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by

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Abstract

In this paper the conditions for integrability of the indirect addi-log budget allocation model are sharpened. This is done by means of the indirect utility function. The conditions for the existence of an analytical expression of the direct utility function are set out and the pertinent analytical expression is presented.

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1. Introduction

The indirect addi-log can be introduced in several ways. In Somermeyer and Wit (1956) it is put forward as:

$$w_k = \frac{d_k \left(\frac{p_k}{C}\right)^{\alpha_k}}{\sum_{h=1}^{K} d_h \left(\frac{p_h}{C}\right)^{\alpha_h}}, \qquad (1)$$

where:

w_k = budget share of good k (= 1, ..., K),

p_k = price of good k,

C = total amount to be spent;

the parameters d_k are always positive. Somermeyer and Wit prove that if all α_k are less than 1 the system (1) meets the well-known four conditions of integrability for consumer demand systems: adding up, homogeneity, symmetry and negativity.

Leser (1941) introduced the model (1) as a model that meets the condition of adding up, i.e. $\sum_h w_h = 1$, and that has the property that the cross price elasticities of all goods k with respect to the price of some good ℓ are equal, meaning that if only price p_ℓ changes the amounts spent on the other items will change proportionately in order that total consumption remains equal to the total budget. Leser used the model as a vehicle for computing price elasticities using budget data plus one additional datum long before Frisch's (1959) well-known alternative.

Houthakker (1960, 1961) discovered the model's indirect utility function. He also gave the model its present name. Neither Leser nor Houthakker got deeply into the subject of optimality.

In this paper we shall set out that Somermeyer's condition (all $\alpha_k < 1$) for integrability can be sharpened. To be precise, we shall show that in order that (1) be the result of maximizing a strictly quasi-concave utility function subject to a budget restriction, at most one α_k may be equal to 1, whereas the other have to be less than 1. We shall do that, starting from the indirect utility function, in section 3. [In Van Driel (1974, section 4.7) the same assertion is proved by analyzing the Slutsky matrix of system (1). There exists some confusion about this borderline case. Some authors even say that the utility functions's quasi-concavity is not strict in this case.]

Second, we shall prove (in section 4) that if one parameter α_k is equal to 1, there exists an analytical expression for the direct utility function. If all α_k are less than 1 than only in some special cases (a set of measure zero in the whole set of possibilities) an analytical expression for the direct utility function can be found.

2. Some lemmas

In this section some lemmas are mentioned. We need them in the next sections. For reason of shortness we only indicate briefly how they can be proved or where a proof can be found.

Lemma 1. If a demand system's indirect utility function is increasing and strictly quasi-convex in its arguments $r_1, \ldots, r_K$ ($r_k = p_k/C$) and differentiable in $r_1, \ldots, r_K$, then the system can be considered as the result of maximizing a strictly quasi-concave direct utility function subject to a budget restriction.

The proof can be given in two steps. First, because the indirect utility function is increasing in C given prices it can be inverted into an expenditure function that is strictly quasi-concave and linear-homogeneous in prices. Second, it can be proved that this expenditure function defines a direct utility function that is strictly quasi-concave in quantities. This can be done in a way similar to that of Varian (1980) p. 39-40, where it is proved how a cost function defines a technology; Varian restricts himself to quasi-concavity but the argument can also be used for strict quasi-concavity.

Lemma 2. If $f(x_1, \ldots, x_N)$ is twice differentiable and quasi-concave then for each $n = 1, \ldots, N$:

$$(-1)^n D_n \geqslant 0, \tag{2}$$

where:

$$D_n = \det \begin{bmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{bmatrix} \tag{3}$$

with $f_n = \partial f/\partial x_n$ and $f_{nn'} = \partial^2 f/\partial x_n \partial x_{n'}$.

Lemma 3. The twice differentiable function f is quasi-concave if for all $n = 1, \ldots, N$:

$$(-1)^n D_n > 0. \tag{4}$$

These two lemmas together form Theorem 5 of Arrow-Enthoven (1961). Note that lemma 2 is a necessary condition for quasi-concavity, whereas lemma 3 is a sufficient condition and that both show a subtle difference. In the next section we use lemma 2 in order to get a first restriction on the parameters of the indirect utility function; the sufficient and necessary conditions for strict quasi-concavity are further determined by using the special properties of the function at stake. Lemma 3 is mentioned for completeness' sake.

