AgEcon SEARCH
RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

The World's Largest Open Access Agricultural & Applied Economics Digital Library

This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.

Help ensure our sustainability.

Give to AgEcon Search

AgEcon Search
http://ageconsearch.umn.edu
aesearch@umn.edu

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

No endorsement of AgEcon Search or its fundraising activities by the author(s) of the following work or their employer(s) is intended or implied.

ERASMUS

8310

GIANNINI FOUNDATION OF
AGRICULTURAL ECONOMICS
LIBRARY
WITHDRAWN
NOV 13 1984

ECONOMETRIC INSTITUTE

THE CUSP CATASTROPHE IN THE URBAN RETAIL MODEL

J.F. KAASHOEK AND A.C.F. VORST

REPORT 8310/M

Erasmus

ERASMUS UNIVERSITY ROTTERDAM - P.O. BOX 1738 - 3000 DR ROTTERDAM - THE NETHERLANDS

THE CUSP CATASTROPHE IN THE URBAN RETAIL MODEL

J.F. Kaashoek and A.C.F. Vorst

Abstract. In this paper we will show that for a suitable choice of the parameters in an urban retail model as developed by Huff and Lakshmanen and Hansen one finds a cusp catastrophe in the surface of equilibrium points. We give some economic consequences of the fact that we have a cusp catastrophe. Furthermore we show that for other choices of the parameters the equilibrium points of the model depend smoothly on the parameters.

Contents.

1 Introduction.

The last years a lot of papers in regional economics have had catastrophe theory and bifurcations as one of their subjects. See for example: Beaumont, Clarke, Wilson (1981a,b), Puu (1979,1981), Harris, Choukroun, Wilson (1982), Rijk and Vorst (1983a) and the references in the book of Wilson (1981). The last book is completely devoted to the subject of catastrophe theory and bifurcation phenomena in urban and regional science. A lot of interesting examples of the fold catastrophe in regional economic models can be found in the above cited literature; a description of the seven elementary catastrophes can be found in some of these papers and the book. As far as we know never a higher order catastrophe than the fold catastrophe showed up in a regional economic model. The main purpose of this paper is to show that the cusp catastrophe is present in the urban retail model as developed by Huff (1964) and Lakshmanen and Hansen (1965). The equations of this model are as follows

$$(1) \qquad D_j = \sum_{i=1}^{m} O_i \frac{W_j^\alpha C_{ij}}{\sum_{\ell=1}^{n} W_\ell^\alpha C_{i\ell}} \qquad j = 1, \ldots, n$$

$$(2) \qquad \text{with } C_{ij} = \exp(-\beta c_{ij})$$

where

O_i is the total expenditure on goods of inhabitants of zone i,
D_j is the total amount spent in zone j,
c_{ij} is the travelling distance between i and j,
W_j is the size of retail facilities in zone j,
$\alpha > 0$ is a parameter that measures consumer's scale economics
$\beta > 0$ is a parameter reflecting ease of travel.

We assume that the cost per unit of retail facility in zone j is k_j; the revenue in zone j equals the costs in zone j if

$$(3) \qquad D_j = k_j W_j \qquad j = 1, \ldots, n$$

The equations (1) and (3) constitute the urban retail model. By a

positive solution or equilibrium we will mean a set of $W_1, \ldots, W_n$ which fulfill the equations (1) and (3) such that $W_j > 0$ for all $j = 1, \ldots, n$.
In section 4 we will show that if one takes $n = m = 2$ and $\alpha > 1$ the cusp catastrophe plays a rôle in this model. From this we can find some interesting catastrophic behaviour of the positive equilibrium due to small changes in the fuel prices or the income distribution.
Often one also associates a system of differential equations to the urban retail model, since it is reasonable to assume that the behaviour of an entrepreneur is dynamic. Doing this it makes sense to talk about observable equilibria (i.e. equilibria which are asymptotically stable for the differential equations). Hence the observable equilibria form a subset of the equilibria of the differential equations. We will assume that the differential equations are of the following form

$$(4) \qquad \dot{W}_j = f_j(W_j)(D_j - k_j W_j) \qquad j = 1, \ldots, n$$

where $f_j(W_j)$ has one of the following forms

$$(5) \qquad f_j(W_j) = \varepsilon W_j,\ \varepsilon,\ \text{or}\ \varepsilon/W_j \text{ with } \varepsilon \text{ constant.}$$

In the following section we will show that there are no catastrophes for $\alpha < 1$ since in this case there is a unique positive equilibrium which depends continuously on all the parameters of the model. Hence there is a big difference between the case $\alpha < 1$ and $\alpha > 1$ which is one of the other main conclusions of this paper.
In section 3 we have some general results for $\alpha > 1$ concerning the finiteness of the number of equilibria and the non-existence of positive observable equilibria when there are less living zones than shopping zones.

2. The case $\alpha < 1$.

In contrast with the case $\alpha > 1$, we have shown in Rijk and Vorst (1983b) that for $\alpha < 1$ there is always a unique positive solution of the equations (1) and (3) (See also Chudzynska and Slodkowski (1983)). Here we will show that this positive equilibrium depends continuously on the parameters of the model and hence we can not have catastrophic behaviour of the equilibrium point due to a small perturbation in one of the parameters. (For example a small energy price change or a small change in the income distribution between the living zones). This can be formulated and proved as follows

Let $\mathcal{A} = \{\alpha \in R \mid 0<\alpha<1\}$, $\mathcal{B} = \{O = (O_i \in R^m \mid O_i \geq 0$ for all $i, O \neq 0\}$ $\mathcal{C} = \{C = (C_{ij}) \in R^{mn} \mid C_{ij} > 0\}$, $\mathcal{D} = \{k = (k_j) \in R^n \mid k_j > 0$ for all $j\}$. and $\mathcal{W} = \{W=(W_j) \in R^n \mid W_j > 0$ for all $j\}$.

Hence $\mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \subset R^{1+mn+m+n}$ is the parameter space.

