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TRANSFER FUNCTIONS AND OPERATOR THEORY

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ABSTRACT

The notion of a transfer function from systems theory serves as an important tool in dealing with certain problems in operator and matrix theory. This is illustrated with a variety of material taken from existing literature. A striking feature of the results is their high degree of explicitness.

INTRODUCTION

A transfer function is an operator or matrix function of a complex variable λ of the form $D + C(\lambda-A)^{-1}B$. This concept has its origin in systems theory and is closely related to that of a realization. The purpose of the paper is to substantiate the claim made in the Abstract. The topics that will be (briefly) discussed can be read off from the section and subsection headings listed below. One of the main sources of inspiration is a general Factorization Principle suggested by the phenomenon of series connection in systems theory. It relates factorizations of transfer functions to matching pairs of invariant subspaces. The results have a high degree of explicitness. Special attention is paid to the finite dimensional (rational) case, but also infinite dimensional situations are considered. At the end of the paper some remarks are made about exponentially dichotomous operators. These are (generally) unbounded operators of a special type that come up naturally in dealing with the (non-rational) infinite dimensional case.

The section and subsection headings are as follows:

1. Transfer Functions and Realizations
2. The Realization Problem
3. Linearization
4. Inversion of Transfer Functions and Inverse Fourier Transforms
5. The Riemann-Hilbert Boundary Value Problem
6. The Factorization Principle
7. Applications of the Factorization Principle
   - 7.1. Minimal Factorization
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   - 7.5. Self-adjoint Rational Matrix Functions
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8. Wiener-Hopf Equation
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10. Model Reduction
11. Generalizations and Recent Developments
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    - 11.2. The Infinite Dimensional Case
    - 11.3. Exponentially Dichotomous Operators.

The paper consists of expanded notes for two talks given at the International Symposium on Operator Theory held in Athens (Greece), August 26-31, 1985. The material has been taken from existing literature. Much of it is joint work with I. Gohberg and M.A. Kaashoek. It is a pleasure to thank A.C.M. Ran for his important share in the writing of Section 10 on Model Reduction.

1. TRANSFER FUNCTIONS AND REALIZATIONS

An expression of the type

$$W(\lambda) = D + C(\lambda I_X - A)^{-1}B \tag{1.1}$$

is called a <u>realization</u>. Here λ is a complex parameter, $W(\lambda)$ is a bounded linear operator on a complex Banach space Y, X is a complex Banach space and $A: X \to X$, $B: Y \to X$ and $C: X \to Y$ are bounded linear operators. The symbol I_X stands for the identity operator on X. Sometimes it is necessary to specify the subset Ω of the resolvent set $\rho(A)$ of A on which (1.1) holds. We then speak of a <u>realization on</u> Ω. If X and Y are both finite dimensional, then $W(\lambda)$, A, B, C and D can be identified with matrices. In that context the term <u>matrix realization</u> will be used.

<u>Notation</u>. If confusion is impossible, we sometimes write I instead of I_X. Also often $\lambda - A$ stands for $\lambda I - A$. The $n \times n$ identity matrix will be denoted by I_n.

To illustrate the connection with systems theory, consider the linear dynamical system

$$(1.2) \qquad \dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t).$$

Taking Laplace transforms (assuming $x(0) = 0$) and eliminating the Laplace transform $\hat{x}(\lambda)$ of x, one obtains $\hat{y}(\lambda) = \left(D + C(\lambda - A)^{-1}B\right)\hat{u}(\lambda)$, which is a direct relationship between the input u and the output y of the system (1.2). In view of this, functions of the form (1.1) are called <u>transfer functions</u>. In operator theory, sometimes the term <u>characteristic operator function</u> is used too.

<u>Literature</u>: [8],[11],[12],[14],[29],[30],[33],[34],[44],[60], [66],[67],[73],[76],[77],[86],[105],[106].

2. THE REALIZATION PROBLEM

If (1.1) is a matrix realization, then by Cramer's rule $W(\lambda)$ is a rational (square) matrix function. Also $W(\lambda)$ is proper (i.e. analytic at ∞) with $W(\infty) = D$. Conversely, if $W(\lambda)$ is a proper rational (square) matrix function, then $W(\lambda)$ admits a matrix realization (1.1) with $D = W(\infty)$. Explicit constructions are available in the literature on systems theory.

In the general (not necessarily finite dimensional) case, the situation is more complicated. There the domain Ω on which (1.1) holds comes into play. Suppose $W(\lambda)$ is a bounded linear operator on a complex Banach space Y depending analytically on λ in an open subset Ω of the complex plane $\mathbb{C}$. Then $W(\lambda)$ admits a realization (1.1) on Ω, provided one of the following conditions is met

(a) Ω is a neighbourhood of ∞ and $W(\lambda)$ is analytic at ∞,

(b) Ω is a bounded subset of $\mathbb{C}$.

Again, explicit constructions are available (see [14]). In case (a), the operator D in (1.1) is necessarily equal to $W(\infty)$; in case (b) one can take for D any bounded linear operator on Y (for instance I_Y). Clearly, analytic operator functions always admit a realization locally.

As a special case from the category (a), consider the situation where Ω is an unbounded Cauchy domain(cf. [107]), hence a neighbourhood of ∞. Assume $W(\lambda)$ is analytic on Ω, continuous on the closure $\overline{\Omega}$ of Ω and analytic at ∞. Let Γ be the (positively oriented) boundary of Ω, and let X be the complex Banach space of all Y-valued continuous functions on Γ endowed with the supremum norm. Define $A: X \to X$, $B: Y \to X$ and $C: X \to Y$ by

$$(Af)(z) = zf(z)$$

$$(By)(z) = y,$$

$$Cf = \frac{1}{2\pi i} \int_{\Gamma} \big(W(\infty)-W(z)\big)f(z)dz,$$

and put $D = W(\infty)$. Then Ω is contained in the resolvent set $\rho(A)$ of A and the realization (1.1) holds on Ω. The proof is based on Cauchy's integral formula.

Literature: [8],[14],[29],[44],[67],[73],[76],[77],[90].