Lemma 4. The system:

$$q_k = \frac{d_k r_k^{\alpha_k - 1}}{\sum_{h=1}^{K} d_h r_h^{\alpha_h}}, \tag{5}$$

with $r_k = p_k/C$ and q_k = quantity consumed of good k (= 1, ..., K), is a bijection of the positive K-dimensional orthant of the vectors $(r_1 \ldots r_K)$ into the positive K-dimensional orthant of the vectors $(q_1 \ldots q_K)$ if all α_k are less than 1.

Lemma 5. If $\alpha_1 = 1$ and $\alpha_k < 1$ for $k = 2, \ldots, K$ then the system (5) is a bijection of the positive K-dimensional orthant of all r-vectors into the subset of the positive orthant of q-vectors such that:

$$\sum_{h=2}^{K} q_h \left(\frac{q_h/d_h}{q_1/d_1}\right)^{\frac{1}{\alpha_h - 1}} < 1. \tag{6}$$

The two lemmas above are proved in Van Driel (1974), section 4.8 and 4.9. They are purely mathematical and allow us to prove in a simple way that the direct utility function is always differentiable.

Lemma 6.

$$\det \begin{bmatrix} 0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_{11} & 0 & \cdots & 0 \\ a_2 & 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \sum_{j=1}^{n} a_j^2 \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} a_{\ell\ell}. \tag{7}$$

The proof of this lemma is elementary.

Lemma 7. The system of N partial differential equations in the positive real numbers $x_1, \ldots, x_N$:

$$\frac{\partial f}{\partial x_n} = a_n x_n^{\alpha_n - 1} . g(x_1, \ldots, x_N) \tag{8}$$

has as general solution:

$$f(x_1, \ldots, x_N) = F\{\sum_{n=1}^{N} a_n x_n^{(\alpha_n)}\}, \tag{9}$$

where F is some differentiable function of one variable and where $x_n^{(\alpha_n)}$ is the so-called Box-Cox transformation of x_n:

$$x_n^{(\alpha_n)} = \frac{x_n^{\alpha_n} - 1}{\alpha_n} \quad \text{if } \alpha_n \neq 0 \tag{10}$$

$$= \log x_n \quad \text{if } \alpha_n = 0,$$

if and only if for all $x_1, \ldots, x_N$:

$$g(x_1, \ldots, x_N) = F'\{\sum_{n=1}^{N} a_n x_n^{(\alpha_n)}\}. \tag{11}$$

[Note that $x_n^{(\alpha_n)}$ is continuous in α_n for each positive value of x_n and that always $dx_n^{(\alpha_n)}/dx_n = x_n^{\alpha_n - 1}$.]

The proof of this lemma can be given by considering the function f with:

$$f(x_1, \ldots, x_N) = G\{\sum_n a_n x_n^{(\alpha_n)}\} \cdot A(x_1, \ldots, x_N) + B(x_1, \ldots, x_N), \qquad (12)$$

where neither A nor B are non-constant functions of $\sum_n a_n x_n^{(\alpha_n)}$ but otherwise arbitrary and where G is some differentiable function of one variable. After substitution into (8) one can conclude that (12) is a solution of (8) if and only if A is a non-zero constant and B is an arbitrary constant. From this (9) and (11) follow easily.

3. The indirect utility function

In order to find the indirect utility function v of the system (5) we use Roy's theorem. For our purposes this theorem can most suitably be formulated as:

$$w_k = \frac{r_k \frac{\partial v}{\partial r_k}}{\sum_h r_h \frac{\partial v}{\partial r_h}}. \qquad (13)$$

Comparing this with (1) yields:

$$\frac{\partial v}{\partial r_k} = d_k r_k^{\alpha_k - 1} \cdot \phi(r_1, \ldots, r_K), \qquad (14)$$

where (because v is decreasing in the r_k) ϕ is a negative function of $r_1, \ldots, r_K$. Lemma 7 learns us that there is no loss of generality if we take as indirect utility function:

$$v = -\sum_{h=1}^{K} d_h r_h^{(\alpha_h)}. \qquad (15)$$

This formulation is a generalization of Houthakker's (1960) expression for v $(= -\sum_h \frac{d_h}{\alpha_h} r_h^{\alpha_h}$ in our notation) because (15) allows for zero values of some α_k; it differs from Houthakker's expression by $\sum_k d_k/\alpha_k$ in case no α_k is equal to zero.