Theorem 1. There exists a continuous function

$G: \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \to \mathcal{W}$ such that for all $(\alpha,O,C,k) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D}$, $G(\alpha,O,C,k)$ is the unique positive solution of the equations (1) and (3) where the parameters are equal to (α,O,C,k).

Proof. In Rijk and Vorst (1983b) it has been shown that there is a unique positive solution of (1) and (3) given the parameters (α,O,C,k) (and this positive equilibrium is asymptotically stable). In fact we only proved the case where all coëfficients of k are equal but the proof can easily be generalized to the case where the k_j's are different. Hence we know that $G: \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \to \mathcal{W}$ is well defined if we take for $G(\alpha,O,C,k)$ this unique positive equilibrium. We still need to show that G is continuous.

Hence take $(\alpha_o,O_o,C_o,k_o) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D}$ and let $W_o = G(\alpha_o,O_o,C_o,k_o)$.

Define

$$F: \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \times \mathcal{W} \to R^n \text{ by}$$

$$(6)\quad F_j(\alpha,O,C,k,W) = \sum_{i=1}^{m} O_i \frac{W_j^{\alpha-1}C_{ij}}{\sum_{\ell=1}^{n} W_\ell^{\alpha}C_{i\ell}} - k_j \qquad (j=1,\ldots,n)$$

Then F is differentiable with continuous derivative. Furthermore in Rijk and Vorst (1983b) it has been shown that

$H(\alpha_o,O_o,C_o,k_o,W_o)$, with

$$(7)\quad (H(\alpha_o,O_o,C_o,k_o,W_o))_{\ell j} = \frac{\partial F_j}{\partial w_\ell}(\alpha_o,O_o,C_o,k_o,W_o)$$

is a negative definite matrix and hence $\det(H(\alpha_o,O_o,C_o,k_o,W_o)) \neq 0$ But then we can apply the implicit function theorem (Chow and Hale (1982) th.2.2.3) which says that there is a continuous function $K: U \subset \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \to \mathcal{W}$ where U is an open neighbourhood of (α_o,O_o,C_o,k_o) such that

$$(8)\qquad F(\alpha,O,C,k,K(\alpha,O,C,k)) = 0$$

for all $(\alpha,O,C,k) \in U$. But then K must be G on U and we have that G is continuous in (α_o,O_o,C_o,k_o), what we wanted to show. Hence for $0 < \alpha < 1$ the positive equilibrium behaves as nice as we like.

3. General results for $\alpha > 1$.

In this section we will give two results for the general model of equations (1) and (3) for $\alpha > 1$. Since the proofs of these results are a little technical we don't give them here but in appendix A.

Theorem 2. Let $\alpha > 1$. If there are more shopping zones than living zones (i.e. $m < n$) there can not be any positive asymptotically stable equilibrium for the urban retail model given by equations (1), (4) and (5).

This result has strong implications. If one tries to model a real situation with more shopping zones than living zones one should keep in mind that this is only possible if $\alpha \leq 1$. If $\alpha > 1$ one will never meet a stable situation with more shopping zones than living zones. If the situation becomes stable this is only possible if a few of the shopping centres disappear.
The following result shows that in most cases we only have finitely many equilibrium points and hence certainly finitely many observable equilibrium points. It is only incidental that we meet a situation with infinitely many solutions of the equations (1) and (3) and a small disturbance in some of the parameters of the model brings us back in a situation with only finitely many equilibrium points. This result can be formally formulated as follows.

Theorem 3. Let $\alpha > 1$. There exists a set M of measure zero in the parameter space $\mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D}$ such that for every combination of parameters which is not lying in M, the system of equations (1) and (3) has only finitely many positive solutions.

4. The cusp catastrophe for n=m=2.

In this section we will restrict ourselves to the case where we have two zones which are both living and shopping zones. Thus $n = m = 2$. This case has been studied in Rijk and Vorst (1983a,b) and Harris, Choukroun and Wilson (1982).
We furthermore assume that $C_{11} = C_{22} > C_{12} = C_{21} > 0$ (i.e. the inner distances in both centres are the same and are smaller than the distance to the other centre) and $k_1 = k_2 = k$. We let

$$C = C_{12} / C_{22} \quad (0 < C < 1) \tag{9}$$

For every solution of (1) and (3) we have that
$W_1 + W_2 = (O_1+O_2)/k$.

By setting:

(10) $\quad (O_1+O_2)/k = 1$ and hence $W_1 = 1-W_2$,

we don't destroy the generality of the model since all results can be easily translated to the case where $(O_1+O_2)/k \neq 1$. For convenience of notation we write

(11) $$Q = O_1/k - \tfrac{1}{2}, \quad -\tfrac{1}{2} \leq Q \leq \tfrac{1}{2}$$

Hence Q measures the deviation from an equal income distribution. Under these restrictions equations (1) and (3) become equivalent with

(12) $$\frac{(Q+\frac{1}{2})W_1^\alpha}{W_1^\alpha+W_2^\alpha C} + \frac{(\frac{1}{2}-Q)W_1^\alpha C}{W_1^\alpha C+W_2^\alpha} - W_1 = 0$$

(13) $$\frac{(Q+\frac{1}{2})W_2^\alpha C}{W_1^\alpha+W_2^\alpha C} + \frac{(\frac{1}{2}-Q)W_2^\alpha}{W_1^\alpha C+W_2^\alpha} - W_2 = 0$$

By adding (12) to (13) we see that (12) and (13) are equivalent to (12) and $W_2 = 1-W_1$. Substituting this last equation in (12) we see that we are searching for solutions of the equation

(14) $$\begin{gathered}(Q+\tfrac{1}{2})\{W_1^{2\alpha}C+W_1^\alpha(1-W_1)^\alpha\} + (\tfrac{1}{2}-Q)\{W_1^{2\alpha}C+W_1^\alpha(1-W_1)^\alpha C^2\} = \\ W_1^{2\alpha+1}C + W_1^{\alpha+1}(1-W_1)^\alpha(1+C^2) + W_1(1-W_1)^{2\alpha}C.\end{gathered}$$

