) $$Q_2(s) = \sum_{i=1}^{n-p} \tau_i(s)U_i,$$

where the U_i's are mutually independent $\chi^2(1)$ random variables. Consider the event $S_2^* \leq s$. Since $S_2 \leq s$ is equivalent to $Q_2(s) \leq 0$ it follows from (D.38) that $S_2^* \leq s$ is equivalent to $Q_2(\frac{k-p}{n-k}s) \leq 0$, where

(D.43) $$Q_2(\frac{k-p}{n-k}s) = (\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - (\frac{k-p}{n-k})s\hat{u}_X'\hat{u}_X/\sigma^2.$$

If we define $W_2(s)$ by

$$W_2(s) = Q_2(\frac{k-p}{n-k}s),$$

it is seen from (D.42) that

(D.44) $$W_2(s) = \sum_{i=1}^{n-p} \alpha_i(s)U_i,$$

where

(D.45) $$\alpha_i(s) = \tau_i(\frac{k-p}{n-k}s),$$

$i = 1, 2, ..., n-p$.
This shows that the event $S_2^* \leq s$ is equivalent to $W_2(s) \leq 0$ with $W_2(s)$ as given in (D.44) and that

(D.46) $$P(S_2^* \leq s) = P(W_2(s) \leq 0).$$

In order to find an approximation to $P(S_2^* \leq s)$, we therefore first derive the limiting distribution of

$$\frac{W_2(s) - \mu_2(s)}{\sigma_2(s)},$$

where $\mu_2(s) = E(W_2(s))$ and $\sigma_2^2(s) = Var(W_2(s))$.

From (D.44), (D.45) and the definition of the coefficients $\tau_i(s)$ we obtain the following expressions for $\mu_2(s)$, $\sigma_2^2(s)$ and the coefficient of skewness $\gamma_2(s)$:

$$\mu_2(s) = -(k-p)(s-1)$$

$$\text{(D.47)} \qquad \sigma_2^2(s) = 2(k-p)\ \beta_n(s)$$

$$\gamma_2(s) = \sqrt{\frac{8}{k-p}}\ \frac{\delta_n(s)}{[\beta_n(s)]^{3/2}}$$

where

$$\beta_n(s) = (\frac{k-p}{n-k})s^2 - 2(\frac{tr-p}{n-k})s + 1$$

$$\delta_n(s) = -(\frac{k-p}{n-k})^2 s^3 + 3\,\frac{(k-p)(tr-p)}{(n-k)^2}s^2 - 3(\frac{tr-p}{n-k})s + 1.$$

It easily follows from (D.48) that

$$\text{(D.49)} \qquad \lim_{n\to\infty} \beta_n(s) = 1, \qquad \lim_{n\to\infty} \delta_n(s) = 1$$

and this implies (see (D.47)) that

$$\text{(D.50)} \qquad \lim_{n\to\infty} \gamma_2(s) = \sqrt{\frac{8}{k-p}} > 0.$$

Since $\gamma_2(s)$ is the coefficient skewness of the random variable $(W_2(s) - \mu_2(s))/\sigma_2(s)$, the latter result shows that this variable does not have a limiting $n(0, 1)$ distribution.

In other words, although $W_2(s)$ can be written as a sum of mutually independent random variables (see (D.44)), we cannot apply Liapounov's theorem in order to find the limiting distribution of $(W_2(s) - \mu_2(s))/\sigma_2(s)$.

We shall now show that this limiting distribution can directly be found from the expression (D.43).

From the definition of $W_2(s)$ we get

$$W_2(s) = (\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - (\frac{k-p}{n-k})s\hat{u}_X'\hat{u}_X/\sigma^2,$$

and by making use of (D.47) we obtain

$$\frac{W_2(s) - \mu_2(s)}{\sigma_2(s)} = \frac{(\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - (\frac{k-p}{n-k})s\hat{u}_X'\hat{u}_X/\sigma^2 + (k-p)(s-1)}{\sqrt{2(k-p)}\sqrt{\beta_n(s)}},$$

which can be rewritten as

$$\text{(D.51)} \quad \frac{W_2(s) - \mu_2(s)}{\sigma_2(s)} = \frac{(\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - (k-p)}{\sqrt{2(k-p)}} / \sqrt{\beta_n(s)}$$

$$- \sqrt{\frac{k-p}{2}}\, s(\frac{\hat{u}_X'\hat{u}_X/\sigma^2}{n-k} - 1)/\sqrt{\beta_n(s)}.$$