The Hessian of the indirect utility function is diagonal. Hence the matrices in (3) of lemma 2 have the form of the matrix in (7) of lemma 6. Instead of v we consider now -v and apply lemma 2 in order to find a first set of restrictions on the parameters α_k. It can now easily be seen, using lemma 6, that for all k = 1, ..., K the determinants D_k of -v are:

$$D_k = -\sum_{j=1}^{k} d_j^2 r_j^{2\alpha_j-2} \prod_{\substack{\ell=1 \\ \ell\neq j}}^{k} c_\ell(\alpha_\ell-1) r_\ell^{\alpha_\ell-2}. \tag{16}$$

and, therefore, $(-1)^k D_k \geq 0$ if all $\alpha_k \leq 1$ and $(-1)^k D_k > 0$ for each k = 1, ..., K if all $\alpha_k \leq 1$ and at most one $\alpha_k = 1$.

Hence for quasi-convexity[1] all α_k have to be at most equal to 1; apparently, in this case quasi-convexity cannot exist without convexity. In the special case at stake, however, strict quasi-convexity is impossible if two or more α_k are equal to 1. Let, say, $\alpha_1 = \alpha_2 = 1$. For each two vectors $(r_1^1, \ldots, r_K^1)$ and $(r_1^2, \ldots, r_K^2)$ that only differ in the first two elements and that yield equal values of indirect utility one can prove that any convex combination of such two vectors yields also that value of indirect utility.

If all α_k are less than 1 then v is even strictly convex and, therefore also strictly quasi-convex. The only remaining case to consider is that in which all α_k are at most equal to one and only one of them is exactly equal to 1. Suppose, therefore, $\alpha_1 = 1$ and $\alpha_k < 1$ for k = 2, ..., K. Let be given two vectors r^1 and r^2 and let $\underline{r}^1$ and $\underline{r}^2$ be the (K-1)-vectors that result from omitting the first elements of r^1 and r^2, respectively.

First, we suppose $\underline{r}^1 \neq \underline{r}^2$. Then:

1) Or, equivalently, quasi-concavity of -v.

$$-v(\lambda r^1 + (1-\lambda)r^2) = d_1(\lambda r_1^1 + (1-\lambda)r_1^2) + \phi(\lambda \underline{r}^1 + (1-\lambda)\underline{r}^2), \tag{17}$$

where λ is a real number between 0 and 1 and where ϕ is a function of K-1 variables that is strictly concave because $\alpha_k < 1$ for k = 2, ..., K. Hence, because $\underline{r}^1 \neq \underline{r}^2$,

$$\phi(\lambda \underline{r}^1 + (1-\lambda)\underline{r}^2) > \lambda\phi(\underline{r}^1) + (1-\lambda)\phi(\underline{r}^2). \tag{18}$$

This means:

$$\begin{aligned} -v(\lambda r^1 + (1-\lambda)r^2) &> \lambda(d_1 r_1^1 + \phi(\underline{r}^1)) + (1-\lambda)(d_1 r_1^2 + \phi(\underline{r}^2)) = \\ &= -(\lambda v(r^1) + (1-\lambda)v(r^2)). \end{aligned} \tag{19}$$

Consequently, in this case:

$$v(\lambda r^1 + (1-\lambda)r^2) < \lambda v(r^1) +(1-\lambda)v(r^2) \leq \max\ (v(r^1),\ v(r^2)). \tag{20}$$

Second, we suppose alterntatively $\underline{r}^1 = \underline{r}^2$ and $r_1^1 \neq r_1^2$. Without loss of generality we can take $r_1^1 > r_1^2$. Then we have:

$$\begin{aligned} v(\lambda r^1 + (1-\lambda)r^2) &= -\lambda d_1 x_1^1 - (1-\lambda)d_1 x_1^2 - \phi(\underline{r}^2) < v(r^2) = \\ &= \max\ (v(r^1),\ v(r^2)). \end{aligned} \tag{21}$$

This means that for the indirect utility function in question always:

$$v(\lambda r^1 + (1-\lambda)r^2) < \max\ (v(r^1),\ v(r^2)) \tag{22}$$

if $r^1 \neq r^2$. Hence v is strictly quasi-convex. This proves the following theorem.