To find catastrophes we must necessarily take $\alpha > 1$. A very convenient case is

(15) $\alpha = 2$,

since then we can make the easiest calculations, although the results are certainly an indication of the behaviour of the

equations if α is in a neighbourhood of 2.
If we write $W_1 = x$ we get that (14) is equivalent with

$$f_{C,Q}(x)x(x-1) = 0 \tag{16}$$

where

$$\begin{aligned} f_{C,Q}(x) &= (C+1)^2x^3-\{Q(1-C^2)+\tfrac{3}{2}(1+C)^2\}x^2 \\ &\quad + \{Q(1-C^2)+\tfrac{1}{2}(1+C)^2+2C\}x - C \end{aligned} \tag{17}$$

The equation $f_{C,Q}(x) = 0$ has 1,2 or 3 solutions depending on the discriminant of $f_{C,Q}$ (see appendix B). Every solution of $f_{C,Q}(x) = 0$ is lying between 0 en 1 since a solution of $f_{C,Q}(x) = 0$ is a solution of (16) ($x = W_1$) which only has solutions for $W_1 \geq 0$ and by replacing W_1 by W_2 we see that $W_2 \geq 0$ and hence $W_1 \leq 1$. Having calculate the discriminant as a function of C and Q, one can calculate the number of solutions of $f_{C,Q}(x) = 0$ depending on C and Q. We made a plot of this in figure 1 where the number at a certain point indicates the number of solutions of $f_{C,Q}(x) = 0$. (The sign 2 in figure 1 doesn't necessarily mean that there are exactly two solutions in this point but it indicates that there is a point in the neighbourhood where we have exactly two solutions). Figure 1 shows us already the cusp which is the set of bifurcation points of a cusp catastrophe.
For every polynomial equation of degree three one can explicitly calculate the solutions (see e.q. Van der Waerden (1966) pg193). How these solutions depend on Q and C can be seen in figure 2a,b where we find the example of the cusp catastrophe as promised in the introduction.
If one considers the system of differential equations (4) and (5) in our case with $n = m = 2$ one can show by the techniques developed in Rijk and Vorst (1983b) that except for the points $W_1 = 0$ or $W_1 = 1$, which are asymptotically stable, only the middle leaf of figure 2 represents asymptotically stable, and hence observable, equilibria. The other points of figure 2 represent boundaries between regions of attraction of different

points. (A region of a attraction of a point W is the set of all initial values from solutions of the differential equation converging to W). In the terminology of Wilson (1981) this implies that we are dealing with a false cusp catastrophe, but since we still have the solutions $W_1 = 0$ or $W_1 = 1$ we have interesting catastrophic behaviour as can be seen from the discussion in section 5. To study the whole equilibrium surface we made some plots. In figure 3 we plotted how the equilibrium points depend on Q while D is kept fixed and in figure 4 we interchanged the rôles of Q and D. Solid lines represent asymptotically stable equilibria, while dotted lines represent the unstable points.

5. Economic consequences of the cusp catastrophe.

In this section we describe the consequences of small changes in the income distribution or fuel prices in the case $n = m = 2$. We use for this discussion figures 2, 3 and 4, which represents the following five cases.

Case A1: fig. 3a, C fixed, with $C \geq 3-2\sqrt{2}$.

For every Q only the points $W_1 = 0$ or $W_1 = 1$ are asymptotically stable solutions. This means that only one centre will serve as a shopping centre and varying Q will not change the situation. Hence we don't have catastrophic behaviour.

Case A2: fig. 3b, C fixed, with $C < 3-2\sqrt{2}$.

For every Q with $a_o < Q < b_o$ we have an asymptotically stable positive equilibrium. Assume the whole system is in such a state with $a_o < Q < b_o$ and $0 < W_1 < 1$ and let zone 1 slowly get a bigger share of total income. Then Q will gradually increase and W_1 with Q, until Q becomes b_o. From then on a further increase in Q will be catastrophic since the positive asymptotically stable equilibrium no longer exists and we are in the region of attraction of the solution $W_1 = 1$ which means that shopping centre 2 will suddenly disappear and all the shopping will be done in centre 1. On the other hand if Q gradually decreases then shopping centre 1 will disappear after Q passed the state a_o.

Case B1: fig. 4a, Q fixed, with $Q < 0$.

For $C < c_o$ we have a positive asymptotically stable equilibrium $W_1 (0<W_1<1)$. Let assume the system is in such a state. Now we can analyze what happens if the fuel prices go down (reflected in a decrease of β). C will gradually increase and we will stay in a positive asymptotically stable equilibrium as long as $C < c_o$, but when C passes c_o then we don't have a positive asymptotically stable equilibrium anymore and since we are in the region of attraction of the solution $W_1 = 1$ the first shopping centre will suddenly disappear. Hence we have catastrophic behaviour.

Case B2: fig. 4c, Q fixed, with $Q > 0$.

We get the same story but now the second shopping centre will disappear. One also sees that an increase in the fuel prices doesn't give catastrophic behaviour if we are already in a positive stable equilibrium position. The only thing which happens with an increase in the fuel prices is that every shopping centre more and more only attracts the customers of its own living zone. This is reflected by the fact that the asymptotically stable positive solution goes to $Q + \frac{1}{2}/\frac{1}{2} - Q$ if $C \to 0$. However, another kind of catastrophic behaviour can happen if the fuel prices increase. If fuel prices are low, but $C < c_o$ and only in the first zone we have a shopping centre (i.e. $W_1 = 1$) then a small disturbance caused by someone who opens a small shop in the second zone fades a way, since we stay in the region of attraction of $W_1 = 1$. But if fuel prices are high i.e. D is very small and also only in the first zone we have a shopping centre, the same small disturbance may bring us in the region of attraction of the middle leaf solution and suddenly the whole system goes to the asymptotically stable positive equilibrium. Hence high fuel prices make the solutions with only one shopping centre less stable.

Case B3: fig. 4b, Q fixed, with $Q = 0$.

