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SOME VERY EASY KNAPSACK/PARTITION PROBLEMS

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Some Very Easy Knapsack/Partition Problems

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Abstract

Consider the problem of partitioning a group of b indistinguishable objects into subgroups each of size at least ℓ and at most u. The objective is to minimize the additive separable cost of the partition, where the cost associated with a subgroup of size j is c(j). In the case that c(.) is convex, we show how to solve the problem in O(log u) steps. In the case that c(.) is concave, we solve the problem in $O(\min(\ell, b/u, (b/\ell)-(b/u), u-\ell))$ steps. This problem generalizes a lot-sizing result of Chand and has potential applications in clustering.

Consider the problem of partitioning a group of b objects into subgroups each of size at least ℓ and at most u. The objective is to minimize an additive cost $\Sigma_{j=\ell}^{u} c(j)x_j$ where c(.) is some real-valued function and x_j is the number of subgroups of size j. This problem may be expressed as the knapsack problem P below.

$$\text{Minimize} \quad \sum_{j=\ell}^{u} c(j)x_j \qquad \text{(P)}$$

$$\text{Subject to} \sum_{j=\ell}^{u} jx_j = b$$

$$x_j \geq 0 \text{ integer for } j = \ell,\ldots,u.$$

It is well known that the problem P may be solved in $O((u-\ell)b)$ steps via dynamic programming recursion. Moreover, if $b \geq u^2$, then P may be solved in $O((u-\ell)u)$ steps because the optimal solution to the associated group problem is feasible for P. (See Garfinkel and Nemhauser [1972] and Denardo and Fox [1979] for further details.)

The purpose of this note is to provide very efficient algorithms for the case that c(.) is either concave or convex. In particular, we show that we can solve the case of P in which c(.) is convex in O(log u) steps. This algorithm extends a previous algorithm by Chand [1972] for a variant of the discrete time EOQ model, as mentioned below.

In the case that c(.) is concave, we show how to solve the knapsack/partition problem in $O(\min(\ell, b/u, (b/\ell)-(b/u), u-\ell))$ steps. It is an open question as to whether the concave case can be solved in a number of steps that is polynomially bounded in log b.

In both the case that c(.) is convex and the case that c(.) is concave, if the number of subgroups in the partition is specified, then the resulting problem is solvable in O(1) steps.

The bounds on the number of steps are based on the computational model of counting each arithmetic operation as one step. If one counts the addition of x and y as log (x+y) steps, then we must add the assumption that c(.) is integral, and the convex case is solvable in $O((\log b)(\log c_{max}))$ steps, whereas the number of steps in the concave case should be multiplied by a factor of $(\log b)(\log c_{max})$.

We assume that the data for the problem is specified by the values (ℓ,u,b), and by an "oracle function" c. Thus the size of the data is $O(\log b)$. Thus the algorithms for the convex case is polynomial in the size of the data, whereas the algorithm in the concave case is not necessarily polynomial.

As an example of the knapsack/partition problem consider the problem of subdividing a group of b people into subcommittees, each consisting of between ℓ and u members. Suppose further that the "value" of a committee with j numbers is c(j), where the function c(.) is concave, reflecting decreasing marginal returns. Here the objective is to maximize the value of the partition of people into subcommittees.

As another example, suppose that a data base includes data on b distinct users, and we are interested in aggregating the b users into subgroups each with a number of users between ℓ and u. Suppose further that the cost of aggregating j users into a subgroup is c(j), reflecting a loss in accuracy but a gain in computational convenience. Then the optimum solution to the knapsack problem gives an estimate of how much aggregation is desirable.

A third example is an application to a variant of the EOQ model as described and solved by Chand. Chand addressed the usual EOQ model with two minor modifications. First, the number of "periods" of the problem is b (rather than ∞ as in the usual model). Secondly, each order interval must be an integer (rather than continuous) number of periods. In our knapsack model, x_j would represent the number of order intervals of j time periods.

The Convex Case

We first consider the case in which c(.) is strictly convex. (If c(.) were convex but not strictly convex we could replace c(.) by $c'(j) = c(j) + \varepsilon^j$ for a suitably small ε.)

Henceforth, we let c_j denote c(j). As a preliminary we define the parametric linear programming problem LP(M).

$$\text{Minimize } \sum_{j=\ell}^{u} c_j x_j \qquad \text{(LP(M))}$$

$$\text{Subject to } \sum_{j=\ell}^{u} j x_j = b$$

$$\sum_{j=\ell}^{u} x_j = M$$

$$x_j \geq 0 \text{ for } j = \ell, \ldots, u.$$

We denote each instance of P as a quadruple $\langle c, \ell, u, b \rangle$, and we denote each instance of LP(M) as a quintuple $\langle c, \ell, u, b, M \rangle$. The following lemma is implicit in the work of Sinha and Zoltners [1979] on the more general multiple choice knapsack problems. The proof in the special case below is more elementary than that of the more general problem.