Further reading: [20],[22],[24].

Standing Assumption. In the remainder of this article, we shall always work with realizations (1.1) for which $D = I$. Partly, this is for simplicity of exposition, partly because in several applications this situation arises in a natural way.

3. LINEARIZATION

An old trick in the theory of analytic operator functions is to make a reduction to a linear pencil of the form $\lambda S-T$ or rather $\lambda I-T$. One approach is modelled after the well-known method of changing a higher order differential equation into a system of first order equations by adding new variables. It has been used to study operator polynomials and even operator power series (see [79],[95],[43],[89]). A second, more sophisticated, approach involves 'linearization by extension' (see [55]). Here the idea is to bring an analytic operator function in a simple form by allowing 'extension' (i.e. direct sums with identity operators) and analytic equivalence. An elementary (but still very informative) result along these lines can be quickly obtained from realization (cf.[14]).

Suppose on an open subset Ω of the complex plane the operator function $W(\lambda)$ is given by $W(\lambda) = I_Y + C(\lambda I_X-A)^{-1}B$. Here $\Omega \subset \rho(A)$. For $\lambda \in \Omega$, put

$$E(\lambda) = \begin{bmatrix} I_Y & -C \\ -B & \lambda I_X-(A-BC) \end{bmatrix}, \quad F(\lambda) = \begin{bmatrix} I_Y & C \\ (\lambda I_X-A)^{-1}B & I_X+(\lambda I_X-A)^{-1}BC \end{bmatrix}.$$

Then $E(\lambda)$ and $F(\lambda)$ are invertible operators depending analytically on λ in Ω and

$$E(\lambda)^{-1} = \begin{pmatrix} W(\lambda) & C(\lambda I_X - A)^{-1} \\ (\lambda I_X - A)^{-1}B & (\lambda I_X - A)^{-1} \end{pmatrix}, \quad F(\lambda)^{-1} = \begin{pmatrix} W(\lambda) & -C \\ -(\lambda I_X - A)^{-1}B & I_X \end{pmatrix}.$$

Moreover,

$$E(\lambda) \begin{pmatrix} W(\lambda) & 0 \\ 0 & I_X \end{pmatrix} F(\lambda) = \begin{pmatrix} I_Y & 0 \\ 0 & \lambda I_X - (A-BC) \end{pmatrix},$$

so $W(\lambda)$ and $\lambda I_X-(A-BC)$ are (analytically) equivalent after (two-sided) extension. Obviously, $W(\lambda)$ and $\lambda I_X - (A-BC)$ have many properties in common: the operator A-BC may be viewed as a 'linearization' of $W(\lambda)$.

As was indicated in Section 2, analytic operator functions admit realizations locally. So locally the reduction to a linear pencil of the form λ-T (spectral theory) is always possible. This can be used to give quick proofs of results like Theorem 5.2 of [9], where necessary and sufficient conditions are given in order that the (pointwise) inverse of an analytic operator function exists in a deleted neighbourhood of a point λ_o in the complexe plane and has a pole at λ_o of order p (cf. [10]). The idea of using equivalence and extension for bringing operators in a simple form plays also an important role in [16], [18], [21] and [24]. The simplification is again made through the use of transfer functions (cf. Section 8 below).

Literature: [9],[10],[14],[27],[40],[43],[55],[62],[79],[88],[89] [90],[91],[94],[95],[97].

Further reading: [3],[11],[12],[16],[18],[21],[24],[60],[65], [68],[69],[70],[72].

Notation. The operator A-BC will play an important role in what follows. It will be denoted by $A^{\times}$. Note that $A^{\times}$ depends not only on A, but also on B and C.

4. INVERSION OF TRANSFER FUNCTIONS AND INVERSE FOURIER TRANSFORMS

Consider the linear dynamical system

$$(4.1) \qquad \dot{x} = Ax + Bu, \qquad y = Cx + u.$$

Interchanging the roles of the input u and output y, one obtains the "inverse system"

$$(4.2) \qquad \dot{x} = A^{\times}x + By, \qquad u = -Cx + y.$$

As one may expect, the transfer functions of (4.1) and (4.2) are each others inverse (pointwise). The precise and complete result reads as follows. If, for $\lambda \in \rho(A)$, the operator $W(\lambda)$ is given by

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B,$$

then $W(\lambda)$ is invertible if and only if $\lambda \in \rho(A^{\times})$, and in that case

$$W(\lambda)^{-1} = I_Y - C(\lambda I_X - A^{\times})^{-1}B,$$

$$(\lambda I_X - A^{\times})^{-1} = (\lambda I_X - A)^{-1} - (\lambda I_X - A)^{-1}BW(\lambda)^{-1}C(\lambda I_X - A)^{-1}.$$

This simple observation can be used to give concrete formulas for certain inverse Fourier transforms.

Let $k \in L_1^{n\times n}(-\infty,\infty)$, i.e., let k be an integrable $n\times n$ matrix function on the real line. Put $W(\lambda) = I_n - \hat{k}(\lambda)$, where

$$\hat{k}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t}k(t)dt$$

is the Fourier tranform of k, and suppose det $W(\lambda)$ does not vanish on the real line. Then, according to a well-known theorem of N. Wiener (see [46], Section 17), there exists $k^{\times} \in L_1^{n\times n}(-\infty,\infty)$ such that

$$W(\lambda)^{-1} = I_n - \hat{k}^{\times}(\lambda) = I_n - \int_{-\infty}^{\infty} e^{i\lambda t} k^{\times}(t) dt.$$

Assume now that $W(\lambda)$ is rational. Since $W(\lambda)$ has no poles on the real line and $W(\infty) = I_n$ (Riemann-Lebesgue), $W(\lambda)$ admits a matrix realization $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$, where A has no real eigenvalues. But then the invertibility condition on $W(\lambda)$ is equivalent to the requirement that $A^{\times}$ has no real eigenvalues too and $W(\lambda)^{-1}$ has the realization $W(\lambda)^{-1} = I_n - C(\lambda I_m - A^{\times})^{-1}B$. In terms of $k^{\times}$ this means that

$$(4.3) \qquad k^{\times}(t) = \begin{cases} iCe^{-itA^{\times}} P^{\times} B, & t < 0, \\ -iCe^{-itA^{\times}} (I_m - P^{\times}) B, & t > 0, \end{cases}$$

where $P^{\times}$ is the spectral projection corresponding to the eigenvalues of $A^{\times}$ located in the (open) upper half plane.