If we have $C < c_o$ then the situation in which both shopping centres are equal is an asymptotically stable positive equilibrium. Assume we are in such a state. Now if C increases (e.q. the fuel prices decrease) we stay in this state untill C passes c_o. Then one of shopping centres disappears, but we cannot predict which centre, because it depends on what kind of small predict which centre, because it depends on what kind of small disturbance appears. But it's sure we will meet a catastrophic change. In Rijk and Vorst (1983b) we assumed $Q = 0$ and now we see that this is a very particular situation since figure 4b is not stable, while 4a and 4c are stable.

6 Conclusions.

We showed that the cusp catastrophe is present in the urban retail model. Using this we could describe some interesting behaviour of our system due to small changes in the fuel prices or income distribution. We only have these catastrophic results for $n = m = 2$, but we hope that by analyzing these small situations one can get a better understanding of the general urban retail model. These catastrophic results are in sharp contrast with our general result for $\alpha < 1$, which says that for $\alpha < 1$ the positive equilibrium depends continuously on all the parameters in the model. So by this mathematical analysis it is made clear that there is a whole lot of a difference between the urban retail model with $\alpha < 1$ and the urban retail model with $\alpha > 1$. Hence the discussion whether $\alpha > 1$ or $\alpha < 1$ in an urban retail model is very fundamental.

APPENDIX A:
Proofs of theorems 2 and 3

Proof of theorem 2.

If x^* is an equilibrium point of the dynamical system $\overset{o}{x} = f(x)$ then x^* is asymptotically stable if the eigenvalues of the square matrix $Df(x^*)$ all have negative real part and x^* is not asymptotically stable if at least one eigenvalue of $Df(x^*)$ has a positive real part (see Hirsch and Smale (1974) p. 180-187). Our dynamical system depends on the choice of $f_j(W_j)$ which we take in equation (5).

Let A, D en $\hat{W}$ be the following matrices

$$(18)\quad A_{\ell j} = \sum_{i=1}^{m} O_i \frac{W_j^{\alpha-1} C_{ij}}{\sum_{k=1}^{n} W_k^{\alpha} C_{ik}} \frac{W_\ell^{\alpha-1} C_{i\ell}}{\sum_{k=1}^{n} W_k^{\alpha} C_{ik}}$$

$$(19)\quad D = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_n \end{pmatrix} \quad \text{and} \quad \hat{W} = \begin{pmatrix} W_1^* & & 0 \\ & \ddots & \\ 0 & & W_n^* \end{pmatrix}$$

Now let a positive solution $W^* = (W_1^*,\ldots,W_n^*)$ of (1) and (3) be an equilibrium point and hence if $f_j(W_j) = \varepsilon/W_j$ in (4) we have

$$(20)\quad \frac{1}{\varepsilon}DF(W^*) = -\alpha A + (\alpha-1)D\hat{W}^{-1} = C.$$

If $f_j(W_j) = \varepsilon$ in (4) we have

$$(21)\quad \frac{1}{\varepsilon}DF(W^*) = -\alpha A\hat{W} + (\alpha-1)D = E.$$

If $f_j(W_j) = \varepsilon W_j$ in (4) we have

$$(22)\quad \frac{1}{\varepsilon}DF(W^*) = -\alpha A\hat{W}^2 + (\alpha-1)D\hat{W} = G.$$

Hence we have to prove that all these three matrices C, E, G have at least one eigenvalue with a positive real part.

Now we have the following lemma

Lemma. (Intermoving eigenvalue lemma). Let B be a symmetric nxn-matrix with at least r positive eigenvalues. Then $B - aa^T$ has at least (r-1) positive eigenvalues for every $a \in \mathbb{R}^n$.

A proof of this result can be given using the techniques of Bellman(1970) pg. 115-118.

Now we observe that $(\alpha-1)D\hat{W}^{-1}$ is a diagonal matrix with all diagonal elements positive. Hence all eigenvalues of $(\alpha-1)D\hat{W}^{-1}$ are positive. But one gets C by substracting m times a matrix of the form aa^T from $(\alpha-1)D\hat{W}^{-1}$. Hence by the above lemma we have that $C = -\alpha A+(\alpha-1)D\hat{W}^{-1}$ has at least n-m positive eigenvalues and we are done in case (20).

Now we turn to the case (21): $E = C\hat{W}$ with C a symmetric matrix with at least one positive eigenvalue. We first consider the case $\det(C) \neq 0$ then $\hat{W} = C^{-1}E$ and $\hat{W} + \hat{W}^T = C^{-1}E + E^TC^{-1}$ is a positive definite symmetric matrix. By an extension of Lyapunov's theorem (Taussky-Todd (1961)) we see that the number of positive eigenvalues of C is equal to the number of eigenvalues with positive real part of E. Hence E has at least one eigenvalue with positive real part, which we had to prove. We only assumed that C is invertible. If C is not invertible one can show by the argument of the proof in case (20) that for very small δ the number of positive eigenvalues of C and $(C-\delta\hat{W}^{-1})$ is the same. Hence $(C-\delta\hat{W}^{-1})$ still has a positive eigenvalue, since C had a positive eigenvalue and we can choose δ such that $(C-\delta\hat{W}^{-1})$ is invertible. Now $E - \delta I = (C-\delta\hat{W}^{-1})\hat{W}$ and by the argument we gave before, we see that $E - \delta I$ has at least one eigenvalue with positive real part, but then this is certainly true for E.

The case of (22) goes similar to (21) by observing that $G = CW^2$ in stead of CW.

Proof of theorem 3. For this proof we use a special version of Sard's theorem (Chow and Hale (1982)). To use this theorem one has to verify some very special conditions. We do not want to explain these conditions and the terminology in which they are stated here, since this would take to much space. The interested reader should first read some part of chapter 2 of Chow and Hale and then he will understand that the conditions which we will verify are enough for proving theorem 3. Consider the map

G: $\mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \times \mathcal{W} \to R^n$ given by

$$G(\alpha,0,C,k,W)_j = \sum_{i=1}^{m} 0_i \frac{W_j^{\alpha-1} C_{ij}}{\sum_{k=1}^{n} W_\ell^{\alpha} C_{i\ell}} - k_j$$

Now $\frac{\partial G}{\partial k} = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$ everywhere and hence non-singular (hence 0 is a regular value of G).