LEMMA 1. Let $\langle c, \ell, u, b, M \rangle$ be an instance of LP(M) such that c is strictly convex with M integral. If $\ell > b/M$ or if $u < b/M$ then there is no feasible solution. Otherwise, let $t = \lfloor b/M \rfloor$. Then the unique optimal solution for $\langle c, \ell, u, b, M \rangle$ is x^* defined as follows:

$$x_j^* = \begin{cases} (t+1)M-b & \text{for } j=t \\ b - tM & \text{for } j=t+1 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. If $\ell > b/M$ or if $u < b/M$ it is clear that there is no feasible solution. If $\ell \leq b/M \leq u$, then it is easy to see that the above solution is feasible by considering separately the case that $t = b/M$ and the case $t > b/M$. (In the former case it is possible that $t+1 > u$.)

If $\ell = b/M$ or $u = b/M$, then the above solution x^* is the unique feasible solution and is thus optimal. Otherwise, Let $\pi_1 = c_{t+1} - c_t$ and let $\pi_2 = (t+1)c_t - c_{t+1}$. Then the reduced cost of c_j is $\bar{c}_j = c_j - j\pi_1 - \pi_2$. It is easy to verify that $\bar{c}_t = \bar{c}_{t+1} = 0$, and by the strict convexity of c it is easy to verify that $\bar{c}_j > 0$ for $j \leq t-1$ or $j \geq t+2$. Thus the basic feasible solution x^* with basic variables x_t and x_{t+1} is the unique optimal solution. □

In the following, we let x(M) denote the unique optimal solution for instance < c, ℓ, u, b, M > of LP(M). We observe that by Lemma 1 the solution is integer valued for each integral value of M. We also let z(M) denote the corresponding objective value.

THEOREM 1. Suppose that < c, ℓ, u, b> is an instance of problem P and that c is a convex function. If [b/u] = [b/ℓ] + 1, then there is no feasible solution to P. Otherwise, let r be chosen so that c(r)/r = min{c(j)/j : $\ell \le j \le u$} and let M' = b/r. Then x([M']) or x([M']) or both are optimal solutions for problem P.

PROOF. We first observe that the optimal solution to P is the best of the solutions {x(j) : $j \ge 1$}. In the case that [b/ℓ] = [b/u] + 1, there is no integer M satisfying $b/u \le M \le b/\ell$ and thus by Lemma 1 there is no feasible solution to P(M) for any M. If $[b/u] \le [b/\ell]$, then $b/u \le b/r \le b/\ell$ and thus P($\lfloor M' \rfloor$) or P($\lceil M' \rceil$) or both have feasible solutions.

The optimal solution to the linear relaxation of P is given by x_r = M' with all other values being 0. The optimality of x([M']) or x([M']) then follow from the convexity of the optimal value function z(M), which is minimized at the (possibly fractional) value M = M'. □

We note that we have dropped the assumption of strict convexity. As before, if c is convex and $c_r/r = \min(c_j/j : \ell \le j \le u)$, then we may perturb c to a strictly convex function c' with $c'_r/r = \min(c'_j/j : \ell \le j \le u)$. An optimal solution for (c', ℓ, u, b) will also be optimal for < c, ℓ, u, b >.

Because the function c(·) is convex, it follows that the function d(j) = c(j)/j for $\ell \le j \le u$ is quasi-convex (see, for example, Avriel [1976]). Therefore, if $d(r) \le d(r+1)$ it follows that $d(r) \le d(j)$ for all $j \ge r+1$. If $d(r) \ge d(r+1)$, it follows that $d(r+1) \le d(j)$ for all $j \le r$. One can thus find the integer value r with $\ell \le r \le u$ which minimizes d(j) in O(log(u-ℓ)) steps by using binary search. (Recall that each arithmetic operation is being counted as one step.) In the case that d is differentiable, then the minimum will occur at r = ℓ or r = u or at [r'] or [r'], where r' is a real number at which the derivative of d is 0. In many cases, finding the 0 of the derivative will be faster than binary search.

The Concave Case

Below we solve the case in which c(.) is strictly concave. We offer two different algorithms, the first of which takes $O(\min(\ell, b/u))$ steps and the second of which takes $O(\min((b/\ell) - (b/u), u-\ell))$ steps. Unfortunately, the author does not know of any algorithm than runs in time polynomial in log b. We do note that for $\ell = 1$ or $u = b$, the first algorithm runs in O(1) steps.