Expressions of the type (4.3) are the (time domain) counterparts of realizations and called (<u>spectral</u>) <u>exponential representations</u>. They play an important role in [15], [16], [18] and [24]. The presence of $e^{-itA^{\times}}P^{\times}$ and $e^{-itA^{\times}}(I_m - P^{\times})$ suggests a more general notion involving exponentially decaying C_0-semigroups. In [20] and [22] this idea is worked out and used to study the non-rational case (see also Section 11.3 below).

<u>Literature</u>: [14],[16],[20],[22],[67].

5. THE RIEMANN-HILBERT BOUNDARY VALUE PROBLEM

When the rational $n \times n$ matrix function $W(\lambda)$ is given by the matrix realization $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$, then $W(\lambda)C(\lambda I_m - A^{\times})^{-1} = C(\lambda I_m - A)^{-1}$. Evaluation at $x \in \mathbb{C}^m$ yields

$$(5.1) \qquad W(\lambda)C(\lambda I_m - A^{\times})^{-1}x = C(\lambda I_m - A)^{-1}x.$$

This identity can be used to analyse the Riemann-Hilbert boundary value problem

$$(5.2) \qquad W(\lambda)\Phi_+(\lambda) = \Phi_-(\lambda), \qquad \lambda \in \Gamma.$$

For the case when the contour Γ is the (extended) real line, the details are as follows.

Suppose A and $A^\times$ have no real eigenvalues. In other words $W(\lambda)$ has no poles on the real line and det $W(\lambda)$ does not vanish for real λ. Let P and $P^\times$ be the spectral projections corresponding to the eigenvalues located in the (open) upper half plane of A and $A^\times$, respectively. Put $M = \operatorname{Im}P$ and $M^\times = \operatorname{Ker}P^\times$. If $x \in M \cap M^\times$, then $\Phi_+(\lambda) = C(\lambda I_m - A^\times)^{-1}x$ is analytic in the open upper half plane, continuous in the closed upper half plane and vanishes at ∞. Analogously $\Phi_-(\lambda) = C(\lambda I - A)^{-1}x$ is analytic in the open lower half plane, continuous in the closed lower half plane and vanishes at ∞. Also it is obvious from (5.1) that (5.2) holds (on the real line). This means that the pair (Φ_+, Φ_-) is a solution of the Riemann-Hilbert boundary value problem (5.2). Letting x range through $M \cap M^\times$ one obtains all solutions of (5.2). Also, different vectors $x \in M \cap M^\times$ correspond to different solutions (Φ_+, Φ_-). For generalizations to the non-rational case, see [18] and [21].

Literature: [18],[21].

Background material: [57],[93],[96],[98].

6. THE FACTORIZATION PRINCIPLE

Suppose the operator functions $W_1(\lambda)$ and $W_2(\lambda)$ are given by

$$W_j(\lambda) = I_Y + C_j(\lambda I_{X_j} - A_j)^{-1}B_j, \qquad \lambda \in \rho(A_j);\ j = 1,2,$$

and put $W(\lambda) = W_1(\lambda)W_2(\lambda)$. Then

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B, \qquad \lambda \in \rho(A_1) \cap \rho(A_2) \subset \rho(A),$$

where $X = X_1 \oplus X_2$ and

$$A = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad C = (C_1 \quad C_2).$$

The systems theoretical background of this "product rule" is series connection or cascade synthesis (cf., e.g., [86]).

Observe that X_1 is an invariant subspace for A and X_2 is one for $A^{\times}$. The latter is clear from

$$A^{\times} = \begin{pmatrix} A_1^{\times} & 0 \\ -B_2C_1 & A_2^{\times} \end{pmatrix}.$$

So X is the direct sum of an invariant subspace for A and an invariant subspace for $A^{\times}$. Further analysis leads to the following counterpart of the product rule.

<u>Factorization Principle</u>. Let

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B, \qquad \lambda \in \rho(A),$$

let M be an invariant subspace for A, let $M^{\times}$ be an invariant subspace for $A^{\times}$ and suppose $X = M \oplus M^{\times}$ ('matching'). Put

$$W_1(\lambda) = I_Y + C(\lambda I_X - A)^{-1}(I_X - \Pi)B, \qquad \lambda \in \rho(A),$$

$$W_2(\lambda) = I_Y + C\Pi(\lambda I_X - A)^{-1}B, \qquad \lambda \in \rho(A),$$

where Π is the projection of X onto $M^{\times}$ along M. Then

$$W(\lambda) = W_1(\lambda)W_2(\lambda), \qquad \lambda \in \rho(A).$$

This Factorization Principle first appeared in [23] and [14]. Earlier, less general, versions may be found in [29], [33], [82], [103] and [106].

The Factorization Principle can be used to construct factorizations of operator or matrix functions having prescribed special features. Options that are available in this context are

(a) the choice of the realization of $W(\lambda)$,

(b) the choice of the matching invariant subspaces M and $M^{\times}$.

Applications will be discussed in the next section.

Literature: [14],[23],[29],[33],[67],[82],[103],[106],[107].

Further reading: [4],[11],[12],[37],[60],[110].