So DG the matrix of all derivatives of G has maximum rank and furthermore we remark that G is p times continuously differentiable for all $p \in \mathbb{N}$. But now we may apply theorem 2.10.3 of Chow and Hale (1982) which says that there is a set M of measure zero in the parameter space $\mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D}$ such that for all $(\alpha,0,C,k)$ / M, 0 is a regular value of $G(\alpha,0,C,k)$. This implies that there cannot be a positive limit point of zero's of $G(\alpha,0,C,k)$. However, one can show by an argument as in the appendix of Rijk and Vorst (1983b) that solutions of $G(\alpha,0,C,k) = 0$ are bounded away from the boundary. Hence there are only finitely many positive solutions of (1) and (3) for all $(\alpha,0,C,k)$ / M. This finishes the proof of theorem 3.

APPENDIX B.

The cusp catastrophe

Let

(23) $\qquad f_{\alpha,\beta,\gamma}(x) = x^3 + \alpha x^2 + \beta x + \gamma = 0$

be a polynomial equation of degree 3.

Write $\delta = (-\frac{1}{3}\alpha^2+\beta)$ and $\varepsilon = (\frac{2}{27}\alpha^3-\frac{1}{3}\alpha\beta+\gamma)$.

Let

(24) $\qquad \Delta = \delta^3 + \frac{27}{4}\varepsilon^2$

be the discriminant of $f_{\alpha,\beta,\gamma}$. If

(25) $\qquad \Delta \begin{array}{l} < 0 \text{ (23) has 3 solutions} \\ = 0 \text{ (23) has 2 solutions,} \\ > 0 \text{ (23) has 1 solution} \end{array}$

while explicit formulas for the solutions can be found in Van der Waerden (1966) page 193.

In the situation of section 4 we have

(26)
$$\begin{aligned} \alpha &= -\{Q(1-C^2) + \tfrac{3}{2}(1+C)^2\}/(1+C)^2 \\ \beta &= \{Q(1-C)^2 + \tfrac{1}{2}(1+C)^2+2C\}/(1+C)^2 \\ \gamma &= -C/(1+C)^2 \end{aligned}$$

In such a situation we can find fold catastrophes if $\Delta = 0$ and cusp catastrophes if $\delta = \varepsilon = 0$. In a fold catastrophe point (23) has a solution which is also a zero of the derivative of $f_{\alpha,\beta,\gamma}$ while in a cusp catastrophe point (23) has exactly one solution which is also a zero of $f'_{\alpha,\beta,\gamma}$ and $f''_{\alpha,\beta,\gamma}$.

In our model we find a fold iff

$$Q^4(1-C)^4 + \frac{1}{4}\{\frac{1}{4}(1-6C+C^2)^2 - \frac{3}{4}(1+6C+C^2)^2\}Q^2(1-C)^2/(1+C)^2 + \tag{27}$$
$$+ \frac{1}{16}(1-6C+C^2)^3/(1+C)^2 = 0$$

and a cusp iff

$$-\frac{1}{3}Q^2(1-C)^2 - \frac{1}{4}(1-6C+C^2) = 0 \text{ and} \tag{28}$$

$$-\frac{2}{27}Q^3(1-C)^3 + \frac{1}{6}Q(1-C)(1+6C+C^2) = 0 \tag{29}$$

With the extra restriction that $-\frac{1}{2} < Q < \frac{1}{2}$ and $0 < C < 1$ the only solution of (28) and (29) is $Q = 0$ and $C = 3-2\sqrt{2}$. Hence this is the only cusp point, which is confirmed by figure 1. The fold points can be calculated explicitly by using (27), and then we get a mathematical description of figure 1.

References.

1. Beaumont, J.R., Clarke, M. and Wilson A.G. (1981a) "The Dynamics of Urban Spatial Structures: Some Exploratory Results using Difference Equations and Bifurcation Theory", Environment and Planning A, 3, 1473-1483.
2. Beaumont, J.R., Clarke, M. and Wilson, A.G. (1981b) "Changing Energy Parameters and the Evolution of Urban Spatial Structure", Regional Science and Urban Economics, 11, 287-316.
3. Bellman, R. (1970), "An Introduction to Matric Analysis", McGraw-Hill, New York.
4. Chudzynska, I. and Slodkowski, Z. (1983), "On Equilibria of a Gravity Demand Model", Preprint, to appear in Environment and Planning A.
5. Chow, S.N. and Hale, J.K. (1982), "Methods of Bifurcation Theory", Springer Verlag, New York.
6. Harris, B., Choukroun, J-M. and Wilson, A.G. (1982), "Economics of Scale and the Existence of supply side Equilibria in a Production constrained spatial interaction model", Environment and Planning A, 14, 823-837.
7. Hirsch, M.W. and Smale, S. (1974), "Differential Equations, Dynamical Systems and Linear Algebra", Academic Press, New York.
8. Huff, D. (1964), "Defining and Estimating a Trading Area", Journal of Marketing, 28, 34-38.
9. Lakshmanen, T.R. and Hansen, W.G. (1965), "A Retail Market Potential Model", Journal of the American Institute of Planners, 31, 134-143.
10. Puu, T. (1979), "Regional Modeling and Structural Stability", Environment and Planning A 11, 1431-1438.
11. Puu, T. (1981), "Catastrophe Structural Change in a continuous regional model", Regional Science and Urban Economics 11, 317-334.
12. Rijk, F.J.A. and Vorst, A.C.F. (1983a), "On the Uniqueness and Existence of Equilibrium Points in an Urban Retail Model", Report Econometric Institute, Erasmus University

Rotterdam. To appear in Environment and Planning A.