As a preliminary, we define the parametric integer program P(M) similarly to the problem LP(M) for the convex case

$$\text{Minimize} \quad \sum_{j=\ell}^{u} c_j x_j \qquad\qquad P(M)$$

$$\text{Subject to} \quad \sum_{j=\ell}^{u} j x_j = b$$

$$\sum_{j=\ell}^{u} x_j = M$$

$$x_j \geq 0 \quad \text{integer for } \ell \leq j \leq u.$$

LEMMA 2. Suppose that $\langle c, \ell, u, b \rangle$ is an instance of the Knapsack problem P and that c is strictly concave. Suppose further that M is an integer such that $b/u \leq M \leq b/\ell$. Then there is a unique optimal solution $\hat{x}$ for P(M) defined as follows:

(i) $\hat{x}_u = [(b - \ell M)/(u - \ell)]$,

(ii) $\hat{x}_\ell = [(uM - b)/(u - \ell)]$,

(iii) If $r \equiv (b - \ell M) \bmod (u - \ell)$ and $1 \leq r \leq u - \ell - 1$ then $\hat{x}_{\ell+r} = 1$,

(iv) $\hat{x}_j = 0$ otherwise.

PROOF. Let x* be an optimum solution to P(M) and let $k = x^*_{\ell+1} + \ldots + x^*_{u+1}$. We first show that $k \leq 1$. Suppose otherwise that $k \geq 2$. Choose s, t so that $\ell + 1 \leq s, t \leq u - 1$ and either (1) $x^*_s, x^*_t \geq 1$ and $s \neq t$ or else (2) $s = t$ and $x^*_s \geq 2$. Let x' be obtained from x* by decrementing both x_s and x_t by 1 (i.e., if s = t we decrement x_s by 2), and incrementing x_{s-1} and x_{t+1} by 1. Let z* and z' be the objective values for x* and x' respectively. Then x' is feasible for P(M). Moreover,

$$z^* - z' = c(s) + c(t) - c(s - 1) - c(t - 1),$$

and $z^* - z' > 0$ by the strict concavity of $c(.)$, contradicting the optimality of x^*. Thus we have proved that $k \leq 1$.

If $k = 0$, then x^*_ℓ and x^*_u are determined uniquely by the equations "$\ell x^*_\ell + u x^*_u = b$", and "$x^*_\ell + x^*_u = M$". Thus $x^*_\ell = (uM - b)/(u - \ell)$ and $x^*_u = (b - \ell M)/(u - \ell)$. Since x^*_u is integral it follows that $(u - \ell)$ is a divisor of $b - \ell M$. Thus $x^* = \hat{x}$.

If $k = 1$, let r be the index such that $1 \leq r \leq u - \ell$ and $x^*_{\ell+r} = 1$. Then $\ell x^*_\ell + u x^*_u = b - \ell - r$ and $x^*_\ell + x^*_u = M - 1$. Solving for x^*_ℓ and x^*_u we get that $x^*_\ell = (Mu - b - (u - \ell - r))/(u - \ell)$ and $x^*_u = (b - M\ell - r)/(u - \ell)$. Since x^*_u is integral it follows that x^* satisfies (iii) and (iv), and thus $x^* = \hat{x}$. □

In the above lemma we have proved that $\Sigma(x_j : \ell+1 \leq j \leq u-1) \leq 1$. Thus, an optimal solution x^* may be determined uniquely as a function of x^*_ℓ and x^*_u. In the next lemma, we show that the optimal solution x^* is almost uniquely determined by the value x^*_u.

LEMMA 3. Suppose that $\langle c, \ell, u, b \rangle$ is an instance of problem P and that c is strictly concave. Suppose further that x^* is optimal for P. Then at least one of (i), (ii), (iii) and (iv) is true.

(i) $x^*_\ell = 0$,

(ii) $x^*_\ell = (b - u x^*_u)/\ell$,

(iii) $x^*_\ell = \lceil (b - u x^*_u - u + 1)/\ell \rceil$,

or (iv) $x^*_\ell = \lfloor (b - u x^*_u - \ell - 1)/\ell \rfloor$.

PROOF. Let $k = x^*_{\ell+1} + \ldots + x^*_{u-1}$. By Lemma 2 we know that $k = 0$ or 1. If $k = 0$ then (ii) holds. Henceforth we consider the case when $k = 1$ and when $x^*_{\ell+r} = 1$ for $1 \leq r \leq u - \ell - 1$. Let $b' = b - u x^*_u$.

