ECONOMETRIC INSTITUTE

OPTIMAL HOUSING MAINTENANCE UNDER UNCERTAINTY

A.C.F. VORST

REPORT 8407/M

Erasmus

ERASMUS UNIVERSITY ROTTERDAM - P.O. BOX 1738 - 3000 DR ROTTERDAM - THE NETHERLANDS

OPTIMAL HOUSING MAINTENANCE UNDER UNCERTAINTY

A.C.F. Vorst
Econometric Institute, Erasmus University Rotterdam
P.O. Box 1738, 3000 DR Rotterdam, The Netherlands

ABSTRACT. In this paper we give the optimal maintenance policy of a building where we let the rent depend on the quality of the building. We assume that the quality of the building can be described by a stochastic process on which the maintenance policy has an influence. Furthermore we give rules for when the landlord has to sell the building or when he has to rent it. From these rules it follows that under uncertainty the landlord is more prepared to rent the building than without uncertainty.

Contents

1. Introduction
2. The model of housing maintenance
3. The solution of the fixed horizon time problem
4. The solution of the free horizon time problem
5. The influence of the stochastic term
6. Some extensions of the model
7. Conclusions
Appendix
References

1. Introduction

In Arnott, Davidson and Pines (1983) the optimal maintenance policy of a building has been studied. In studying this optimal maintenance policy it was necessary to solve an optimal control problem. The difficulty with these optimal control problems always lies in the fact that it is very hard to find an explicit solution of such a problem. This is certainly true if one doesn't specify all functions involved explicitly, but only some of their properties. In that case the best possible result seems to be about the asymptotic behaviour of the problem. These are exactly the kind of results which can be found in the study of Arnott, Davidson and Pines. They describe very precisely nice general results about their system.

In this paper we will be more specific about the functions involved in the optimal maintenance problem. On one hand this has the disadvantage of being less general, but on the other hand we give a very precise recipe for the optimal maintenance policy. One may hope that, if the actual model is slightly different from ours, one can apply the same kind of analysis.

Concerning another point our study is more general, because we do not consider a deterministic model but a stochastic optimal control problem. However, this stochastic optimal control model can be very easily converted to a deterministic model as we will show. The importance of a stochastic component in the model lies in the fact that the quality of a building is not only influenced by the maintenance policy but also by exogenous variables such as the weather conditions (rain, snow, too much sun), inhabitants, calamities and other factors. Hence it seems necessary to put a stochastic term to the equation describing the quality of a building. Even if we add this stochastic term to our model, it is still possible to give an explicit recipe for the optimal maintenance policy with the help of the theory of stochastic differential equations and stochastic optimal control.

We also consider the problem whether a landlord should rent and maintain a building or whether he should sell it and make an instant profit. Also this problem will be studied by a stochastic

model.
The theory of stochastic differential equations and stochastic optimal control which will be used, is rather technical. In this paper we will try to avoid these technicalities as much as possible and will only show the main techniques. But we can imagine that the whole concept of stochastic differential equations might be hard to understand for an interested reader. In that case one can easily skip everywhere the terms involving σdz and neglect the expectation operator E. Then one has a deterministic optimal maintenance problem and we will give an explicit solution of this problem. We will also give an advice about when selling or when renting is most profitable for the landlord in this deterministic case. On the other hand we will show that if the real model is stochastic and one uses a deterministic model to find the optimal maintenance and renting or selling policy, it might be possible that one decides to sell the building, while it is better to rent it. In section 5 we will be more explicit about this difference between a stochastic and deterministic model. In the next section we will describe our model, while in section 3 we will give the optimal maintenance policy, if it is known that the building will be sold at some fixed time T in the future. In section 4 we will drop this assumption of the fixed selling time and will build in the possibility that the landlord can decide on renting or selling the building.
In section 6 we give some extensions of our model while the last section is devoted to some conclusions and comments.

2. The model of housing maintenance

In this section we will describe the stochastic optimal control problem concerning the maintenance expenditure which the landlord has to solve. Except for the stochastic influence this model has a lot in common with the model of Arnott, Davidson and Pines (1983).
In what follows we will use the following notation for the main variables of our model

q	the quality of the building
m	the maintenance expenditures (the control variable)
p(q)	the rent on a building of quality q
r	discount rate
T	horizon time
C(q)	construction costs for a building of quality q at time 0
P(q,T)	price of a building of quality q at the final time T

Since we want to model stochastic influences on the quality of the building we can not use an ordinary differential equation to describe the quality, but we must use the theory of stochastic differential equations (see, for example, Arnold (1974), McKean (1969), or for economic applications Malliaris and Brock (1982)). We assume that the quality of the building fulfils the following stochastic differential equation

$$\frac{dq}{q} = (am^{\alpha-\rho}-c)dt + \sigma dz \qquad (1)$$

where dz stands for a Wiener process, $0 < \alpha, p < 1$ and $0 < a, c$ are constants and $\sigma^2 q^2$ gives the instantaneous variance of the quality of the building per unit of time.
In such a setting we have that q(t) is a random variable at every instant. If the landlord doesn't spend any money on the maintenance of the building, the expected quality of the building given the quality at time zero $E\{q(t)|q(0)\}$ is given as follows

$$E\{q(t)|q(0)\} = q(0)\exp(-ct-\tfrac{1}{2}\sigma^2 t) \qquad (2)$$

while

$$\mathrm{var}\{\log(q(t)/q(0))\} = \sigma^2 t \qquad (3)$$

Hence without any maintenance expenditure the building will detoriate. Since we have $0 < \alpha < 1$ we see that more maintenance expenditure has a positive influence but this positive influence becomes less by more and more expenditure, in fact the maintenance technology is concave. The factor q^{-p} makes that if

the same amount of maintenance has been spent on a low and high quality building, the growth of quality for the low quality building is greater.
The problem with a stochastic differential equation as (1) might be that there is positive chance that q becomes zero or even negative. Without the disturbance term this is not possible. Whether there exists such a positive chance, depends of course on the maintenance level. However, later on we will show that for the optimal maintenance policy the probability that q becomes zero or negative is zero. Hence the model as specified in (1) is correct in the sense that we will not have a building of negative quality, since this would imply by what follows that the rent of the house would be negative.
We first assume that the landlord wants to rent his building till time T and then to sell it. The expected discounted profit or loss for the landlord at time zero will be:

$$E\{\int_0^T e^{-rt}(p(q(t))-m(t))dt+e^{-rT}P(q(T),T)\} \tag{4}$$

Now the landlord faces the problem of maximizing (4) by choosing a maintenance expenditure policy.
We will further assume that P(q,T) does not depend on T explicitly and take the following functional forms for p and P

$$p(q) = bq^{\beta} \tag{5}$$

$$P(q) = Bq^{\gamma} \tag{6}$$

with $0 < \beta,\gamma < 1$ and $0 < b,B$ constants.
All in all we have the following stochastic optimal control problem

$$\max_{m(t)} E\{\int_0^T e^{-rt}(bq^{\beta}-m)dt+e^{-rT}Bq(T)^{\gamma}\} \tag{7}$$

under $$dq = (am^{\alpha-\rho}-c)qdt + \sigma qdz \tag{8}$$

This optimal control problem has a fixed horizon time T. One can not sell the house before T. Of course it is also interesting to solve the problem where it is allowed to sell earlier. This will be done in section 4 using the solution of the fixed horizon time problem which will be given in the next section.