13. Rijk, F.J.A. and Vorst, A.C.F. (1983b), "Equilibrium Points in an Urban Retail Model and their Connection with Dynamical Systems", Report Econometric Institute, Erasmus University Rotterdam. To appear in Regional Science and Urban Economics.

14. Taussky-Todd, O. (1961), "A Generalization of a Theorem of Lyapunov", SIAM J. of Applied Mathematics 9, 640-643.

15. Van der Waerden, B.L. (1966), "Algebra", Erster Teil, Springer Verlag, New York.

16. Wilson, A.G. (1981), "Catastrophe Theory and Bifurcation: Applications to Urban and Regional Systems", Croom Helm, London.

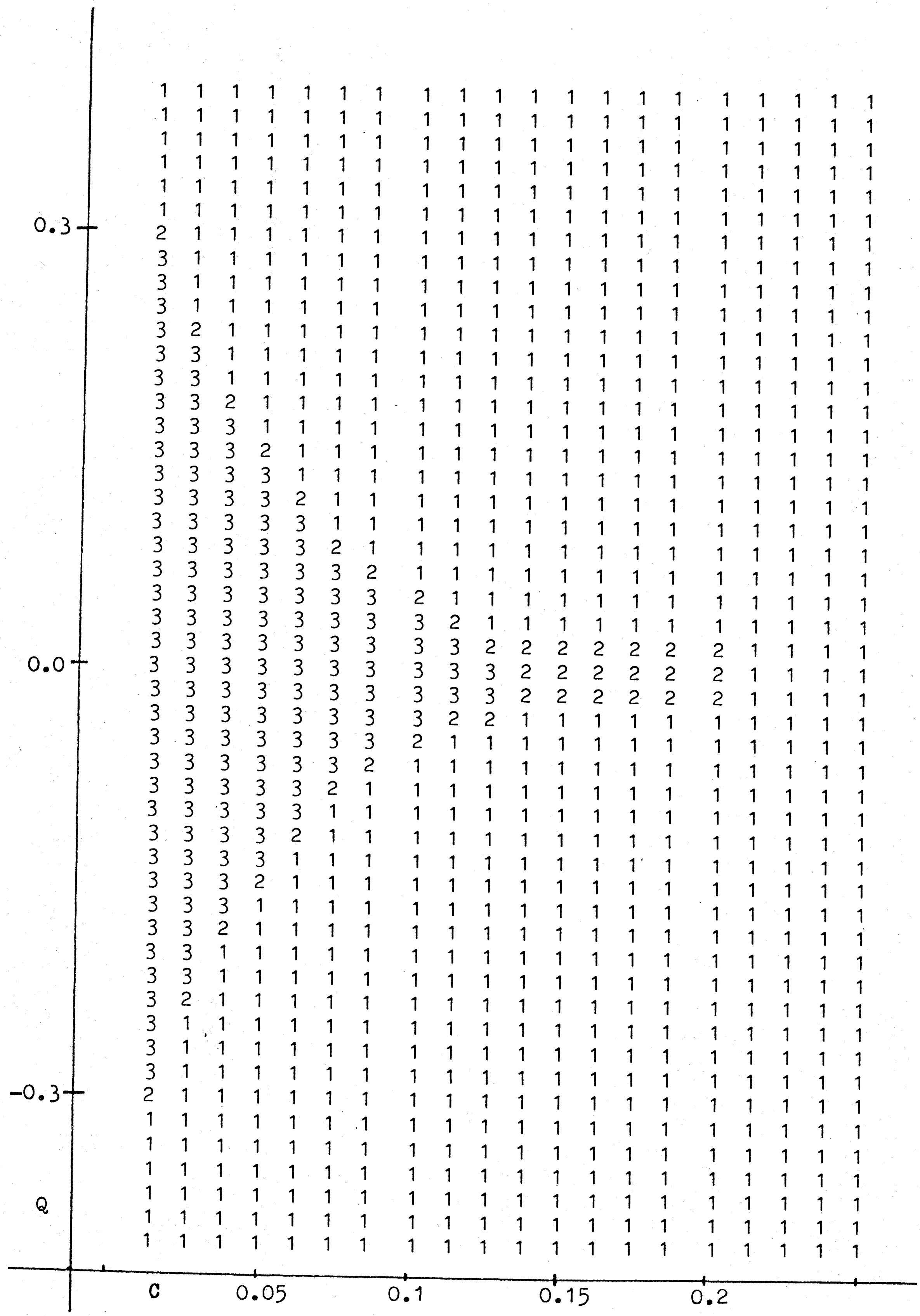

Fig. 1.