It can easily be verified that under the hypothesis (D.4) we have for all n

$$\text{(D.52)} \quad \begin{aligned} &(\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 \sim \chi^2(k-p) \\ &\hat{u}_X'\hat{u}_X/\sigma^2 \sim \chi^2(n-k) \end{aligned}$$

Hence, $(\hat{u}_X'\hat{u}_X/\sigma^2)/(n-k)$ converges in probability to 1 if $n\to\infty$. Since $\beta_n(s) \to 1$, if follows that the second term at the right-hand side of (D.51) converges in probability to 0 if $n\to\infty$.
This means that $(W_2(s) - \mu_2(s))/\sigma_2(s)$ has the same limiting distribution as the first term at the right-hand sie of (D.51).
Again since $\beta_n(s) \to 0$ it is seen that

$$\frac{(\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - (k-p)}{\sqrt{2(k-p)}} / \sqrt{\beta_n(s)}$$

has the same limiting distribution as

$$\frac{(\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - (k-p)}{\sqrt{2(k-p)}}.$$

However, as can be seen from (D.52) the latter random variable has a distribution which is independent of n. This shows that the limiting distribution of $(W_2(s) - \mu_2(s))/\sigma_2(s)$ is equal to the distribution of

$$\frac{(\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 - (k-p)}{\sqrt{2(k-p)}}.$$

Since $(\hat{u}_Z'\hat{u}_Z - \hat{u}_G'\hat{u}_G)/\sigma^2 \sim \chi^2(k-p)$ this variable has mean k-p and variance 2(k-p) and we have shown that

$$\text{(D.53)} \qquad \frac{W_2(s) - \mu_2(s)}{\sigma_2(s)} \xrightarrow{F} \frac{\chi^2(k-p) - (k-p)}{\sqrt{2(k-p)}}$$

if $n \to \infty$.

That is, $(W_2(s) - \mu_2(s))/\sigma_2(s)$ converges in distribution to a standardized $\chi^2(k-p)$ distribution, i.e.,

$$\text{(D.54)} \qquad P\left[\frac{W_2(s) - \mu_2(s)}{\sigma_2(s)} \leq x\right] \to G_{k-p}[k-p + x\sqrt{2(k-p)}]$$

if $n \to \infty$, where

$$G_r(x) = \int_0^x \frac{t^{\frac{r}{2}-1} e^{-\frac{1}{2}t}}{\Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} dt.$$

The convergence in (D.54) holds true for $-\infty < x < \infty$, except when k-p = 1, 2, then the points $x = -\frac{1}{2}\sqrt{2}$ and $x = -1$, respectively, are excluded.

This result enables us to approximate $P(S_2^* \leq s)$ for large n and s > 0. From (D.46) we have

$$P(S_2^* \leq s) = P(W_2(s) \leq 0) = P\left[\frac{W_2(s) - \mu_2(s)}{\sigma_2(s)} \leq \frac{-\mu_2(s)}{\sigma_2(s)}\right]$$

and by making use of (D.54) it follows that

(D.55) $$P(S_2^* \leq s) \approx G_{k-p}[k-p - \frac{\mu_2(s)}{\sigma_2(s)} \sqrt{2(k-p)}]$$

when n is large.
Substitution of $\mu_2(s)$ and $\sigma_2(s)$ from (D.47) into the right-hand side of (D.55) yields for $s > 0$

(D.56) $$P(S_2^* \leq s) \approx G_{k-p}[(k-p)(1 + \frac{s-1}{\sqrt{\beta_n(s)}})]$$

when n is large, where $\beta_n(s)$ is as given in (D.48).
It should be observed that $1 + (s-1)/\sqrt{\beta_n(s)} > 0$ for all $s > 0$.
Note that since $\beta_n(s) \to 1$ if $n\to\infty$, we have

$$G_{k-p}[(k-p)(1 + \frac{s-1}{\sqrt{\beta_n(s)}})] \to G_{k-p}[(k-p)s]$$

if $n\to\infty$. This agrees with the fact that $S_2^* \overset{F}{\to} \chi^2(k-p)/(k-p)$ if $n\to\infty$.
It should also be observed that the coefficient of skewness $\gamma_2(s)$ of $(W_2(s) - \mu_2(s)/\sigma_2(s)$ converges to the coefficient of skewness $\sqrt{\frac{8}{k-p}}$ of

$$\frac{\chi^2(k-p) - (k-p)}{\sqrt{2(k-p)}},$$

if $n\to\infty$ (see (D.50)).
With the aid of (D.56) we can find a approximation to the value of $s_\alpha \geq 0$ which satisfies $P(S_2^* < s_\alpha) = \alpha$.
For large n this value can be approximated by the value of x which satisfies