7. APPLICATIONS OF THE FACTORIZATION PRINCIPLE

7.1. Minimal factorization

Let $W(\lambda)$ be a proper rational $n\times n$ matrix function with $W(\infty) = I_n$. The smallest possible non-negative integer m for which $W(\lambda)$ admits a (matrix) realization

$$W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B \tag{7.1}$$

(so with the size of A equal to $m\times m$) is called the McMillan degree of $W(\lambda)$ and denoted by $\delta(W)$. It is equal to the total number of poles of $W(\lambda)$ counted according to pole multiplicity. The latter notion is defined as follows. If λ_0 is a pole of $W(\lambda)$, the pole multiplicity (or local degree) of λ_0 is the rank of the block Toeplitz matrix

$$\begin{pmatrix} W_{-p} & W_{-p+1} & \cdots & W_{-1} \\ 0 & W_{-p} & \cdots & W_{-2} \\ \vdots & & & \vdots \\ 0 & & \cdots\ 0 & W_{-p} \end{pmatrix},$$

where $(\lambda-\lambda_o)^{-1}W_{-1}+\ldots+(\lambda-\lambda_o)^{-p}W_{-p}$ is the principal part of the Laurent expansion of $W(\lambda)$ at λ_o.

The realization (7.1) is said to be minimal if $m = \delta(W)$. An equivalent requirement is that the realization is both observable and controllable in the sense of systems theory. By the well-known state space isomorphism theorem, minimal realizations are essentially unique: If (8.1) is a minimal realization of $W(\lambda)$, then all possible minimal realizations of $W(\lambda)$ can be obtained by replacing A, B and C by (respectively) $S^{-1}AS$, $S^{-1}B$ and CS, where S is any invertible $m\times m$ matrix. (For a thorough discussion of minimality in a more general setting, see [50] - [54] and [84].)

From the product rule discussed in Section 6 it is clear that the McMillan degree enjoys a sublogarithmic property: If $W(\lambda) = W_1(\lambda)W_2(\lambda)$, then $\delta(W) \leqq \delta(W_1) + \delta(W_2)$. In network theory one is interested in factorizations for which equality holds (no 'pole/zero cancellations'). Such factorizations are called minimal.

The Factorization Principle can be used to describe all possible minimal factorizations of a given proper rational matrix function $W(\lambda)$ (with $W(\infty)$ equal to I or at least invertible). The situation is as follows (see [14] and [23]). If (7.1) is a minimal realization, then there is a one-to-one correspondence between all possible minimal factorizations of $W(\lambda)$ and all possible pairs of subspaces $M, M^\times$ with

(7.2) $\quad AM \subset M, \quad A^\times M^\times \subset M^\times, \quad \mathbb{C}^m = M \oplus M^\times,$

and given such a pair, the factors of the associated minimal factorization are as described in the Factorization Principle. (Here no distinction is made between the factorizations $W(\lambda) = W_1(\lambda)W_2(\lambda)$ and $W(\lambda) = W_1(\lambda)T.T^{-1}W_2(\lambda)$.)

The problem of how to compute minimal factorizations in practice (using the above result) has been discussed in [23]. The accuracy of the method presented there depends on the minimal angle between the invariant subspaces M and $M^{\times}$ (small angles are bad). Related questions concerning stability and perturbation problems are analysed in [14]. In this context the notion of a stable invariant subspace (of a matrix) plays an essential role (see [13], [14] and [35]).

The result about minimal factorization formulated in the third paragraph of this subsection implicitly gives a criterion for irreducibility. We call $W(\lambda)$ <u>irreducible</u> if $W(\lambda)$ does not admit a non-trivial minimal factorization. If (7.1) is a minimal realization, then $W(\lambda)$ is irreducible if and only if the only pairs of subspaces satisfying (7.2) are (0), $\mathbb{C}^m$ and $\mathbb{C}^m$, (0). For example, let m be a positive integer and write

$$W(\lambda) = \begin{pmatrix} 1 & \lambda^{-m} \\ 0 & 1 \end{pmatrix}.$$

The McMillan degree of $W(\lambda)$ is equal to the pole multiplicity of the origin as a pole of $W(\lambda)$, that is m. Define the $m\times m$ matrix A, the $m\times 2$ matrix B and the $2\times m$ matrix C by

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $W(\lambda) = I + C(\lambda - A)^{-1}B$ is a minimal realization. Clearly

$BC = 0$, and so $A^{\times} = A$. Since the nilpotent Jordan block A is unicellular, it follows that $W(\lambda)$ is irreducible (regardless of how big one takes m).

Literature: [13],[14],[23],[35],[38],[67].

Further reading: [2],[4],[37],[38],[39],[42],[50],[51],[52],[53],[54],[84],[100],[101],[108],[109],[110],[112].

Background material: [8],[26],[48],[73],[76],[77].

7.2. Complete factorization

Suppose the rational $n\times n$ matrix function $W(\lambda)$ is given by (7.1) and assume that A or $A^{\times}$ is a diagonable matrix. Then $W(\lambda)$ admits a factorization of the form

$$(7.3) \qquad W(\lambda) = \left(I_n + \frac{1}{\lambda-\lambda_1}R_1\right)\cdots\left(I_n + \frac{1}{\lambda-\lambda_m}R_m\right),$$

where $\lambda_1,\dots,\lambda_m$ are the eigenvalues of A (counted according to multiplicity) and $R_1,\dots,R_m$ are $n\times n$ matrices of rank at most one. If, in addition, the realization (7.1) is minimal, then all the matrices $R_1,\dots,R_m$ have precisely rank one, $\lambda_1,\dots,\lambda_m$ are the poles of $W(\lambda)$ counted according to pole multiplicity and the factorization (7.3) is minimal. In general, such 'complete factorizations' are not unique and the order in which the poles $\lambda_1,\dots,\lambda_m$ appear may vary. On the other hand, examples can be constructed in which the poles can appear in one (or some) specific order(s) only.

The proof of the result described above may be found in [14]. It is based on a repeated application of the Factorization Principle and the following simple observation about matrices which is interesting in its own right. Suppose M and N are $m\times m$ matrices, M diagonable and N arbitrary. Then there exists an invertible $m\times m$ matrix S such that $S^{-1}MS$ is an upper triangular and $S^{-1}NS$ is a lower triangular matrix.

Literature: [14],[39],[103].

