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A SUFFICIENT CONDITION IN OPTIMAL CONTROL THEORY

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A sufficient condition in optimal control theory

by

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Abstract. In this paper a sufficient condition for local optimality is given. From this condition for local optimality a sufficient condition for global optimality is deduced.

Contents.

1. Introduction.

First we formulate the optimal control problem, which we shall consider.
Consider a system of differential equations and initial conditions

$$\dot{x} = f(x,u) \qquad (1.1)$$

$$x(t_o) = x_o;\ t_o \text{ and } x_o \text{ given} \qquad (1.2)$$

where x, the state vector, and f are n-vectors and u is an m-vector.
The m-dimensional vector u(t) which is called the control function, or control for the system, is defined on an interval $[t_o,t_u]$, where the right endpoint t_u may depend on the function u(t); we assume that u(t) satisfies the following conditions:
1) u(t) is piecewise continuous on $[t_o,t_u]$ and u(t) is left continuous at each point of discontinuity $\tau \in (t_o,t_u)$.
2) For all $t \in [t_o,t_u]$, $u(t) \in U$, where $U \subset \mathbb{R}^m$ is a prescribed set, called the control set.

Controls which satisfy the conditions 1) and 2) above are called permissible.

A function x(t) is called a corresponding trajectory of the permissible control u(t), $t \in [t_o,t_u]$, if x(t) has the following properties: x(t) is defined and continuous on an interval of the form $[t_o,t_o+\delta]$, $\delta > 0$; $x(t_o) = x_o$ and $\dot{x}(t) = f(x(t),u(t))$ for all t with $t_o \leq t \leq \min\{t_u,t_o+\delta\}$ and t not a point of discontinuity of u(t).

We denote the components of the vector f(x,u) by $f_i(x,u)$, $i = 1,2, \ldots, n$. The functions $f_i(x,u)$ are defined and continuous on $\mathbb{R}^n \times U$; moreover, we assume that the functions

$\frac{\partial f_i(x,u)}{\partial x_j}$, $i,j = 1,2, \ldots, n$, exist and are continuous on $\mathbb{R}^n \times U$.

We are given a non-empty set $T \subset \mathbb{R}^n$

We are also given a function f_o: $R^n \times U \to \mathbb{R}$, which enjoys the same properties as the functions f_i.

<u>Definition</u>. Let Ω be an open subset of $\mathbb{R}^n$ such that $\bar{\Omega} \cap T \neq \emptyset$. ($\bar{\Omega}$ denotes the closure of Ω). We call a permissible control $u(t)$, $t \in [t_o,t_u]$, Ω-feasible if the corresponding trajectory $x(t)$ is defined throughout the interval $[t_o,t_u]$ and satisfies the following conditions

1) $x(t_u) \in T$
2) $x(t) \in \Omega$ for all $t \in [t_o,t_u)$

If $\Omega = \mathbb{R}^n$ we say simply "feasible" instead of "Ω-feasible".

Fixing a set $\Omega \subset \mathbb{R}^n$ with Ω open and $\bar{\Omega} \cap T \neq \emptyset$, the optimal control problem which we shall consider, may now be stated as follows:

Find among the Ω-feasible controls that control which minimizes the functional

$$J(u) = \int_{t_o}^{t_u} f_o(x(t),u(t))dt \qquad (1.3)$$

(Of course, the function $x(t)$ in (1.3) is the corresponding trajectory of $u(t)$).

An Ω-feasible control $u^*(t)$, with corresponding trajectory $x^*(t)$, which yields the solution of the above problem, is called an Ω-optimal control; $x^*(t)$ is called the corresponding Ω-optimal trajectory. We shall refer to the pair $(u^*(t),x^*(t))$ as an Ω-optimal pair. In the case $\Omega = \mathbb{R}^n$ we speak simply about "(global)optimal" instead of "$\mathbb{R}^n$-optimal".

We conclude this introduction with some remarks on the notation which we shall use. A vector will be either a column or a row vector, as it will be clear from the context how the vector is to be considered. Thus we shall write the inner product of two n-vectors x and y simply as xy. The partial differential operator

$$\left[\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_n}\right]$$

will be denoted by ∇.

2. A sufficient condition for Ω-optimality.

The following theorem gives a sufficient condition for Ω-optimality.

Theorem 1. Let $u^*(t), t \in [t_o, t_{u^*}]$, be an Ω-feasible control with corresponding trajectory $x^*(t)$. Suppose there exists a continuously differentiable function $g: \Omega \to \mathbb{R}$ which has the following properties:

(1) $\nabla g(x)f(x,u) + f_o(x,u) \geq 0$ for all $(x,u) \in R^n \times U$

(2) $\nabla g(x^*(t))f(x^*(t),u^*(t))+f_o(x^*(t),u^*(t)) = 0$ for all $t \in [t_o, t_{u^*})$

(3) For every Ω-feasible control $u(t)$, $t \in [t_o, t_u]$, with corresponding trajectory $x(t)$ we have

$$\lim_{t \to t_u} g(x(t)) \leq \lim_{t \to t_{u^*}} g(x^*(t)) = 0$$

Then $(u^*(t), x^*(t))$ is an Ω-optimal pair.