Theorem 1. The indirect utility function (15) of the addilog budget allocation model is strictly quasi-convex if and only if all $\alpha_k \leq 1$ and at most one α_k is equal to one.

4. The direct utility function

According to theorem 1 and lemma 1 and because of the indirect utility function's differentiability there exists a strictly quasi-concave direct utility function such that the addi-log budget allocation model can be considered as the result of maximizing this direct utility function subject to a budget restriction. According to the lemmas 4 and 5 the addi-log model can be inverted into a system of K differentiable equations:

$$r_k = \psi_k(q_1, \ldots, q_K).$$

Inserting this into the indirect utility function results in the direct utility function which is, therefore, differentiable.

Hence we have proved:

Theorem 2. The addi-log budget allocation model (5) has a differentiable strictly quasi-concave direct utility function if and only if all $\alpha_k \leq 1$ for all k and $\alpha_k = 1$ for at most one k.

If all α_k are less than 1 it is virtually impossible to perform the inversion (23) analytically. If some α_k are irrational it is completely impossible; if all α_k are rational then the inversion boils down to solving a polynomial equation. If the degree of that equation is less than 5 it can be solved analytically; otherwise such solutions are possible in some rare cases only.

If one α_k is equal to 1, however, then we can perform the inversion analytically. Let $\alpha_1 = 1$. Dividing the expressions for q_k and q_1 according to (5) results in:

$$\frac{q_k}{q_1} = \frac{d_k}{d_1} \left(\frac{p_k}{C}\right)^{\alpha_k - 1}, \qquad (24)$$

for k = 2, ..., K. Hence for these values of k we have:

$$\frac{p_k}{C} = \left(\frac{q_k/d_k}{q_1/d_1}\right)^{\frac{1}{\alpha_k - 1}} \qquad (25)$$

Inserting this in (5) for k = 1 gives:

$$q_1 = \frac{d_1}{d_1 \cdot \frac{p_1}{C} + \sum_{k=2}^{K} d_k \left(\frac{q_k/d_k}{q_1/d_1}\right)^{\frac{\alpha_k}{\alpha_k - 1}}} \tag{26}$$

or

$$\frac{p_1}{C} = \frac{1}{q_1} - \sum_{k=2}^{K} \frac{d_k}{d_1} \left(\frac{q_k/d_k}{q_1/d_1}\right)^{\frac{\alpha_k}{\alpha_k - 1}} . \tag{27}$$

We insert (25) and (27) into the indirect utility function. For ease of exposition we restrict ourselves to the case that each $\alpha_k \neq 0$. The indirect utility function can then be written as:

$$v = - \sum_{k=1}^{K} \frac{d_k}{\alpha_k} \left(\frac{p_k}{C}\right)^{\alpha_k} . \tag{28}$$

Note that $\alpha_1 = 1$. The direct utility function now becomes:

$$u = - \frac{d_1}{q_1} + \sum_{k=2}^{K} d_k \cdot \frac{\alpha_k - 1}{\alpha_k} \left(\frac{q_k/d_k}{q_1/d_1}\right)^{\frac{\alpha_k}{\alpha_k - 1}} . \tag{29}$$

For k > 1 the marginal utility is:

$$\frac{\partial u}{\partial q_k} = d_k \left(\frac{q_k/d_k}{q_1/d_1}\right)^{\frac{1}{\alpha_k - 1}} \cdot \frac{1/d_k}{q_1/d_1} > 0, \tag{30}$$

whereas for k = 1:

$$\frac{\partial u}{\partial q_1} = \frac{d_1}{q_1^2} [1 - \sum_{k=2}^{K} q_k \left(\frac{q_k/d_k}{q_1/d_1}\right)^{\frac{1}{\alpha_k - 1}}], \tag{31}$$

which is positive if the expression between the square brackets is positive. This is, however, just the case if q is in the range of the mapping (5); see lemma 5.

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