7.3. Canonical Wiener-Hopf factorization

Consider the rational n×n matrix function given by the matrix realization (7.1). Suppose A has no eigenvalues on a given contour in the Riemann sphere, for definiteness the (extended) real line, say. Also assume that det $W(\lambda)$ does not vanish on the real line or, equivalently, $A^\times$ has no real eigenvalues. Let M and $M^\times$ be the spectral subspaces corresponding to the eigenvalues of A and $A^\times$ located in the upper and lower half plane, respectively. Then $AM \subset M$ and $A^\times M^\times \subset M^\times$. If $\mathbb{C}^m = M \oplus M^\times$ (matching), one can apply the Factorization Principle to get a factorization

$$(7.4) \qquad W(\lambda) = W_-(\lambda)W_+(\lambda)$$

with rational factors $W_-(\lambda)$ and $W_+(\lambda)$. In view of the special choice of M and $M^\times$, these factors have the following properties: $W_-(\lambda)$ has no poles and takes invertible values in the closed lower half plane (including the point ∞) and $W_+(\lambda)$ has no poles and takes invertible values in the closed upper half plane (including ∞). Thus the factorization (7.4) obtained in this way is a so called (right) canonical Wiener-Hopf factorization of $W(\lambda)$ with respect to the real line.

Further analysis leads to the following result (see [14]). Suppose $W(\lambda)$ is given by the matrix realization (7.1), where A has no eigenvalues on the real line. Then $W(\lambda)$ admits a canonical Wiener-Hopf factorization (with respect to the real line) if and only if

(1) $A^\times$ has no real eigenvalues (i.e. det $W(\lambda)$ does not vanish on the real line),

(2) $\mathbb{C}^m = M \oplus M^\times$,

where M and $M^{\times}$ are the spectral subspaces introduced above. In that case, the Wiener-Hopf factors $W_{-}(\lambda)$ and $W_{+}(\lambda)$ are given by

$$W_{-}(\lambda) = I_n + C(\lambda I_m - A)^{-1}(I_m - \Pi)B,$$

$$W_{+}(\lambda) = I_n + C\Pi(\lambda I_m - A)^{-1}B,$$

and their inverses by

$$W_{-}(\lambda)^{-1} = I_n - C(I_m - \Pi)(\lambda I_m - A^{\times})^{-1}B,$$

$$W_{+}(\lambda)^{-1} = I_n - C(\lambda I_m - A^{\times})^{-1}\Pi B,$$

where Π is the projection of $\mathbb{C}^m$ onto $M^{\times}$ along M.

An analogous theorem holds of course for left canonical Wiener-Hopf factorization. For a generalization to the non-rational case, see [17], [19] and [22].

Literature: [14],[17],[19],[22],[67].

Further reading: [102].

Background material: [36],[49],[57],[59],[98],[111].

7.4. Non-canonical Wiener-Hopf factorization

Every rational n×n matrix function whose determinant does not vanish on a given contour in the Riemann sphere admits a possibly non-canonical Wiener-Hopf factorization with respect to that contour. The difference between canonical and (possibly) non-canonical Wiener-Hopf factorization is that in the latter there is an additional 'middle term' between the factors $W_{-}(\lambda)$ and $W_{+}(\lambda)$:

$$W(\lambda) = W_{-}(\lambda)D(\lambda)W_{+}(\lambda).$$

This 'middle term' is of a very special kind. In the case (considered here) when the contour is the (extended) real line, it is a diagonal n×n matrix function of the form

$$D(\lambda) = \begin{pmatrix} \left(\frac{\lambda-i}{\lambda+i}\right)^{\kappa_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \quad \left(\frac{\lambda-i}{\lambda+i}\right)^{\kappa_n} \end{pmatrix},$$

with $\kappa_1 \leqq \kappa_2 \leqq \cdots \leqq \kappa_n$. The integers $\kappa_1,\ldots,\kappa_n$ are unique (the factors $W_-(\lambda), W_+(\lambda)$ not) and they are called the <u>factorization indices</u>. Of course canonical Wiener-Hopf factorization corresponds to the situation where all these factorization indices happen to be zero. Except for special cases, the traditional methods (algorithms) for finding Wiener-Hopf factorizations (see, for instance, [36]) do not yield explicit formulas.

Assume $W(\lambda)$ has the matrix realization (7.1) and suppose A and $A^\times$ have no real eigenvalues. Then one can construct a Wiener-Hopf factorization of $W(\lambda)$ in terms of A, B, C and the spectral subspaces M and $M^\times$ introduced in subsection 7.3. The construction is based on a detailed analysis of $M \cap M^\times$ and $\mathbb{C}^m/M+M^\times$ and provides explicit formulas for the factors and the factorization indices. For example, if

$$t = \dim \frac{M \cap M^\times}{M \cap M^\times \cap \operatorname{Ker} C}$$

and α_k is the number of non-negative integers p such that

$$\dim \frac{M \cap M^\times \cap \operatorname{Ker} C \cap \operatorname{Ker} CA \cap \ldots \cap \operatorname{Ker} CA^{p-1}}{M \cap M^\times \cap \operatorname{Ker} C \cap \operatorname{Ker} CA \cap \ldots \cap \operatorname{Ker} CA^{p}} \geqq k,$$

then $-\alpha_1,\ldots,-\alpha_t$ are the (strictly) negative factorization indices of $W(\lambda)$. There is an analogous description of the positive factorization indices involving the subspaces $M+M^\times + \operatorname{Im} B + \operatorname{Im} AB + \ldots + \operatorname{Im} A^q B$. For details and infinite dimensional generalizations, see [17] and [19].

Literature: [17],[19],[36],[67].

Further reading: [71],[101].

Background material: [49],[57],[59],[98].

7.5. Self-adjoint rational matrix functions

A rational $n\times n$ matrix function $W(\lambda)$ is called self-adjoint if $W(\lambda) = W(\bar{\lambda})^*$. Minimal factorization of such matrix functions having various symmetry properties can be studied in great detail with the help of the Factorization Principle (see [100] and [101]). An important tool in this context is the theory of finite dimensional indefinite inner product spaces as presented by I. Gohberg, P. Lancaster and L. Rodman in [61]. The reason for this lies in the following observation (see [28] and [45]). Suppose (7.1) is a minimal realization of the self-adjoint matrix function $W(\lambda)$. Then $W(\lambda) = I_n + B^*(\lambda I_m - A^*)C^*$ is a minimal realization too. By the state space isomorphism theorem there exists a unique invertible $m\times m$ matrix H such that $A^* = HAH^{-1}$, $C^* = HB$ and $B^* = CH^{-1}$. Taking adjoints and using the unicity of H, one sees that H is self-adjoint. The operators A and $A^\times$ are self-adjoint with respect to the indefinite inner product induced by H.