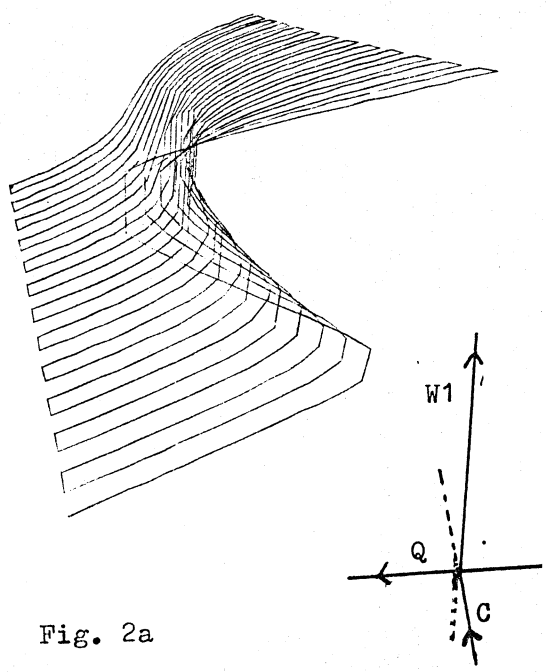

Fig. 2a

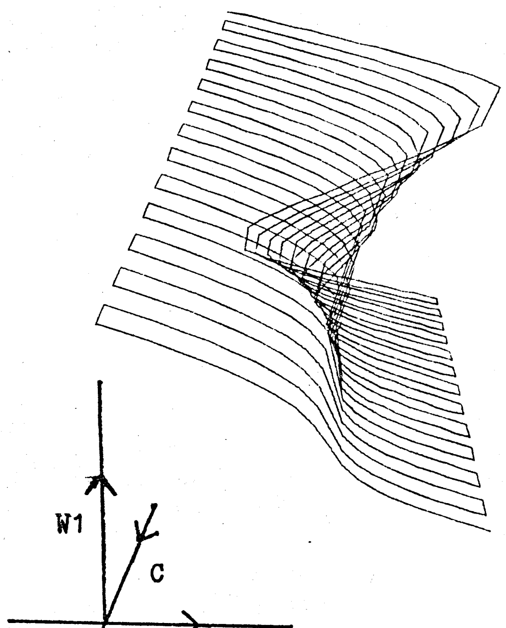

Fig. 2b

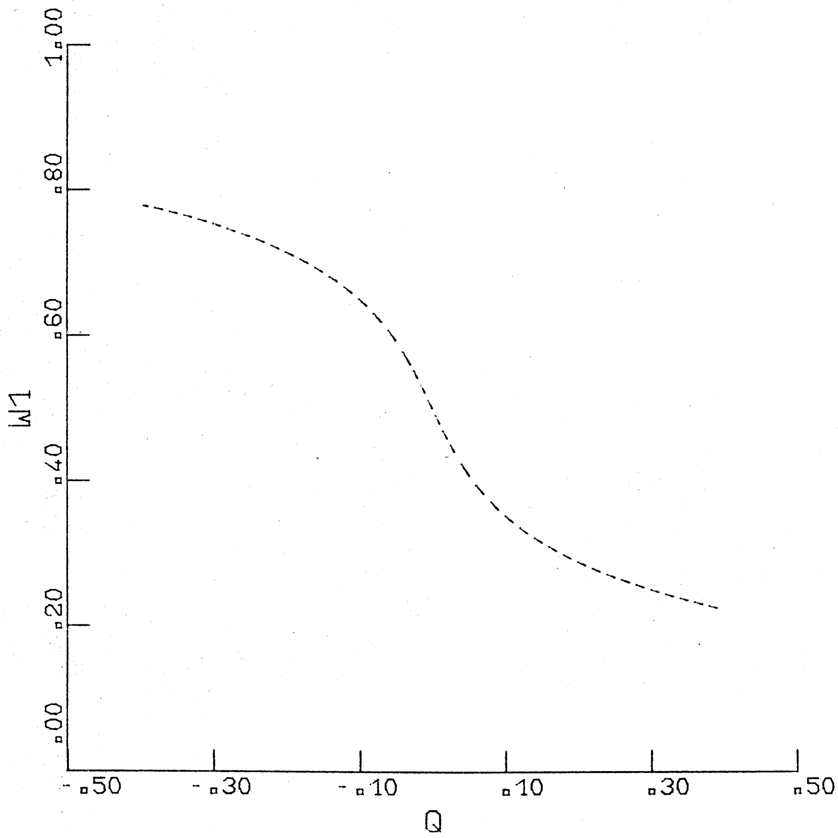

FIG.3A (C = 0.25)

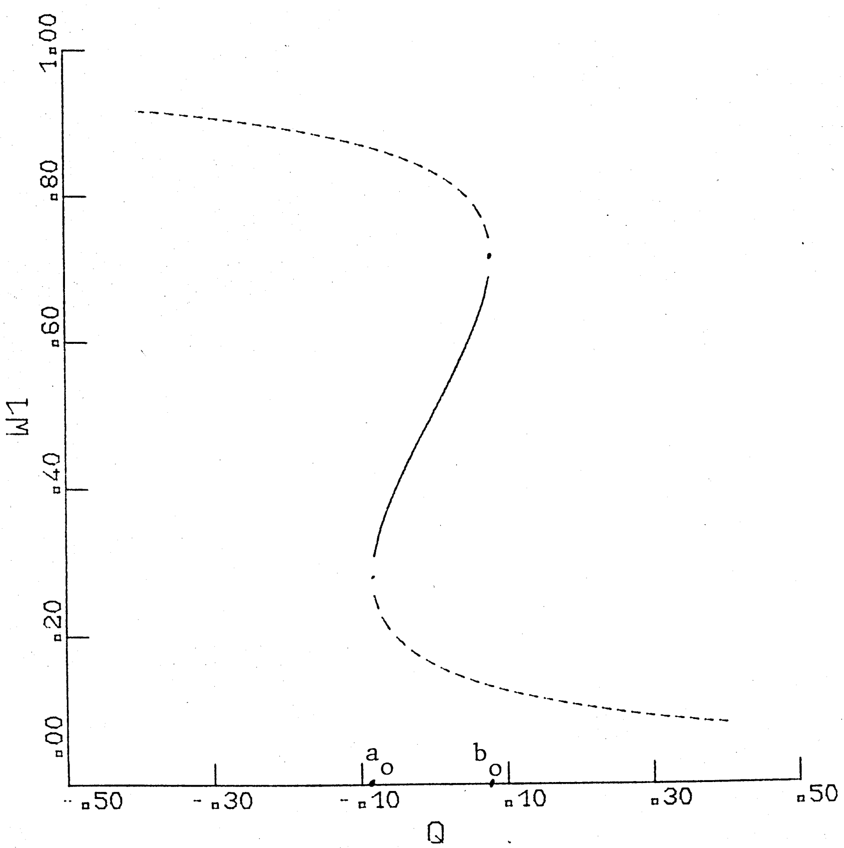

FIG.3B (C = 0.08)