3. The solution of the fixed horizon time problem

How one solves a stochastic optimal control problem such as given by (7) and (8) can be found in for example Malliaris and Brock (1982) and in a mathematically more rigorous setting in Fleming and Rishel (1975). Under certain conditions one has to solve a dynamic programming equation, which is a second order partial differential equation and find a control which fits into this solution. In our case this goes as follows.
For $q \geq 0$, $0 \leq s \leq T$ let

$$W(s,q) = e^{rs} \max E\left\{\int_s^T e^{-rt}(bq^{\beta}-m)dt + e^{-rT}Bq(T)^{\gamma}\right\} \qquad (9)$$

under (8), where we can only use $m(t)$, $0 \leq t \leq T$ as control. Hence $W(s,q)$ is the maximum profit at time s if the quality of the building at that time is q, multiplied by e^{rs}.
Now if the maximum profit function is differentiable then $W(s,q)$ has to be a solution of the following dynamic programming equation

$$-W_s + rW = \max_m \left[(bq^{\beta}-m) + W_q(am^{\alpha}q^{-\rho}-c)q + \tfrac{1}{2}\sigma^2 q^2 W_{qq}\right] \qquad (10)$$

with $\quad W(T,q) = Bq^{\gamma} \qquad (11)$

and $\quad W(s,0) = 0 \qquad (12)$

If one has a differentiable solution of (10), (11) and (12), a control which maximizes the right hand side of (10) and $q(t)$, a solution of the stochastic differential equation (8) for these values of the control, then the control is also optimal. So we then have a sufficient condition. An equation as (10) has also been called a Hamilton-Jacobi-Bellman equation of stochastic optimal control.

One finds the optimal control by maximizing the left hand side of (10) at every instant. As long as $W_q > 0$ this maximum is found for

$$m = (a\alpha W_q q^{1-\rho})^{\frac{1}{1-\alpha}} \tag{13}$$

and hence (10) becomes

$$0 = W_s - rW + bq^{\beta} - cqW_q + \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\left(aq^{-\rho}W_q\right)^{\frac{1}{1-\alpha}} + \frac{1}{2}\sigma^2 q^2 W_{qq} \tag{14}$$

Equation (14) is a second order partial differential equation of parabolic type. In general it is very hard to solve such a partial differential equation explicitly (see, for example, Friedman (1964)) and most of the time one has to use a numerical method to solve the equation (see e.g. Fritz John (1978). These methods are known as finite difference methods. Since we want to stress the influence of the stochastic term on the optimal control decisions, we are mainly interested in controls which have a closed analytic form. Hence we will give some of the coefficients a particular value in such a way that we get a partial differential equation which has a solution in closed analytic form. For this form we can give economic interpretations of the optimal control solutions, as we will show in the next section. On the other hand we believe that our restrictions on the parameters are not to severe and by continuity assumptions one may hope that if these restrictions are relaxed a little, we still get more or less the same results.

The restrictions are

$$B = \gamma, \ \alpha = \tfrac{1}{2}, \ \rho = \tfrac{1}{2}\beta \tag{15}$$

The first restriction says that the shape of the price function and the rent function are the same and that these functions differ by a constant. This seems a reasonable assumption. The $\alpha = \frac{1}{2}$ is just in the middle of the domain of α and by this