The existence of (non-trivial) minimal factorizations can be established here by showing that there is 'automatic matching' between certain maximal invariant subspaces satisfying a positivity condition. Also a symmetric version of Wiener-Hopf factorization, sometimes called Nikolaicuk-Spitkovski factorization, can be constructed explicitly. For details, see [101] and [71].

Literature: [28],[45],[61],[71],[99],[100],[101].

7.6. Invariant subspaces and characteristic operator functions

In the finite dimensional context of Subsection 7.1, there is a one-to-one correspondence between pairs of matching invariant subspaces (of matrices) and minimal factorizations (of rational matrix functions of the form (7.1)). There do exist infinite dimensional versions of this result, particularly in the theories of characteristic operator functions as developed in [29], [33], [34] and [106]. We shall give some details concerning the Livsic-Brodskii theory [29] and discuss an application to the Volterra integral operator on $L_2[0,1]$.

The Livsic-Brodskii characteristic operator function has the form

$$(7.6.) \qquad W(\lambda) = I_Y + 2iB^*(\lambda I_X - A)^{-1}B,$$

where X and Y are complex Hilbert spaces, B^* is the adjoint of B and BB^* is the imaginary part of A. For such functions one can introduce a notion of 'regular factorization' which generalizes that of a minimal factorization (see [29], Section I.5; cf. also [14], Section 4.3). Observe that $A^\times = A - 2iB^\times B = A^*$ and so, for a given invariant subspace M of A one has 'automatic matching' with the invariant subspace $M^\times = M$ of $A^\times = A^*$. Suppose the realization (7.6) is minimal in the sense that ImB + ImAB + $\mathrm{Im}A^2B + \ldots$ is dense in X. Then there is a one-to-one correspondence between the invariant subspaces of A and the regular factorizations of $W(\lambda)$. The proof of this uses an infinite dimensional version of the state space isomorphism theorem ([29], Theorem 3.2).

As an application, we sketch the proof given in [25] of the well-known fact that the Volterra integral operator V on L_2 [0,1],

$$(Vf)(t) = \int_0^t f(s)ds,$$

is unicellular (see also [31] and [32]). One can 'inbed' V, or rather $A = iV$, in a realization of the type discussed in the preceding paragraph by letting B be the operator from $\mathbb{C}$ into $L_2[0,1]$ which assigns to $z \in \mathbb{C}$ the constant function with value $z/\sqrt{2}$. The 'characteristic function' of A, that is the (scalar) function defined by (7.6), is then $e^{i/\lambda}$. With the help of an integral representation theorem for non-negative harmonic functions, it can be shown that the regular factorizations of this characteristic function are of the form $e^{i/\lambda} = e^{i(1-\alpha)/\lambda} . e^{i\alpha/\lambda}$ with $0 \leqq \alpha \leqq 1$. This corresponds to the fact that M is an invariant subspace of V if and only if there exists $\alpha \in [0,1]$ with

$$M = \{f \in L_2[0,1] \mid f = 0 \text{ a.e. on } [0,\alpha]\}.$$

The unicellularity of V (that is, the total ordering of its lattice of invariant subspaces) is now obvious. A different proof, based on Titchmarsh's theorem on convolutions, may be found in [74]. See also [75], and the references given there.

<u>Literature</u>: [14],[25],[29],[31],[32],[33],[34],[67],[74],[75], [105],[106].

<u>Further reading</u>: [11],[12],[30],[44],[58],[60],[66],[87].

8. WIENER-HOPF EQUATIONS

Realizations (and their time domain counterparts, (spectral) exponential representations) are useful tools in dealing with various classes of integral equations and their discrete analogues. We shall illustrate this here for the (vector valued) Wiener-Hopf integral equation. Details and material about other equations (singular integral equations, block Toeplitz equations, convolution equations on a finite interval etc.), may be found in [14], [15], [16] and [18].

Consider the (vector-valued) Wiener-Hopf integral equation

$$(8.1) \qquad \phi(t) - \int_0^\infty k(t-s)\phi(s)ds = f(t), \qquad t \geqq 0,$$

with $k \in L_1^{n\times n}(-\infty,\infty)$ and $f,\phi \in L_p^n[0,\infty)$. Here n and p are fixed, $1 \leqq p \leqq \infty$. The <u>symbol</u> of (8.1) is the n×n matrix function $W(\lambda) = I_n - \hat{k}(\lambda)$, where $\hat{k}(\lambda)$ denotes the Fourier transform of k (cf. Section 4). We say that (8.1) is <u>uniquely solvable</u> (in $L_p^n[0,\infty)$) if for every f in $L_p^n[0,\infty)$ the equation (8.1) has a unique solution ϕ in $L_p^n[0,\infty)$. This means that the integral operator I-K on $L_p^n[0,\infty)$ associated with (8.1) is invertible.

From [57] it is known that (8.1) is uniquely solvable if and only if the symbol $W(\lambda)$ admits a (right) canonical Wiener-Hopf factorization (with respect to the real line). Also, in that case, the solution of (8.1) can be described in terms of the Wiener-Hopf factors of $W(\lambda)$. In general, however, the results obtained in this way are not very explicit.

Assume now that $W(\lambda)$ is rational and let $W(\lambda) = I_n + C(\lambda I_m - A)^{-1}B$ be a matrix realization such that A has no real eigenvalues (cf. Section 4). Combining the results of [57] with those outlined in Subsection 7.3., one gets that (8.1) is uniquely solvable if and only if

(1) $A^\times$ has no real eigenvalues (i.e., det $W(\lambda)$ does not vanish on the real line),

(2) $\mathbb{C}^m = M \oplus M^\times$.