(D.57) $$G_{k-p}[(k-p)(1 + \frac{x-1}{\sqrt{\beta_n(x)}})] = \alpha.$$

Let $f_\alpha = G_{k-p}^{-1}(\alpha)$, then (D.57) is equivalent to

(D.58) $$\frac{x-1}{\sqrt{\beta_n(x)}} = \frac{f_\alpha}{k-p} - 1 \overset{\text{def.}}{=} q_\alpha.$$

Since $q_\alpha < 0$ if and only if $\alpha < G_{k-p}(k-p)$, we first consider the case where $\alpha < G_{k-p}(k-p)$ and try to find the solution x of (D.58) which satisfies $0 \le x < 1$. Note that $q_\alpha \ge -1$ and that $q_\alpha = -1$ if and only if $\alpha = 0$.

Further we also note that $0.683 = G_1(1) > G_2(2) > G_3(3) > \dots$ and $G_r(r) \to \frac{1}{2}$ if $r\to\infty$.

Now (D.58) can be rewritten as

$$f(x) = (x-1)^2 - q_\alpha^2\beta_n(x) = 0,$$

and with the aid of (D.48) this becomes,

$$f(x) = a_n x^2 - 2b_n x + c = 0, \tag{D.59}$$

where

$$a_n = 1 - \frac{q_\alpha^2(k-p)}{n-k}$$

$$b_n = 1 - \frac{q_\alpha^2(tr-p)}{n-k} \tag{D.60}$$

$$c = 1 - q_\alpha^2$$

From $k-p < n-k$ (n is large), $p \le tr < k$, $-1 \le q_\alpha < 0$ it follows that

$$0 \le c < a_n < b_n \le 1. \tag{D.61}$$

This shows that

$$\begin{aligned} &b_n^2 > a_n c \\ &a_n - 2b_n + c < 0 \end{aligned} \tag{D.62}$$

Note that for $\alpha < G_{k-p}(k-p)$ we have $c = 0$ if and only if $\alpha = 0$ and also $b_n = 1$ if and only if $tr = p$.

If we solve (D.59), we get

(D.63)
$$x_1 = \frac{b_n - \sqrt{b_n^2 - a_n c}}{a_n}$$

$$x_2 = \frac{b_n + \sqrt{b_n^2 - a_n c}}{a_n}$$

From $b_n^2 > a_n c$ it follows that x_1 and x_2 are real and different. Further it is seen from $a_n > 0$, $f(0) = c \geq 0$ and $f(1) = a_n - 2b_n + c < 0$ that $0 \leq x_1 < 1$ and $x_2 > 1$. Since we are looking for a solution in the interval [0, 1) the above results show that, for large n, the value of s_α which satisfies $P(S_2^* \leq s_\alpha) = \alpha$ can be approximated by

(D.64)
$$s_\alpha \approx x_1 = \frac{b_n - \sqrt{b_n^2 - a_n c}}{a_n},$$

where the coefficients a_n, b_n and c are as given in (D.60) and where $0 \leq \alpha < G_{k-p}(k-p)$.
Next we consider the case where $\alpha > G_{k-p}(k-p)$.
Since $q_\alpha > 0$ if and only if $\alpha > G_{k-p}(k-p)$, it follows from (D.58) that we are looking for a solution x which satisfies $x > 1$. Here we have to consider two subcases:

(i) $\alpha \leq G_{k-p}[2(k-p)]$

(ii) $\alpha > G_{k-p}[2(k-p)]$

Note that for $\alpha > G_{k-p}(k-p)$ we have $c = 0$ if and only if $\alpha = G_{k-p}[2(k-p)]$ and also $b_n = 1$ if and only if $tr = p$.
Under the assumption (i) we have $0 < q_\alpha \leq 1$ and it is easily seen that a_n, b_n and c satisfy (D.61) and (D.62). It follows that $0 \leq x_1 < 1$ and $x_2 > 1$.
Hence the approximation of s_α becomes