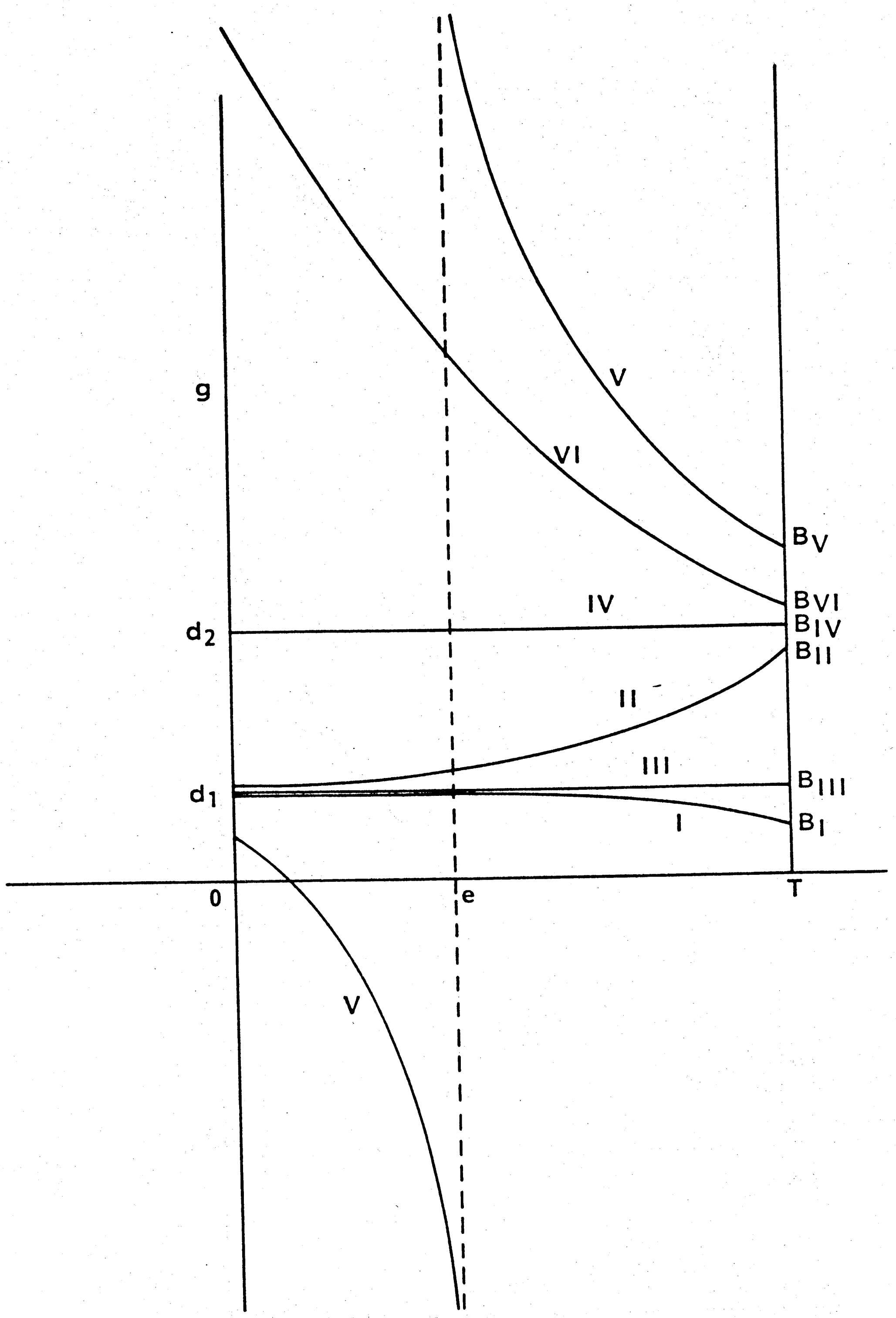

Figure 1: g(s) for D>0, formula (25), for several values of B

restriction we find that (14) has only quadratic or linear terms in W_q. The last restriction is for technical reasons. Using these restrictions, equation (14) becomes

$$0 = W_s - rW + bq^{\beta} - cqW_q + \frac{1}{4}a^2 q^{2-\beta} W_q^2 + \frac{1}{2}\sigma^2 q^2 W_{qq} \qquad (16)$$

To find a solution of this partial differential equation we try the following functional form for W

$$W(s,q) = g(s)q^{\beta} \qquad (17)$$

Then (16) reduces to

$$0 = \{g'(s) - rg(s) + b - \beta cg(s) + \frac{1}{4}a^2\beta^2 g^2(s) + \frac{1}{2}\sigma^2\beta(\beta-1)g(s)\}q^{\beta} \qquad (18)$$

and we only have to solve the ordinary differential equation between the brackets.

$$g'(s) = -\frac{1}{4}a^2\beta^2 g^2(s) + (r + \beta c - \frac{1}{2}\sigma^2\beta(\beta-1))g(s) - b \qquad (19)$$

with boundary condition

$$g(T) = B \qquad (20)$$

For notational purposes we write

$$a_2 = \frac{1}{4}a^2\beta^2, \quad a_1 = r + \beta c - \frac{1}{2}\sigma^2\beta(\beta-1) \text{ and } a_o = b \qquad (21)$$

We see that $a_i > 0$ for $i = 0,1,2$.
Now (19) becomes

$$g'(s) = -a_2 g^2(s) + a_1 g(s) - a_o \qquad (22)$$

Let

$$D = a_1^2 - 4a_2 a_o \qquad (23)$$

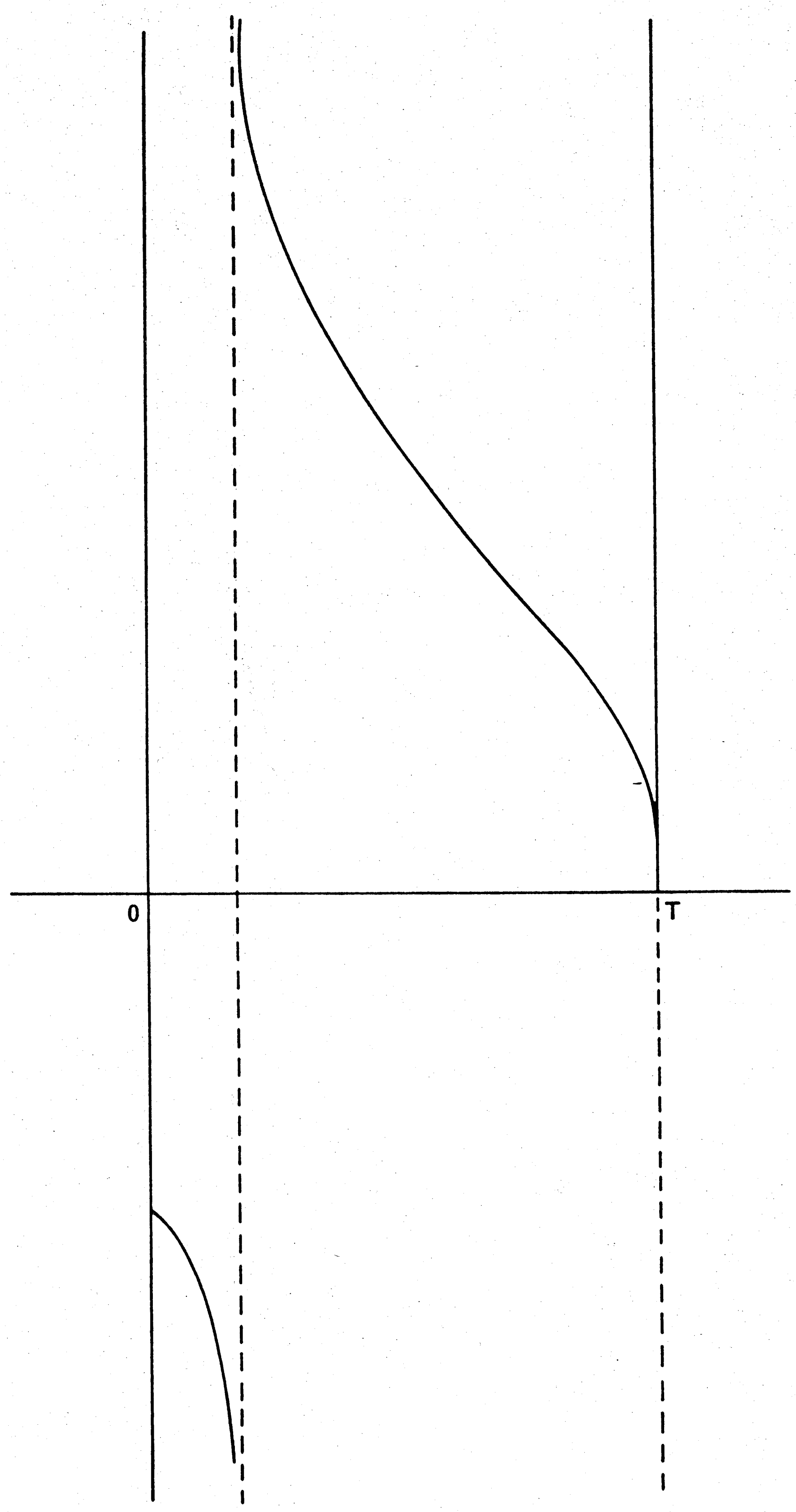

Figure 2: g(s) for D<0, formula(28).