Here (as in Subsection 7.3) M and $M^\times$ are the spectral subspaces corresponding to the eigenvalues of A and $A^\times$ located in the upper and lower half plane, respectively. Also, under these circumstances, the unique solution of (8.1) is given by

$$\phi(t) = f(t) - \int_0^\infty \gamma(t,s)f(s)ds, \qquad t \geqq 0,$$

$$\gamma(t,s) = \begin{cases} -iCe^{-itA^\times}\Pi e^{isA^\times}B, & s < t, \\ iCe^{-itA^\times}(I_m-\Pi)e^{isA^\times}B, & s > t, \end{cases}$$

where Π is the projection of $\mathbb{C}^m$ onto $M^\times$ along M.

Unique solvability of (8.1), in other words invertibility of the operator $I - K$ associated with (8.1), is of course a very special situation. It is known from [57] that $I - K$ is a Fredholm operator if and only if det $W(\lambda)$ does not vanish on the real line (i.e., (1) is satisfied). In principle, the result on non-canonical Wiener-Hopf factorization alluded to in Subsection 7.4 (together with those of [57]) can then be used to obtain a detailed Fredholm theory for $I - K$. There are, however, more efficient ways of achieving this. We have in mind the coupling method presented in [18] and the reduction to linear systems with appropriate boundary conditions described in [15].

Assume that (1) is satisfied and introduce the operator $J^\times = P^\times|M\colon M \to \operatorname{Im} P^\times$. Here M is as above and $P^\times$ is the spectral projection corresponding to the eigenvalues of $A^\times$ lying in the upper half plane. Applying the coupling method, one proves that $J^\times$ and $I-K$ are 'matricially coupled', i.e., they satisfy a 'coupling relation' of the type

$$\begin{pmatrix} I-K & * \\ * & * \end{pmatrix}^{-1} = \begin{pmatrix} * & * \\ * & J^\times \end{pmatrix}.$$

This implies (cf. [18], Section I.1) that $I - K$ and $J^\times$ are equivalent after (two-sided) extension with suitable identity operators (see also [24]). But then the Fredholm properties of $I - K$ and $J^\times$ are the same. It follows, for instance, that $\dim \operatorname{Ker}(I-K) = \dim(M \cap M^\times)$ and $\operatorname{codim} \operatorname{Im}(I-K) = \operatorname{codim}(M + M^\times)$.

So far about the coupling method. The reduction to linear systems (see [15]) is based on the observation that in the situation considered here, (8.1) is equivalent to the linear system

$$(8.2) \qquad \dot{x}(t) = -iA^{\times}x(t) + iBf(t), \quad \phi(t) = Cx(t) + f(t), \quad t \geqq 0,$$

with boundary condition

$$(8.3) \qquad x(0) \in M.$$

In other words, the integral operator I - K associated with (8.1) appears as the 'transfer operator' of the time-invariant linear system (8.2) with boundary condition (8.3). For a systematic study of transfer operators of possibly time varying systems with well-posed boundary conditions, see [50].

In the preceding paragraphs it was assumed that the symbol $W(\lambda)$ is rational. This was done for simplicity; the restriction is not essential. Allowing for infinite dimensional realizations (with bounded operators), everything works for symbols that are analytic in a neighbourhood of the extended real line (see [18] and [24]). An even more general situation, involving symbols that are analytic in a strip around the real line (but not necessarily at ∞), is treated in [21]. Here unbounded operators of a special type enter the picture (cf. Subsection 11.3 below).

Literature: [14],[15],[16],[18],[21],[24],[50],[57],[67].

Further reading: [56],[102].

Background material: [36],[49],[59],[83],[93],[96],[98], [104],[111].

9. TRANSPORT THEORY

Under certain assumptions the neutron (energy) transport equation

$$(9.1) \qquad \mu\frac{\partial\Psi(x,\mu)}{\partial x} + \Psi(x,\mu) = \int_{-1}^{+1} k(\mu,\mu')\Psi(x,\mu')d\mu', \quad x \geqq 0$$

can be reduced to a Wiener-Hopf integral equation with an operator valued kernel (cf. [41], [14], [91]). This equation can then be solved by constructing a canonical Wiener-Hopf factorization of its (operator valued) symbol. Here again the Factorization Principle comes into play and the problem is reduced to the question whether or not two specific spectral subspaces of two given self-adjoint operators match. One of these operators is the multiplication operator T on $L_2[-1,1]$, which is self-adjoint in the standard inner product on $L_2[-1,1]$, the other is a compact perturbation of T, which turns out to be self-adjoint with respect to an equivalent inner product. In the situation considered in [14], Chapter 6 the two spectral subspaces match indeed. This was proved by R.J. Hangelbroek and C.G. Lekkerkerker ([63], [64]; see also [78]) in connection with another method for solving the transport equation. More general circumstances in which the same type of matching occurs have been described by C. van der Mee ([91], [92]; see also [80], [81]). A non-trivial technical complication in the present context is that the factorization contour (here the imaginary axis) does not split the spectra of the operators involved (cf. Section 8.3. below).

Literature: [14],[41],[63],[64],[67],[78],[80],[81],[91],[92].

10. MODEL REDUCTION

The model reduction problem which comes up in many engineering applications is that of approximating a stable rational matrix function by one with smaller McMillan degree. Approximation is taken here in the Hankel norm sense. A more precise formulation reads as follows. Let $G(\lambda)$ be a stable rational matrix function of McMillan degree m. Here stable means that all poles of $G(\lambda)$ are in the open left half plane. Fix a positive real number σ and a natural number k, $k < m$. What one needs is:

(i) a description of all rational matrix functions $\hat{G}(\lambda) + F(\lambda)$, where $\hat{G}(\lambda)$ is rational with McMillan degree k, $F(\lambda)$ is anticausal (i.e., analytic and bounded on the closed left half plane) and

$$\sup_{-\infty<t<\infty} \| G(it) - \hat{G}(it) - F(it) \| \leqq \sigma,$$

(ii) necessary and sufficient conditions in order that such functions exist.