(D.65)
$$s_\alpha \approx x_2 = \frac{b_n + \sqrt{b_n^2 - a_n c}}{a_n}$$

When α satisfies (ii) we get $q_\alpha > 1$ and therefore $c < 0$. Since n is large we make the additional assumption that $q_\alpha < \sqrt{\frac{n-k}{k-p}}$, which is equivalent to

$$\alpha < G_{k-p}[k-p + \sqrt{(k-p)(n-k)}].$$

This is no serious restriction on α since

$$G_{k-p}[k-p + \sqrt{(k-p)(n-k)}] \to G_{k-p}(\infty) = 1,$$

if $n \to \infty$.
Now we get

$$\text{(D.66)} \qquad c < 0 < a_n < b_n \leq 1$$

and it follows again that (D.62) is satisfied. The roots x_1 and x_2 are therefore real and different and since $a_n > 0$, $f(0) = c < 0$ and $f(1) = a_n - 2b_n + c < 0$ we get $x_1 < 0$ and $x_2 > 1$. Hence the approximation of s_α is again given by (D.65).
The above results show that, when

$$G_{k-p}(k-p) < \alpha < G_{k-p}[k-p + \sqrt{(k-p)(n-k)}],$$

the value of s_α which satisfies $P(S_2^* \leq s_\alpha) = \alpha$ can be approximated by

$$\text{(D.67)} \qquad s_\alpha \approx x_2 = \frac{b_n + \sqrt{b_n^2 - a_n c}}{a_n},$$

provided that n is large.
Finally we consider the case $\alpha = G_{k-p}(k-p)$.
Now $q_\alpha = 0$ if and only if $\alpha = G_{k-p}(k-p)$ and it is seen from (D.59) and (D.60) that $s_\alpha \approx x_1 = x_2 = 1$ when $\alpha = G_{k-p}(k-p)$.
It should be observed that x_1 and x_2 in (D.64) and (D.67), respectively, satisfy

$$x_1 \to \frac{f_\alpha}{k-p} \quad \text{and} \quad x_2 \to \frac{f_\alpha}{k-p},$$

if $n \to \infty$. This follows from the fact that $a_n \to 1$ and $b_n \to 1$ when $n \to \infty$ (see (D.60)).

Now it remains to find a large sample approximation to the distribution of the statistic S_3^* as defined in (D.1), under the hypothesis (D.4). Here we assume that $\ell > p$, since otherwise S_3^* is not defined.
We have

$$(D.68)\qquad P(S_3^* \leq s) = 1 - P(\frac{1}{S_3^*} \leq \frac{1}{s}),$$

where

$$(D.69)\qquad \frac{1}{S_3^*} = \frac{\hat{u}_X'\hat{u}_X - \hat{u}_G'\hat{u}_G}{\hat{u}_Z'\hat{u}_Z}\,\frac{n-\ell}{\ell-p}.$$

If we compare the latter expression with the definition of S_2^* it follows that we can find a large sample approximation for

$$P(\frac{1}{S_3^*} \leq \frac{1}{s})$$

from (D.56) through replacing k by ℓ and s by $\frac{1}{s}$.
We get

$$(D.70)\qquad P(\frac{1}{S_3^*} \leq \frac{1}{s}) \approx G_{\ell-p}[(\ell-p)(1 + \frac{\frac{1}{s} - 1}{\sqrt{\beta_n(\frac{1}{s})}})],$$

where

$$\beta_n(\frac{1}{s}) = (\frac{\ell-p}{n-\ell})(\frac{1}{s})^2 - 2(\frac{tr-p}{n-\ell})(\frac{1}{s}) + 1,$$

as can be seen from (D.48).
Since

$$\beta_n(\frac{1}{s}) = \frac{1}{s^2}[s^2 - 2(\frac{tr-p}{n-\ell})s + (\frac{\ell-p}{n-\ell})],$$

we can rewrite (D.70) in the following way

$$(D.71)\qquad P(\frac{1}{S_3^*} \leq \frac{1}{s}) \approx G_{\ell-p}[(\ell-p)(1 + \frac{1-s}{\sqrt{\beta_n(s)}})],$$

where now $\beta_n(s)$ is defined by

(D.72) $\beta_n(s) = s^2 - 2(\frac{tr-p}{n-\ell})s + (\frac{\ell-p}{n-\ell})$.