We first treat the case $D > 0$ since this has the most interesting features. In this case we denote

$$d_1 = \frac{a_1}{2a_2} - \frac{\sqrt{D}}{2a_2},\ d_2 = \frac{a_1}{2a_2} + \frac{\sqrt{D}}{2a_2} \qquad (24)$$

By the signs of the a_i we know that $0 < d_1 < d_2$.
Now we can solve (22) by separation of variables and we get the following solution

$$g(s) = \frac{d_2 - d_1 C\exp(-a_2(d_2-d_1)s)}{1-C\exp(-a_2(d_2-d_1)s)}, \qquad (25)$$

where the constant C is determined by the boundary condition (20).
In figure 1 we have depicted g(s) for several caracteristic values of the boundary condition. Remind that $e^{-rs}W(s,q_o) = e^{-rs}g(s)q_o^{\beta}$ gives the maximal expected discounted profit (MEDP) for a house of quality q_o, if the price function of the house at time T has the form $P(q) = Bq^{\beta}$. Hence if for example $B = B_I$ we see that the MEDP is given by the intersection point of the graph I with the g-axis, miltiplied by q_o^{β}, since then $g(0)q_o^{\beta} = W(0,q_o)e^{-ro}$. The other points of the graph I have the same interpretation but always the MEDP is discounted to time zero. The same remarks hold for the graphs II, III and IV and we see that d_1 is a locally asymptotically stable level of the MEDP if we let the horizon time T grow. By this we mean that if we make T bigger the MEDP approaches d_1 for values of B between 0 and d_2. If $B = d_1$ the horizon time doesn't have an influence on the MEDP. The same is true if $B = d_2$ but this is a unstable value in the sense that a small disturbance of M from d_2 drives the system in situation II or V which have a completely different level for the MEDP.
The graph of V illustrates a very nice technical point of the dynamic programming approach. As we already said, when we stated the dynamic programming equation (10), a solution of it only gives the optimal performance if this solution is differentiable. From the picture we see that g(s) is not differentiable at e and

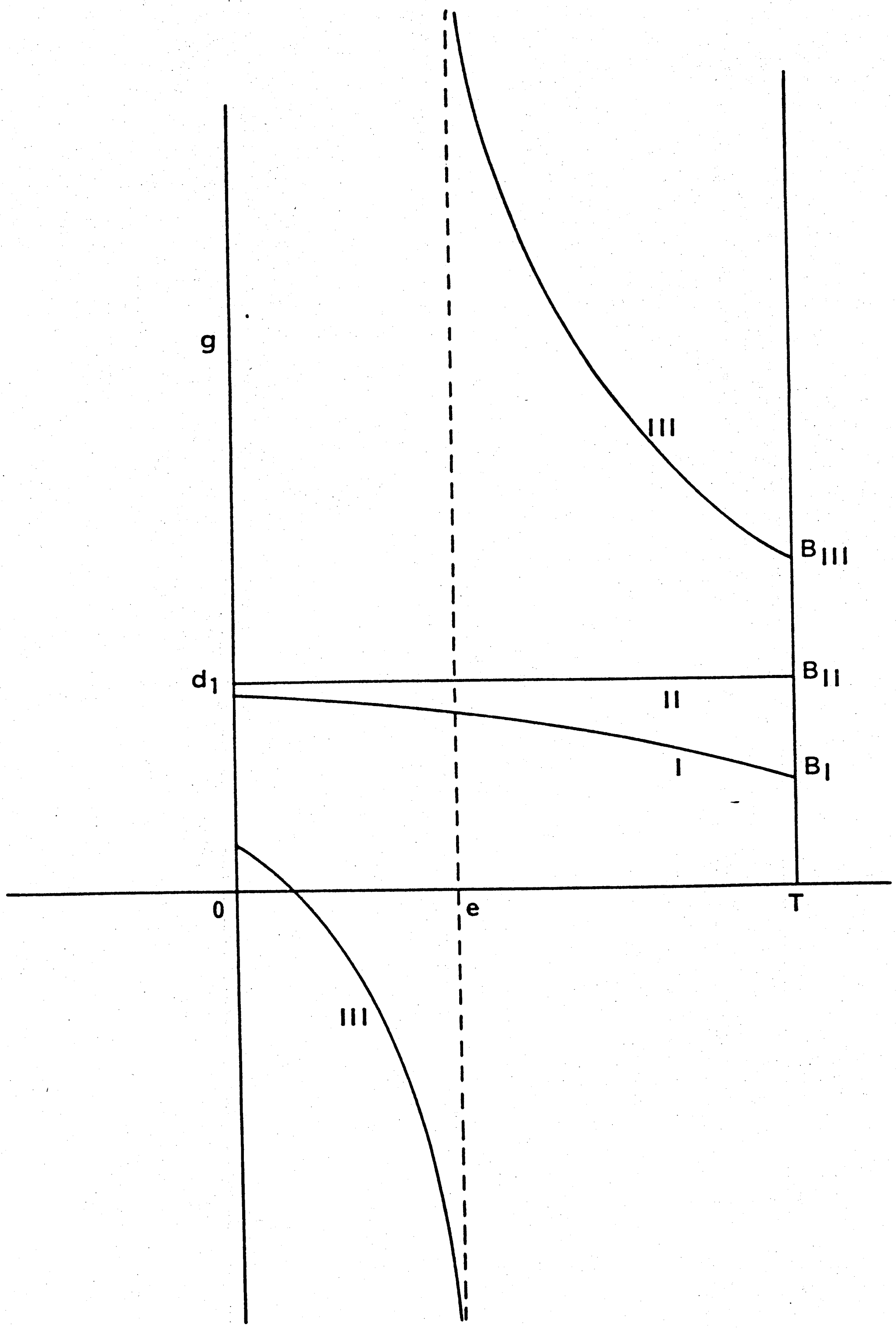

Figure 3: g(s) for D=0, formula(29), for several values of B.

hence the intersection of V with the g-axis doesn't have the same interpretation as in the other cases. This is also clear from the following argument. Consider graph VI. The intersection point of the g-axis and graph VI, multiplied by q_o^β gives the MEDP, because g(s) is differentiable on the relevant interval. Now let B_{VI} grow then the intersection point will increase to infinity and hence the same will be true for the MEDP. From a certain point on we will find a point $e \in [0,T]$ as in graph V. The optimal maintenance policy in this case will be the following. Till time e the maintenance policy doesn't influence the final result, but after e apply the optimal policy. In this way one can make an arbitrary large profit.