This problem and generalizations of it have been a central issue in several articles (cf. [1], [5], [6], [7], [47], [85]). It was K. Glover [47] who in this context used the notion of realization for the first time, thereby obtaining the most explicit results.

An alternative way of getting some of Glover's main results has been described by J.A. Ball and A.C.M. Ran. They use ideas of J.A. Ball and J.W. Helton [5] to reduce the model reduction problem to a Wiener-Hopf factorization problem of the following type: Given a left canonical Wiener-Hopf factorization $W(\lambda) = V_+(\lambda)V_-(\lambda)$, find necessary and sufficient conditions for the existence of a right canonical Wiener-Hopf factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$, and give explicit formulas for the factors $W_+(\lambda)$ and $W_-(\lambda)$ in terms of the matrices appearing in realizations of $V_-(\lambda)$ and $V_+(\lambda)$. The solution of Ball and Ran ([6], [7]) is based on the Factorization Principle and actually uses the material described in Subsection 7.3. A more general version of the problem involving the Nikolaicuk-Spitkovski factorization (cf. Subsection 7.5) will be treated by Ball and Ran in a forthcoming publication.

Literature: [1],[5],[6],[7],[47],[85].

11. GENERALIZATIONS AND RECENT DEVELOPMENTS

11.1. Non-invertible D

The 'standing assumption' in Section 2, holding that in the realization (1.1) always the identity operator (or matrix) is taken for D, was made partly for simplicity of exposition, partly because in some applications this situation arises in a natural way (Wiener-Hopf equations, inverse Fourier transforms). The situation when D is not the identity operator (or matrix) but still invertible does not differ essentially from the case $D = I$. Considerable new difficulties arise when the invertibility condition is dropped. Matrix realizations (1.1) where D is not invertible, or even non-square, have been investigated by N. Cohen [37] and P. Van Dooren [110].

The extreme case $D = 0$ comes up naturally in the theory of operator and matrix polynomials. Indeed, the inverse of a monic operator or matrix polynomial is of the form $C(\lambda-A)^{-1}B$, where A, B and C are subject to additional conditions. This fact plays an important role in the theory of operator or matrix polynomials as developed by I. Gohberg, P. Lancaster and L. Rodman (see [60], and the references given there).

Literature: [11],[12],[37],[60],[110].

11.2. The infinite dimensional case

In Sections 2 and 3, the realization problem and linearization were already discussed in a general, possibly infinite dimensional, context. Also we saw in Subsection 7.6 that certain aspects of minimal factorization have analogues in an infinite dimensional Hilbert space setting. Transport theory (Section 9) is situated in an infinite dimensional context too. Finally we note that much of the material concerning matrix realizations presented above allows for a generalization to the (non-rational) operator valued case.

For example, let $W(\lambda)$ be an operator function having a realization $W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B$, where A, B and C are bounded linear operators and A has no spectrum on the real line. Such a realization exists if and only if $W(\lambda)$ is analytic on a neighbourhood of the extended real line (cf. Section 2). It turns out (see [17], [19]) that $W(\lambda)$ admits a Wiener-Hopf factorization with respect to the real line (infinite dimensional version) if and only if

(1) $A^{\times}$ has no spectrum on the real line (i.e., $W(\lambda)$ is invertible for all real λ),

(2) $\dim(M \cap M^{\times}) < \infty, \quad \dim(X/M+M^{\times}) < \infty,$

where M and $M^{\times}$ are the spectral subspaces corresponding to the parts of the spectra of A and $A^{\times}$ lying in the open upper half plane. In other words, $M = \operatorname{Im}P$ and $M^{\times} = \operatorname{Ker}P^{\times}$, where P and $P^{\times}$ are the appropriate Riesz projections of A and $A^{\times}$, respectively. Just as was indicated in Subsections 7.3 and 7.4 for the rational case, explicit expressions for the Wiener-Hopf factors and the factorization indices can be given. Canonical Wiener-Hopf factorization occurs if and only if the spectral subspaces M and $M^{\times}$ match, i.e., $X = M \oplus M^{\times}$.

Literature: [14],[15],[16],[17],[18],[19],[24],[67].

11.3. Exponentially dichotomous operators

Until now all operators were supposed to be bounded. However, extending the original notion defined in Section 1, one can also consider realizations

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B \tag{8.4}$$

involving unbounded linear operators. This introduces considerable technical difficulties. For instance, in studying Wiener-Hopf factorizations, one encounters spectral splitting problems due to the presence of ∞ in the extended spectrum of an unbounded operator.

At the moment a satisfactory theory is available for the case when Y is finite dimensional, B and C are bounded and $-iA$ is an <u>exponentially dichotomous</u> operator. The latter means that $-iA$ is the direct sum of two (possibly unbounded) operators S_+ and S_- such that S_+ and $-S_-$ are infinitesimal generators of exponentially decaying C_0-semigroups. The projection associated with the direct sum decomposition is unique and called the <u>separating projection</u> for $-iA$. This terminology originates from the fact that the imaginary axis splits the finite spectrum of $-iA$. In general, though, the extended imaginary axis does not split the extended spectrum of $-iA$.

Basic for dealing with realizations of the type described above is the following lemma (see [20],[22]). Suppose $W(\lambda)$ is given by (8.1) with Y finite dimensional, B and C bounded and $-iA$ exponentially dichotomous. Then the following statements are equivalent:

(i) $\det W(\lambda) \neq 0, \quad \lambda \in \mathbb{R}$,

(ii) $A^\times$ has no (finite) real spectrum,

(iii) $-iA^\times$ is exponentially dichotomous.

The equivalence of (i) and (ii) is straightforward (cf. Section 4) and it is also trivial that (iii) implies (ii). The difficulty is in proving that (iii) is a consequence of (i). Indeed, one has to find a separating projection for $-iA^\times$ and this involves the spectral splitting of a possibly connected extended spectrum.

One final remark. The time domain counterparts of the realizations considered here are the (spectral) exponential representations involving C_0-semigroups alluded to at the end of Section 4.

<u>Literature</u>: [20],[21],[22],[67].

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