Then $x^*_\ell = (b' - \ell - r)/\ell$. If $1 \leq r \leq \ell - 1$, then (iv) holds, If $u - 2\ell \leq r \leq u - \ell - 1$, then (iii) holds. Let us now assume that $\ell \leq r \leq u - 2\ell - 1$ and that $x^*_\ell \geq 1$ and we will derive a contradiction.

Let x' be obtained from x^* by incrementing x_ℓ by 1, incrementing x_r by 1, and decrementing $x_{r+\ell}$ by 1. Let x'' be obtained from x^* by decrementing x_ℓ and $x_{r+\ell}$ by 1 and incrementing $x_{r+2\ell}$ by 1. Let z', z'' and z^* be the objective values for x', x'' and x^* respectively. Then

$$2z^* - z' - z'' = 2c(r + \ell) - c(r) - c(r + 2\ell),$$

and thus

$$(z^* - z') + (z^* - z'') > 0$$

by the strict concavity of $c(.)$. It follows that $z' < z^*$ or $z'' < z^*$, contradicting the optimality of z^*. Thus the lemma is true.

Lemma 2 suggests the following method for solving problem P : solve P(M) for all integral M such that $b/u \leq M \leq b/\ell$ and choose the best of these solutions. Lemma 3 suggests the following method for solving P : for each integral value s with $0 \leq s \leq b/u$ let x^s be the best of the solutions (i), (ii), (iii) and (iv) of Lemma 3 with $x_u^s = s$. Then choose the best of the solutions of $\{x^1,\ldots,x^{b/\ell}\}$.

In order to improve the computational bounds of these two procedures, we show that the range of values for M and s can be limited further.

LEMMA 4. Suppose that $\langle c, \ell, u, b \rangle$ is an instance of problem P and that $c(.)$ is strictly concave. Suppose further that x^* is an optimal solution and that $M = x_\ell^* + \ldots + x_u^*$. Then

(i) If $c_\ell/\ell < c_u/u$, then $0 \leq x_u^* \leq \ell - 1$ and $-(u - \ell) + (b/\ell) < M \leq b/\ell$.

(ii) If $c_\ell/\ell > c_u/u$, then $[(b + \ell + 1)/u] - \ell - 1 \leq x_u^* \leq b/u$ and $(b/u) \leq M < (u - \ell) + (b/u)$.

PROOF. We note first that in any feasible solution $(b/u) \leq M \leq (b/\ell)$ and $0 \leq x_u^* \leq b/u$. If $c_\ell/\ell < c_u/u$ and $x_u^* \geq \ell$, then we can find an improved solution x' by decrementing x_u^* by ℓ and incrementing x_ℓ^* by u, contradicting the optimality of x^*. If $x_u^* \leq \ell - 1$, then by (i) of Lemma 2 it follows that $M > -(u - \ell) + (b/\ell)$.

If $c_\ell/\ell > c_u/u$ and $x_\ell^* \geq u$, we can find an improved solution x' by decrementing x_ℓ^* by u and incrementing x_u^* by ℓ. Thus $x_\ell^* \leq u - 1$ and by (ii) of Lemma 2 it follows that $M < (u - \ell) + (b/u)$. In addition, since

$$b = ux_u^* + (\ell + r)x_{\ell+r}^* + \ell x_\ell^* \leq ux_u^* + (u - 1) + \ell(u - 1),$$

it follows that

$$x_u^* \geq \lceil (b + \ell + 1)/u \rceil - (\ell + 1).$$ □

We observe that if $c_\ell/\ell = c_u/u$, then there may be multiple optimum. In this case, there is an optimum solution so that the conclusion (i) of Lemma 4 holds and "another" optimum solution so that the conclusion (ii) of Lemma 4 holds. The proof of this fact follows from the same "interchange" argument as in the proof of Lemma 4.

THEOREM 2. Suppose that $< c, \ell, u, b >$ is an instance of problem P and that c(.) is strictly concave. Then we may solve P in $O(\min(\ell, b/u, (b/\ell) - (b/u), u - \ell))$ steps.

PROOF. The first method is to solve P(M) for $b/u \leq M \leq b/\ell$ and choose the best of these solutions. Moreover, by Lemma 4 we may further restrict our search to a range of at most $u - \ell$ consecutive integers. Thus this procedure is $O(\min((b/\ell) - (b/u), u - \ell))$ steps.

The second method is to consider the four solutions determined upon setting $x_u = s$ as provided by Lemma 5. The best of these x^s is calculated in O(1) steps. Moreover, we can restrict our search to at most $\min(\ell, b/u)$ values of x_u by Lemma 4. Thus determining the best solution x^s for this range of the parameter s takes $O(\min(\ell, b/u))$ steps, completing the proof.

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