Summarizing we have that $g(s)q_o^\beta$ gives us the MEDP if $0 < B \leq d_2$ while if $B > d_2$ we have to see whether $e > 0$ or $e \leq 0$. The value of e of course depends on B.

In our calculations we did not only found the MEDP but also the optimal maintenance policy which is given by the equations (13) and (17) and depends on the time and the quality q of the building

$$m(s,q) = \tfrac{1}{4}a^2\beta^2 q^\beta g^2(s) \tag{26}$$

This optimal maintenance policy depends on the value of q, which is subject to a stochastic proces, hence also m(s,q) is a stochastic proces and one can not stipulate the maintenance policy in advance, but has to see how the quality of the house develops. This is of course not surprising, because one wants the maintenance policy to depend on the influence of the wheather or other exogeneous variables.

Substituting the maintenance policy (26) in the stochastic differential equation (1) gives us a stochastic differential equation for the quality of the building.

$$dq = (\tfrac{1}{2}a^2\beta g(s)-c)qdt+\sigma qdt \tag{27}$$

Now we can treat the problem whether there is a positive chance that q becomes zero or negative. In the appendix we will show

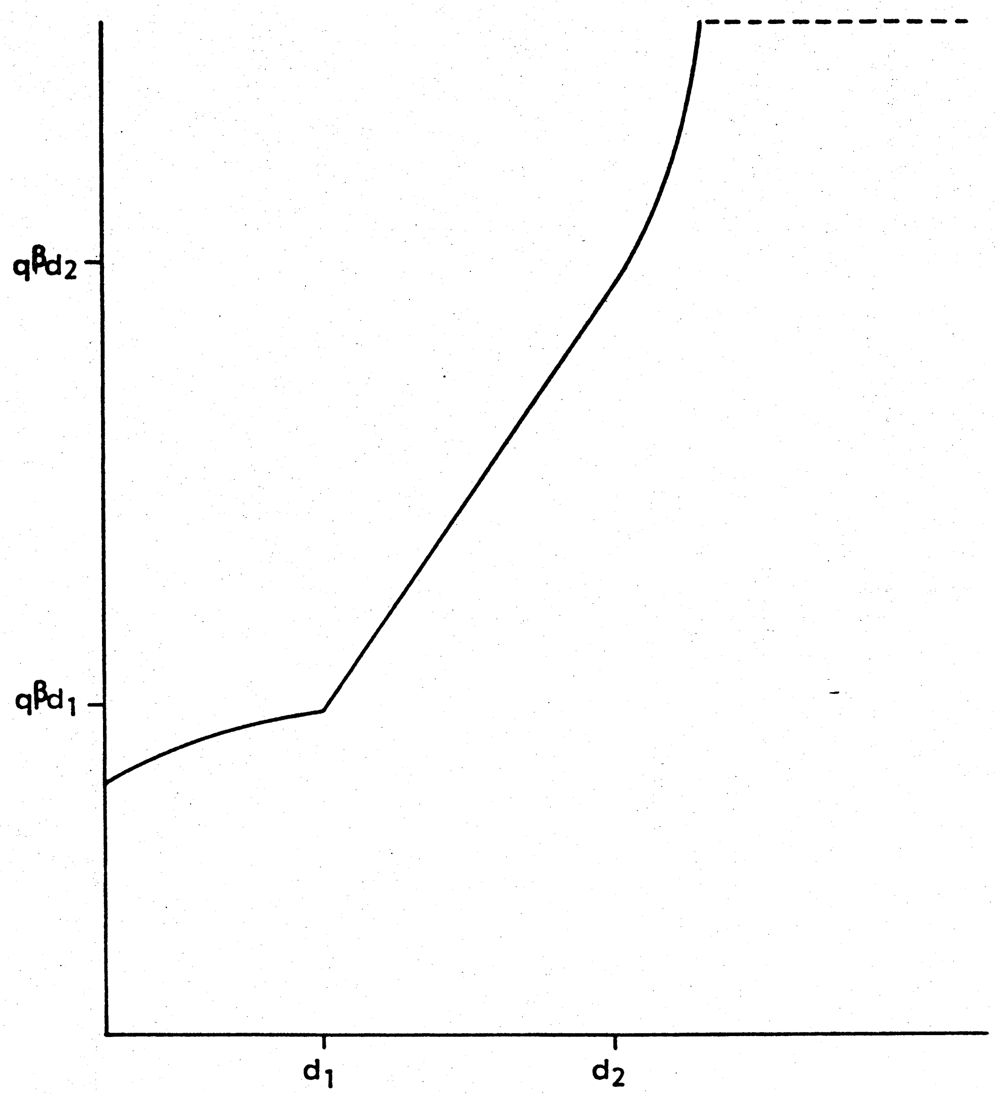

Figure 4: MEDPS as a function of B.

that this chance is zero and hence that q will always be positive with probability one. In the appendix we will also show that as long as g(t) doesn't go to infinity also the probability that the quality goes to infinity is zero. Hence we will only get a building with infinite quality in case V.
In the case $D < 0$ the solution of (22) becomes

$$g(s) = \frac{\sqrt{-D}}{2a_2} \tan\left(\frac{-\sqrt{-D}}{2}s+C\right) + \frac{a_1}{2a_2}, \qquad (28)$$

where C is again determined by the boundary condition (20).
We have plotted g(s) in figure 2. One doesn't get essentially different situations for different B as we had in figure 1. In all cases we observe the same phenomena as in case V of figure 1. By which we mean that we can make an infinite profit if $e > 0$ and the MEDP is given by the intersection point of the g-axis and the graph, multiplied by q^{β} if $e < 0$.
In the case $D = 0$ the solution of (22) becomes

$$g(s) = \frac{1}{a_2 s-c} + d_1, \qquad (29)$$

where C is again determined by the boundary condition (20). A plot of g(s) for this case is given in figure 3. We now have 3 possibilities depending on the value of B and we can give the same comments as in the earlier cases. In this case, however, we have another problem. The system given by (22) is not structurally stable if $D = 0$. By this is meant in our setting the following. If we have a small change in one of the parameters of our model this will influence a_2, a_1 or a_o and hence we will no longer have $D = 0$, so our system will behave as in figure 1 or 2 and no longer as in figure 3. Hence we get a completely different picture. This is not true if $D < 0$ or $D > 0$, because then after a very small change we will still have $D < 0$ or $D > 0$ and hence still have more or less the same picture. Hence the case $D = 0$ is not too interesting to study. This concept of structural stability is not the same as the asymptotic stability about which we talked before because asymptotic stability has to do with a special solution, while structural stability has to do with all solutions. As in the case $D > 0$ we also have in the other cases that the optimal maintenance policy is given by (26), where we have to take for $g(s)$ the appropriate function as given by (28) or (29).