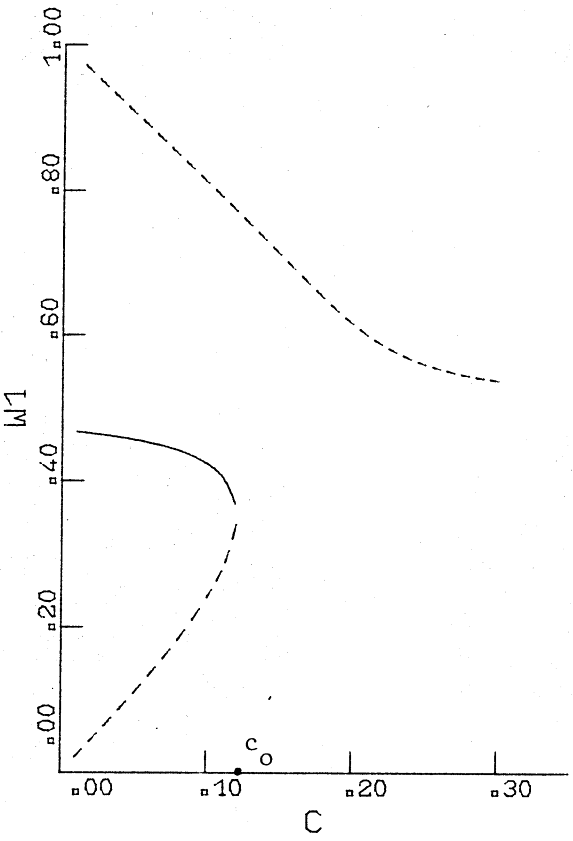

FIG. 4A (Q = -0.03)

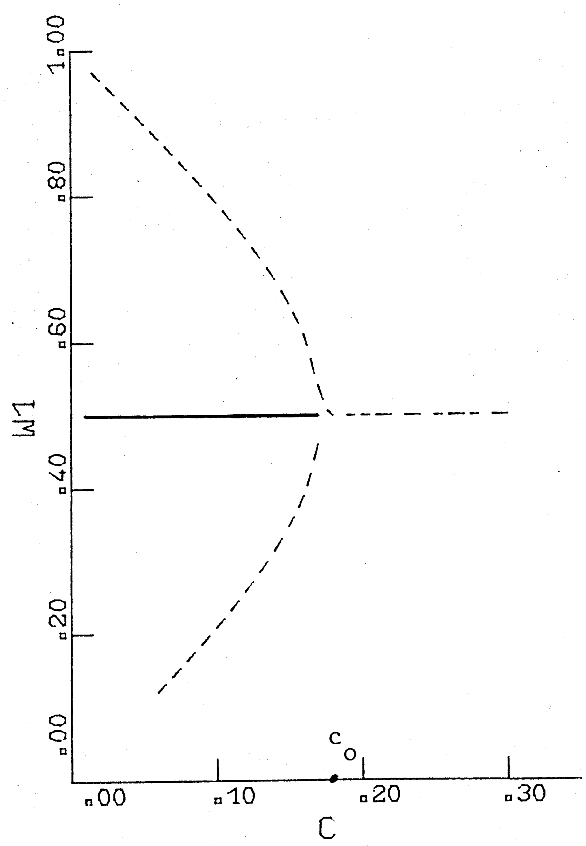

FIG. 4B (Q = 0.0)

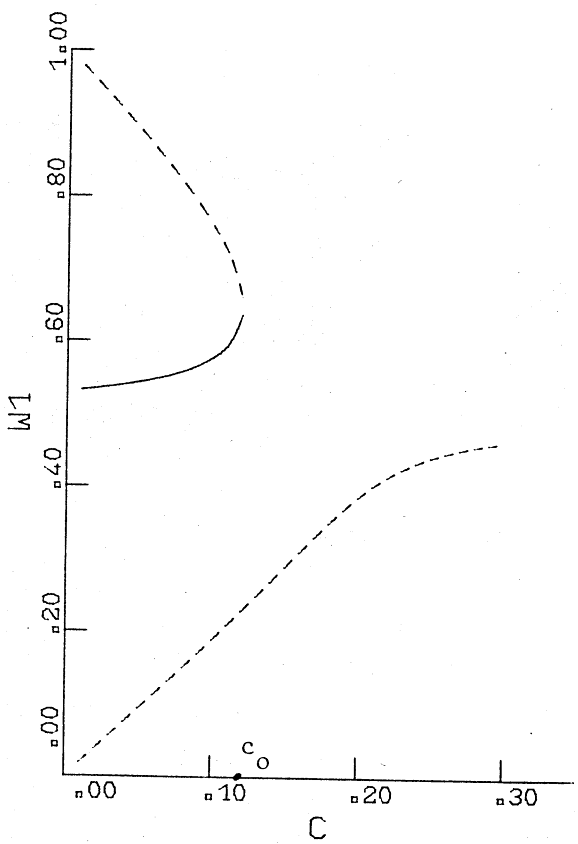

FIG. 4C (Q = 0.03)

LIST OF REPORTS 1983

8300 "Publications of the Econometric Institute Second Half 1982: List of Reprints 300-334, Abstracts of Reports".

8301/S "Constraint Analysis and Hypothesis Generation: An example of systematic preliminary investigation", by J.C.F.L. Groen, A.R. Thurik and D. van der Wijst.

8302/S "Dominated Variation and Related Concepts and Tauberian Theorems for Laplace Transforms", by L. de Haan and U. Stadtmüller.

8303/S "Further Study on the Use of Retail Floorspace and its Efficiency", by A.R. Thurik and J. Koerts.

8304/E "On the Utility Functions of the Indirect Addi-Log Budget Allocation Model", by J. van Daal.

8305/M "On the Difference between Perfect Powers", by J.W.M. Turk.

8306/S "Regression Coëfficient Estimation in Small Linear Models with Unknown Autocorrelation: Evaluation and Recommendation", by C. Dubbelman and E.A. de Groot.

8307/O "A Hierarchical Scheduling Problem with a Well-Solvable Second Stage", by J.B.G. Frenk, A.H.G. Rinnooy Kan and L. Stougie.

8308/M "The Serre Problem for Discrete Hodge Algebras", by A.C.F. Vorst.

8309/S "A Spectral Representation for Max-Stable Processes", by L. de Haan.

8310/M "The Cusp Catastrophe in the Urban Retail Model", by J.F. Kaashoek and A.C.F. Vorst.