Proof. It will be convenient to write t^* for t_{u^*}. Let $u(t), t \in [t_o, t_u]$, be an Ω-feasible control with corresponding trajectory $x(t)$. Using equations (1.1) we see that

$$\nabla g(x(t))f(x(t),u(t)) = \nabla g(x(t))\dot{x}(t) = \frac{d}{dt}[g(x(t))]$$

Hence, for all $\tau \in [t_o, t_u)$ we have

$$\int_{t_o}^{\tau} \nabla g(x(t))f(x(t),u(t))dt = g(x(\tau)) - g(x(t_o)) = g(x(\tau))-g(x_o) \qquad (2.1)$$

From this we conclude that

$$\lim_{\tau\to t_u} \int_{t_o}^{\tau} \nabla g(x(t))f(x(t),u(t))dt \qquad (2.2)$$

exists.

Denoting the limit (2.2) by α and using (2.1) and (3) of the theorem, we see that

$$\alpha \leq -g(x_o) \qquad (2.3)$$

Using (2) and (3) of the theorem, we deduce analogously

$$\int_{t_o}^{t^*} f_o(x^*(t),u^*(t))dt = g(x_o) \qquad (2.4)$$

As a consequence of (1) of the theorem and (2.3), we obtain

$$\int_{t_o}^{t_u} f_o(x(t),u(t))dt \geq - \int_{t_o}^{t_u} \nabla g(x(t))f(x(t),u(t))dt = -\alpha \geq g(x_o) \qquad (2.5)$$

Comparing (2.3) and (2.4) we obtain

$$\int_{t_o}^{t^*} f_o(x^*(t),u^*(t))dt \leq \int_{t_o}^{t_u} f_o(x(t),u(t))dt$$

which is what we wanted to prove.

3. A sufficient condition for global optimality.

Let G be the set of all starting points x_o which can be steered to a point in T by a permissible control, i.e.

$G := \{x_o \in R^n \mid$ There exists a permissible control $u(t)$, $t \in [t_o,t_u]$, such that the corresponding trajectory has the following properties: $x(t)$ is defined throughout $[t_o,t_u]$ and $x(t_u) \in T\}$.

If the set Ω has the additional property that

$$T \cup \Omega \supset G \qquad (3.1)$$

then it is plain that Theorem 1 gives us a sufficient condition for global optimality. Thus we have

Theorem 2. Let Ω be an open subset of $\mathbb{R}^n$ such that $\bar{\Omega} \cap T \neq \emptyset$ and $T \cup \Omega \supset G$. Let $u^*(t), x^*(t)$ and g be as in Theorem 1 and let g have the properties (1), (2), (3) of Theorem 1. Then $(u^*(t), x^*(t))$ is a global optimal pair.

4. A remark on the function g; an example.

In applying Theorem 1, it is of course necessary to determine the function g. Which choice to make for g is suggested by (2.4). Let y be an arbitrary point of Ω. Suppose that $(\tilde{u}(t), \tilde{x}(t))$, $t_o \in [t_o, \tilde{t}]$ is an Ω-optimal pair for the control problem which we considered, but with (1.2) replaced by $x(t_o) = y$. Now define

$$g(y) := \int_{t_o}^{\tilde{t}} f_o(\tilde{x}(t), \tilde{u}(t))dt \qquad (4.1)$$

If the function g, defined by (4.1), happens to be continuously differentiable on Ω, then it is the candidate to use in the application of theorem 1.
We end up with an example in which we apply the sufficient conditions given by the two theorems.

Example.

$t_1 \to \min!$
subject to

$\dot{x}_1(t) = u_1(t)$

$\dot{x}_2(t) = u_2(t)$

$x_1(0) = x^o$, $x_2(0) = x_2^o$; x_1^o and x_2^o given

$$x_1(t_1) = x_2(t_1) = 0$$

$$U = \{(u_1,u_2) \in \mathbb{R}^2 \,|\, u_1^2+u_2^2 = 1\}$$

In this example, $T = \{(0,0)\}$. Using Pontryagin's maximum principle (see [1]), we find $u_1^*(t) = -1 - \frac{x_1^o}{t_1^*}$,

$u_2^*(t) = \frac{-x_2^o}{t_1^*}$ with $t_1^* = -\frac{(x_1^o)^2+(x_2^o)^2}{2x_1^o}$ as extremal control; the corresponding extremal trajectory is

$$x_1^*(t) = -t - \frac{x_1^o t}{t_1^*} + t_1^* + x_1^o, \quad x_2^*(t) = \frac{-x_2^o t}{t_1^*} + x_2^o.$$

The corresponding value of the performance functional t_1 is

$$t_1^* = -\frac{(x_1^o)^2+(x_2^o)^2}{2x_1^o} \qquad (4.2)$$

The set G for this problem turns out to be

$$G = \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 < 0\} \cup \{(0,0)\}$$

To apply Theorem 1, we take $\Omega := \{(x_1,x_2) \in \mathbb{R}^2 \,|\, x_1 < 0\}$. In view of (4.2) we define $g\colon \Omega \to \mathbb{R}$ by

$$g(x_1,x_2) = -\frac{x_1^2+x_2^2}{2x_1}$$

It is rather easy to verify that g meets all the conditions of Theorem 1. In fact, since $\Omega \cup T = G$, the extremal pair $(u^*(t), x^*(t))$ is global minimizing by Theorem 2.

Reference.

[1] Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishenko, E.F.: The Mathematical Theory of Optimal Processes Pergamon Press 1964.

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