4. The solution of the free horizon problem.

In the previous section we assumed that the building can and must be sold at the time t. We now want to relax this assumption, but we still want the building to be eventually sold, for example before the landlord retires. Hence we want the building to be sold before time T_E. Now we can ask: what is the optimal policy for the landlord?

We first consider again the case $D > 0$, since this is the most interesting one. For illustrating our further remarks it is necessary to calculate C explicitly and to substitute this in $g(s)$. Doing this we find

$$g_T(s) = \frac{d_2 - d_1\left(\frac{B-d_2}{B-d_1}\right)\exp(a_2(d_2-d_1)(T-s))}{1-\left(\frac{B-d_2}{B-d_1}\right)\exp(a_2(d_2-d_1)(T-s))} \tag{30}$$

where $g_T(s)$ is the solution of the fixed horizon time T problem, if we sell at time T. First assume that $0 < B < d_1$, then $g_T(0)$ is maximal if we take $T = T_F$. This can also be seen from figure 1, because if we shift 0 to the left (i.e. make the difference to the time of selling larger) then the intersection of graph I with the g-axis will be higher and hence we have a higher MEDP. So in this situation it is most profitable to rent and maintain the building till the final time T_F and then sell it. This is because of the price of the building is relatively low compared with the rent of the building.

If $B = d_1$, it does not matter when one sells the building as long as one sticks to the optimal maintenance policy before one sells the building. The MEDP is always $d_1 q^{\beta}$. If $d_1 < B < d_2$ then $g_T(0)$ is maximal for $T = 0$ as follows from (30), or as can be seen as before from figure 1. Hence the optimal policy is to sell the

building immediately. The reason for this seems to be that the price of the building is more than enough to off-set the earnings from rent and losses from maintenance. In this case the optimal profit is Bq^{β}.
If $B = d_2$, then it does not matter which action one takes as long as the price stays stable at B. But a small change in M can bring the landlord in position II or position V. In V he can make a very large profit, as will be explained in a moment, while in II he immediately after the change has to sell his building, because otherwise his position is deteriorating. If $M > d_2$ the landlord has to keep the building as long as possible and finally has to sell it. It might be possible that his profit is infinite as in figure 1. As can be calculated from (30), this will occur for higher values of B. It may also be that he has to sell the house at time T_F, which is before the time that the house reaches an infinite quality. The reason behind this policy is that the price of the house is so high that the landlord wants to get his house an infinite quality, before he wants to sell it. We have depicted the optimal profit as a function of B for a house of quality q_o in figure 4, where the dotted line means infinite profits.
Summarizing we have the following result, where MEDPS is the maximal expected discounted profit, with the possibility of selling the building.

Theorem 1. For the maintenance problem as described above the optimal policy and MEDPS for a building of quality q_o at time zero are as follows:
If $0 < B < d_1$, rent the house and maintain according to (26) and sell it at time T_F. MEDPS = $q_o^{\beta} g_{T_F}(0)$
If $d_1 < B < d_2$, sell the house immediately. MEDPS = Bq_o^{β}.
If $B > d_2$, rent the house and maintain it according to (26) and sell it at time T_F or earlier if the quality is high enough
MEDPS = $q_o^{\beta} g_{T_F}(0)$ or ∞ ,according to whether one sells at T_F or earlier.
If $B = d_1$ or $B = d_2$ one can sell the house whenever one wants, but has to maintain it before that time according to (26).
MEDPS = $q_o^{\beta} d_1$ resp. $q_o^{\beta} d_2$.

One may argue that the situation with $B > d_3$ is not realistic or even $B > d_2$. This may be true, but then we still have a realistic result for $0 < B < d_3$ or $0 < \beta < d_2$.Theorem 1 shows how the prices influence the optimal decisions.
We let our optimal policy depend on B, but all the other parameters of our model are reflected in d_1 and d_2. Hence one may also try to make a figure like figure 4, where the optimal profit depends on b in stead off B. This is certainly possible.
In the case that $D < 0$ or $D = 0$ one can follow the same procedure as before to solve the free horizon time problem. We will not do this here, because these cases are less interesting. This is because if $D < 0$ one can make an infinite profit and if $D = 0$ we do not have a structurally stable system.

5. The influence of the stochastic term

In most studies one does not take a stochastic term in consideration (see e.g. Arnott, Davidson and Pines (1983)). In this section we want to see what the consequences are, of not using a stochastic term in the description of the quality of the building. As stated in the introduction we can convert our model to a nonstochastic model by putting $\sigma = 0$ in the stochastic differential equation (1). Then (1) just becomes an ordinary differential equation and we can skip the expection operator in (4), since we no longer have uncertainty. We then have an ordinary optimal control problem, as can be seen from (7) and (8), if we leave out the E in (7) and $\sigma q dz$ term in (8). We can do the same calculations as we did in section 3 for the stochastic problem, but now for the deterministic problem. Even better, put $\sigma = 0$ everywhere in section 3 and we find an explicit solution of our deterministic optimal control problem. We should remark that, if one wants to solve the deterministic version of the optimal control problem (7) and (8) by Pontryagin's Maximum Principle, one has to solve the following system of equations and differential equations

$$-e^{-rt} + \tfrac{1}{2}a\lambda(t)m(t)^{-\frac{1}{2}}q(t)^{1-\rho} = 0 \qquad (31)$$

$$\dot{q}(t) = am(t)^{\frac{1}{2}}q(t)^{1-\rho} - cq(t) \qquad (32)$$

$$\dot{\lambda}(t) = -e^{-rt}\beta bq(t)^{\beta-1} + \lambda(t)(am(t)^{\frac{1}{2}}(1-\rho)q(t)^{-\rho}-c) \qquad (33)$$

$$\lambda(T) = e^{-rT}B\beta q(T)^{\beta-1} \qquad (34)$$

It is clear, that these equations are very hard to solve in general and hence it seems that the dynamic programming approach, which we used for solving this optimal control problem, has its advantage above the usual Pontryagin Maximum Principle approach. Now we want to study the influence of the stochastic term on the decisions. Again we assume that we are in a situation with $D > 0$.