Substitution of (D.71) into (D.68) yields for $s > 0$

(D.73) $P(S_3^* \leq s) \approx 1 - G_{\ell-p}[(\ell-p)(1 + \frac{1-s}{\sqrt{\beta_n(s)}})]$,

when n is large, where $\beta_n(s)$ is as given in (D.72). It should be observed that $1 + (1-s)/\sqrt{\beta_n(s)} > 0$ for all $s > 0$.
Note that since $\beta_n(s) \to s^2$ if $n\to\infty$, we have

$$1 - G_{\ell-p}[(\ell-p)(1 + \frac{1-s}{\sqrt{\beta_n(s)}})] \to 1 - G_{\ell-p}(\frac{\ell-p}{s}),$$

if $n\to\infty$. This agrees with the fact that $S_3^* \overset{F}{\to} \frac{\ell-p}{\chi^2(\ell-p)}$ if $n\to\infty$.
Again, from (D.73) we can find a large sample approximation of the value of s_α which satisfies $P(S_3^* \leq s_\alpha) = \alpha$.
That is, s_α can be approximated by the solution x of the equation

$$1 - G_{\ell-p}[(\ell-p)(1 + \frac{1-x}{\sqrt{\beta_n(x)}})] = \alpha,$$

or equivalently,

(D.74) $\frac{1-x}{\sqrt{\beta_n(x)}} = \frac{f_{1-\alpha}}{\ell-p} - 1 \overset{\text{def.}}{=} q_\alpha$,

where $f_{1-\alpha} = G_{\ell-p}^{-1}(1-\alpha)$.
Equation (D.47) can be rewritten as

(D.75) $f(x) = ax^2 - 2b_nx + c_n = 0$,

where

$$a = 1 - q_\alpha^2$$

(D.76) $b_n = 1 - \frac{q_\alpha^2(tr-p)}{n-\ell}$

$$c_n = 1 - \frac{q_\alpha^2(\ell-p)}{n-\ell}$$

Similar to the previous case we obtain the following solutions.
When

$$1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}] < \alpha < 1 - G_{\ell-p}(\ell-p)$$

the value of s_α which satisfies $P(S_3^* \leq s_\alpha) = \alpha$ is approximated by

(D.77)
$$s_\alpha \approx x_1 = \frac{b_n - \sqrt{b_n^2 - ac_n}}{a}, \text{ if } \alpha \neq 1 - G_{\ell-p}[2(\ell-p)]$$
$$s_\alpha \approx x_1 = \frac{c_n}{2b_n}, \text{ if } \alpha = 1 - G_{\ell-p}[2(\ell-p)]$$

Note that

$$\frac{c_n}{2b_n} = \lim_{a \to 0} \frac{b_n - \sqrt{b_n^2 - ac_n}}{a}$$

and that for $\alpha < 1 - G_{\ell-p}(\ell-p)$ we have $a = 0$ if and only if $\alpha = 1 - G_{\ell-p}[2(\ell-p)]$.
The restriction

$$\alpha > 1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}]$$

is not serious for large n since

$$1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}] \to 1 - G_{\ell-p}(\infty) = 0,$$

if $n \to \infty$.
If $1 - G_{\ell-p}(\ell-p) < \alpha \leq 1$, we have

(D.78) $$s_\alpha \approx x_2 = \frac{b_n + \sqrt{b_n^2 - ac_n}}{a}.$$

For the case $\alpha = 1 - G_{\ell-p}(\ell-p)$ we get $s_\alpha \approx x_1 = x_2 = 1$. It should be observed that $0.317 \approx 1 - G_1(1) < 1 - G_2(2) < 1 - G_3(3) < \ldots$ and $1 - G_r(r) \to \frac{1}{2}$ if $r \to \infty$.

Further we note that x_1 and x_2 in (D.77) and (D.78), respectively, satisfy

$$x_1 \to \frac{\ell-p}{f_{1-\alpha}} \quad \text{and} \quad x_2 \to \frac{\ell-p}{f_{1-\alpha}},$$

if $n\to\infty$. This is easily seen from the fact that $b_n \to 1$ and $c_n \to 1$ if $n\to\infty$ (see (D.76)).