σ^2 only appears in the term a_1 and we see that $\frac{\partial a_1}{\partial \sigma} > 0$ and from (23) and (24) it can be seen that $\frac{\partial d_1}{\partial a_1} < 0$ and $\frac{\partial d_2}{\partial a_1} > 0$. Hence we have

$$\frac{\partial d_1}{\partial \sigma} < 0 \text{ and } \frac{\partial d_2}{\partial \sigma} > 0 \qquad (35)$$

If $d_1 < M < d_2$, we immediately have to sell the house, as we argued before.Hence the interval in which we have to sell the house for a model with stochastic term, is larger than for the model without a stochastic term. Hence under uncertainty one is more willing to keep the house than to sell it. This result can also be stated as follows.

Theorem 2. If (1) is the correct model for the housing maintenance and detoriation and one makes decisions on a model as (1) without a stochastic term, then for certain price functions it might seem optimal to sell the house immediately, while in fact it is optimal to rent and maintain the house.

Of course this is the most dramatic mistake one can make, but if one does not make this mistake, it is still possible, that one

does not take the correct maintenance level. This can be seen from equation (26), because if one does not use a stochastic term, where it is appropriate, one may find the wrong solution $g(s)$ of equation (22).
Another difference between the stochastic and deterministic case is, that in the second case one may postulate the maintenance expenditure over the whole period as given by (26), so one knows how much one has to maintain at every instant in advance. For the stochastic model one has always to observe, how $q(s)$ develops and then to decide with the help of (26) what is the optimal maintenance policy at every instant.

6. Some extensions of the model.

Arnott, Davidson and Pines (1983) not only considered the maintenance level and selling time as control variables, but also the construction quality of the building. So they assume that the building still has to be constructed, while we assumed that the building already exists. We can, however, easily generalize our model to include their model too. The problem then becomes

$$\max_{m(t),q_o,T} \{\int_o^T e^{-rt}(bq^{\beta}-m)dt+e^{-rt}Bq(T)^{\beta}-C(q_o)\} \tag{36}$$

$$\text{under } dq = (am^{\alpha}q^{-p}-c)qdt + \sigma q dz \tag{8}$$

$$q(0) = q_o, \; T \leq T_F \tag{37}$$

To solve this problem one can proceed as follows, if we assume that we have again $D > 0$ and that B is given. Calculate for every q_o the MEDPS as given in theorem 1. Call this MEDPS(q_o) and then maximize the function

$$R(q_o) = \text{MEDPS}(q_o) - C(q_o) \tag{38}$$

The maximum of $R(q_o)$ gives the optimal construction quality of the building.

If for example $C(q_o) = Dq_o^\delta$ with $\delta > \beta$ and $0 < B < d_1$ then the optimal construction quality is given by:

$$q_o = \left(\frac{\beta g_{T_F}(0)}{D\delta}\right)^{\frac{1}{\delta-\beta}} \qquad (39)$$

Another way in which we can extend our model, is by assuming that the price function of the house is time dependent, for example by stating

$$P(q,T) = B(T)q^\beta, \qquad (40)$$

where B(T) is monotonically non-increasing. This means that a constant function is still allowed. This kind of price function might be necessary if one wants to use our study in a vintage model of the housing market. If we now want to solve the optimal maintenance and renting or selling policy, we proceed as before by first solving the fixed horizon time problem, i.e. the problem where the date of selling is determined in advance. This can be solved as before because at a fixed time, B(T) is just a constant and hence the MEDP in this case if given by

$$MEDP = g_T^*(0)q^B \qquad (41)$$

where $g_T^*(0)$ is given by the following obvious generalization of (30):

$$g_T^*(s) = \frac{d_2-d_1\left(\frac{B(T)-d_2}{B(T)-d_1}\right)\exp\left(a_2(d_2-d_1)(T-s)\right)}{1-\left(\frac{B(T)-d_2}{B(T)-d_1}\right)\exp\left(a_2(d_2-d_1)(T-s)\right)} \qquad (42)$$

To solve the problem where one is free to sell the building whenever one wants, one just needs to maximize $g_T^*(0)$ over all $0 \leq T \leq T_F$. Hence

$$MEDPS = \{\max_{T \leq T_o} g_T^*(0)\}q^\beta$$

However, we may remark that if $d_1 < B_o \leq d_2$, then it is always profitable to sell the building immediately because since B(T) is decreasing, one can only jump to a lower curve in figure 1. Hence one is always loosing, if one holds the building still longer. If B(T) is decreasing fast enough, it is possible that it is never profitable to hold the building till the final time T_F. In that case one has in fact solved the infinite horizon time problem. These results could be used in studies like those of Sweeny (1974) and Ohls (1975) where the depriciation depends on the maintenance level. Other generalizations could be to make the autonomous deterioration rate c dependent of the time or to make b, the rent price time dependent or to make a time dependent or even a combination of these generalizations. The problem would be that the coefficients in equation (22) would be time dependent and hence it would be much harder to solve (22), but it might not be impossible for certain specific functions c(t), b(t) and a(t).

7. Conclusions.

In this paper we studied the optimal policy for a landlord, where he can decide on selling or renting the building and on the maintenance level of the building, if he is renting it. We used a specific model which has the advantage of getting specific formulas, for how the optimal policy should be. Our model also involves a stochastic term, which might take care of the influences of the weather or other exogenous variables on the quality of the building. Our main points are that, we can give specific rules for when one has to sell or when one has to rent the house and specific rules for the optimal maintenance level. In fact we showed that for a low price level, it is profitable to rent the house, while for a somewhat higher price level one can better sell the house immediately. For a still higher price level, it is again profitable to rent the house first, because this gives the possibility of increasing the quality of the building. Furthermore we showed that if one does not use a stochastic term in the model, where there are exogenous disturbances, one might follow a non optimal policy. In fact under the influence of the stochastic term one must be less willing to sell the building.

In the preceeding section we gave some possible generalizations of our model and to us it seems worth to make a further study of these generalizations, because the may give a still better description of the real world phenomena.

Acknowledgements.

We like to thank Drs. J.F. Kaashoek for having stimulating discussions about the material of this paper

Appendix

The main result of this appendix is the following.