Summarizing the results of this appendix, we have the following large sample approximations to $P(S_i^* \leq s)$, $s > 0$ and to the value of s_α which satisfies $P(S_i^* \leq s_\alpha) = \alpha$, $i = 1, 2, 3$:

(I) $$P(S_1^* \leq s) \approx \Phi\left[\frac{\sqrt{n-\ell}(s-1)}{\sqrt{2\,\beta_n(s)}}\right], \quad s \neq 1$$

$$s_\alpha \approx x_1 = \frac{b_n - \sqrt{b_n^2 - a_n c_n}}{a_n}, \quad \text{if } \Phi\left[-\sqrt{\frac{n-\max(k,\ \ell)}{2}}\right] < \alpha \leq \frac{1}{2}$$

$$s_\alpha \approx x_2 = \frac{b_n + \sqrt{b_n^2 - a_n c_n}}{a_n}, \quad \text{if } \frac{1}{2} \leq \alpha < \Phi\left[\sqrt{\frac{n-\max(k,\ \ell)}{2}}\right],$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, dt$$

$$\beta_n(s) = \left(\frac{n-\ell}{n-k}\right)s^2 - 2\left(\frac{n-k-\ell+tr}{n-k}\right)s + 1$$

$$a_n = 1 - \frac{2t_\alpha^2}{n-k}$$

$$b_n = 1 - \frac{2t_\alpha^2(n-k-\ell+tr)}{(n-k)(n-\ell)}$$

$$c_n = 1 - \frac{2t_\alpha^2}{n-\ell}$$

$$t_\alpha = \Phi^{-1}(\alpha).$$

(II) $$P(S_2^* \leq s) \approx G_{k-p}[(k-p)(1 + \frac{s-1}{\sqrt{\beta_n(s)}})]$$

$$s_\alpha \approx x_1 = \frac{b_n - \sqrt{b_n^2 - a_n c}}{a_n}, \text{ if } 0 \leq \alpha \leq G_{k-p}(k-p)$$

$$s_\alpha \approx x_2 = \frac{b_n + \sqrt{b_n^2 - a_n c}}{a_n}, \text{ if } G_{k-p}(k-p) \leq \alpha < G_{k-p}[k-p + \sqrt{(k-p)(n-k)}],$$

where

$$G_r(x) = \int_0^x \frac{t^{\frac{r}{2}-1} e^{-\frac{1}{2}t}}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} dt$$

$$\beta_n(s) = (\frac{k-p}{n-k})s^2 - 2(\frac{tr-p}{n-k})s + 1$$

$$a_n = 1 - \frac{q_\alpha^2(k-p)}{n-k}$$

$$b_n = 1 - \frac{q_\alpha^2(tr-p)}{n-k}$$

$$c = 1 - q_\alpha^2$$

$$q_\alpha = \frac{f_\alpha}{k-p} - 1$$

$$f_\alpha = G_{k-p}^{-1}(\alpha).$$

(III) $$P(S_3^* \leq s) \approx 1 - G_{\ell-p}[(\ell-p)(1 + \frac{1-s}{\sqrt{\beta_n(s)}})]$$

$$s_\alpha \approx x_1 = \frac{b_n - \sqrt{b_n^2 - ac_n}}{a},$$

if $1 - G_{\ell-p}[\ell-p + \sqrt{(\ell-p)(n-\ell)}] < \alpha \leq 1 - G_{\ell-p}(\ell-p)$ provided that $\alpha \neq 1 - G_{\ell-p}[2(\ell-p)]$,

$$s_\alpha \approx x_1 = \frac{c_n}{2b_n}, \text{ if } \alpha = 1 - G_{\ell-p}[2(\ell-p)]$$

$$s_\alpha \approx x_2 = \frac{b_n + \sqrt{b_n^2 - ac_n}}{a}, \text{ if } 1 - G_{\ell-p}(\ell-p) \leq \alpha \leq 1,$$

where

$$G_r(x) = \int_0^x \frac{t^{\frac{r}{2}-1} e^{-\frac{1}{2}t}}{\Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} dt$$

$$\beta_n(s) = s^2 - 2(\frac{tr-p}{n-\ell})s + (\frac{\ell-p}{n-\ell})$$

$$a = 1 - q_\alpha^2$$

$$b_n = 1 - \frac{q_\alpha^2(tr-p)}{n-\ell}$$

$$c_n = 1 - \frac{q_\alpha^2(\ell-p)}{n-\ell}$$

$$q_\alpha = \frac{f_{1-\alpha}}{\ell-p} - 1$$

$$f_{1-\alpha} = G_{\ell-p}^{-1}(1-\alpha).$$

The above approximations can easily be computed with the aid of the $n(0, 1)$ and χ^2 distributions.
For the given matrices X and Z we only have to compute

$$p = \dim(M(X) \cap M(Z))$$

$$tr = tr[X'X)^{-1}X'Z(Z'Z)^{-1}Z'X] = tr[(Z'Z)^{-1}Z'X(X'X)^{-1}X'Z].$$

In most applications p will be equal to the number of columnvectors that X and Z have in common.

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