Theorem A1. If the maintenance policy is given by (26), $D > 0$ and $\beta \leq d_2$ (i.e. the most natural case to consider for our model), then the probability that q becomes zero, negative or infinite in the interval $[0,T]$ is zero.

Proof. For the proof we will use the following result of Feller (see Mandl (1968) p.24 or McKean (1969) p.65).

Theorem (Feller). Let $x(t)$ be a process given by the following stochastic differential equation

$$dx = b(x)dt + a(x)^{\frac{1}{2}}dz \tag{A1}$$

Let $x(0) = x_o \in (,\infty)$ be the starting value.
Now define

$$W(x) = \exp\left(-2 \int_{x_o}^{x} \frac{b(u)}{a(u)}du\right) \tag{A2}$$

Then the probability that $x(t)$ becomes 0 is zero if

(i) $(a(x)W(x))^{-1}$ integrable over $(0,x_o]$ implies that $W(x)$ is not integrable over $(0,x_o]$ and

(ii) $(a(x)W(x))^{-1}$ is not integrable over $(0,x_o]$ implies that

$$W(x) \int^{x}(a(s)W(s))^{-1}ds \tag{A3}$$

is not integrable over $(0,x_o]$.

The probability that $x(t)$ becomes ∞ is zero if (i) and (ii) hold with $(0,x_o]$ replaced by $[x_o,\infty)$.

For the proof of theorem A1 we will first assume that $g(s)$ is constant. Hence we take for all $s \in [0,T]$

$$g(s) = g \tag{A4}$$

First we will show that q will with probability one never become zero. Now if we put

$$e = \frac{1}{2} a^2 \beta g - c \tag{A5}$$

then our stochastic differential equation (27) becomes

$$dq = eqdt + \sigma qdz \tag{A6}$$

Hence in the notation of Feller's theorem we get

$$W(q) = E_1 q^{-2e/\sigma^2} \tag{A7}$$

$$(a(q)W(q))^{-1} = \frac{1}{E_1 \sigma^2} q^{2e/\sigma^2 - 2} \tag{A8}$$

$$W(q)\int^q (a(s)W(s))^{-1} ds = \begin{cases} E_2 q^{-1} & \text{if } 2e/\sigma^2 \neq 1 \\ E_3 q^{-1} \ell nq & \text{if } 2e/\sigma^2 = 1 \end{cases} \tag{A9}$$

where E_1, E_2 and E_3 are constants of integration.
Now we remark that (A9) is never integrable on $[0,q_o]$ and hence condition (ii) is always satisfied. For condition (i) we remark that $(a(q)W(q))^{-1}$ is integrable over $[0,q]$ iff $2e/\sigma^2-2 > -1$ iff $-2e/\sigma^2 < -1$. But $-2e/\sigma^2 < -1$ implies that $W(q)$ is not integrable over $(0,q_o]$. Hence also (i) is satisfied. If q can not become zero, it can also not become negative and hence we only have to consider the problem that q might become ∞. Again (A9) is never integrable over $[q_o,\infty)$ and hence condition (ii) is always satisfied. For condition (i) we do the same argument as before where we only reverse the inequatity signs. This finishes the proof under the assumption that g(s) is constant.
The function g(s) which we have by the assumptions in our theorem

is not constant, but there are constants g_1 and g_2 such that

$$0 < g_1 < g(s) < g_2 < \infty \tag{A10}$$

for all $s \in [0,1]$

Now the probability, that q becomes zero for the proces given by (27), is clearly smaller, than the probability that q becomes zero for the proces given by

$$dq = (\tfrac{1}{2}a^2\beta g_1-c)qdt + \sigma qdz, \tag{A11}$$

since always

$$(\tfrac{1}{2}a^2\beta g_1-c) \leq (\tfrac{1}{2}a^2\beta g(s)-c). \tag{A12}$$

But the probability, that q becomes zero for the proces given by (A11), is zero by the argument given above and hence the probability that the proces given by (27) becomes zero is also zero. For infinity we use the same argument but now with g_2 instead off g_1. This finishes the proof of theorem A1.

In the other case where $D \leq 0$ or $D > 0$ and $B > d_2$ we can give the same result as long as g(s) does not go off to infinity on the interval [0,T]. If g(s) goes to infinity it is clear that also the quality will go to infinity. Nevertheless, even if g(s) goes to infinity the probability that q becomes zero is still zero if we take a maintenance policy as described by (27) with g(s) always bounded away from zero and positive.

References

Arnold, L (1974), Stochastic Differential Equations, Theory and Applications, John Wiley & Sons, New York.

Arnott, R., Davidson, R. and Pines, D. (1983), "Housing Quality, Maintenance and Rehabilitation", Review of Economic Studies L, 467-494.

Fleming, W.H. and Rishel, R.W. (1975), Deterministic and Stochastic Optimal Control, Springer-Verlag, Berlin, Heidelberg, New York.

Friedman, A. (1964), Partial Differential Equations of Parabolic Type,Prentice Hall, Englewood Cliffs, N.J.

John, F. (1978), Partial Differential Equations, Springer Verlag, Berlin, Heidelberg, New York.

Malliaris, A.G. and Brock, W.A. (1982), Stochastic Methods in Economics and Finance, North Holland Publ. Cy., Amsterdam, New York, Oxford.

Mandl, P. (1968), One Dimensional Markov Processes, Grundlehren der Mathematischen Wisschenschaften, Springer-Verlag, Prague.

McKean, H.P. Jr. (1969), Stochastic Integrals, Academic Press, New York.

Ohls, J.C. (1975), "Public Policy Towards Low Income Housing and Filtering in Housing Markets", Journal of Urban Economics, 2, 144-171.

Sweeney, J.L. (1974), "A Commodity Hierarchy Model of the Rental Housing Market", Journal of Urban Economics, 1, 288-323.

LIST OF REPORTS 1984

8400 "Publications of the Econometric Institute Second Half 1983: List of Reprints 344-363, Abstracts of Reports".

8401/S "Point processes and central order statistics", by B. Bode.

8402/O "New directions in scheduling theory", by J.K. Lenstra and A.H.G. Rinnooy Kan.

8403/S "Testing nonnested linear hypothesis I: reduction by invariance considerations", by J. Bouman.

8404/S "Towards a micro-economic theory on spatial shopping behaviour: the effect of distance on consumer's choices", by R. Jansen and J. Koerts.

8405/O "Nonlinear network optimization as a Maxmin problem", by V. Stern.

8406/I "An intelligent computing system", by A. Stern and V. Stern.

8407/M "Optimal housing maintenance under uncertainty", by A.C.F. Vorst.