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A sufficient condition in optimal control theory

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<u>Abstract</u>. In this paper a sufficient condition for local optimality is given. From this condition for local optimality a sufficient condition for global optimality is deduced.

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1. Introduction.

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First we formulate the optimal control problem, which we shall consider.

Consider a system of differential equations and initial conditions

$$\dot{x} = f(x, u)$$
 (1.1)

$$x(t_{o}) = x_{o}; t_{o} \text{ and } x_{o} \text{ given}$$
 (1.2)

where x, the state vector, and f are n-vectors and u is an m-vector.

The m-dimensional vector u(t) which is called the control function, or control for the system, is defined on an interval $[t_0, t_u]$, where the right endpoint t_u may depend on the function u(t); we assume that u(t) satisfies the following conditions: 1) u(t) is piecewise continuous on $[t_0, t_u]$ and u(t) is left continuous at each point of discontinuity $\tau \in (t_0, t_u)$. 2) For all t $\in [t_0, t_u]$, u(t) \in U, where U $\subset \mathbb{R}^m$ is a prescribed set, called the control set.

Controls which satisfy the conditions 1) and 2) above are called permissible.

A function x(t) is called a corresponding trajectory of the permissible control u(t), $t \in [t_0, t_u]$, if x(t) has the following properties: x(t) is defined and continuous on an interval of the form $[t_0, t_0 + \delta]$, $\delta > 0$; x(t_0) = x_0 and $\dot{x}(t) = f(x(t), u(t))$ for all t with $t_0 \leq t \leq \min\{t_u, t_0 + \delta\}$ and t not a point of discontinuity of u(t).

We denote the components of the vector f(x,u) by $f_i(x,u)$, i = 1,2, ..., n. The functions $f_i(x,u)$ are defined and continuous on $\mathbb{R}^n \times U$; moreover, we assume that the functions

 $\frac{\partial f_i(x,u)}{\partial x_j}$, i, j = 1, 2, ..., n, exist and are continuous on $\mathbb{R}^n \times \mathbb{U}$.

We are given a non-empty set $T \subset \mathbf{K}^n$

We are also given a function $f_0: \mathbb{R}^n \times U \to \mathbb{R}$, which enjoys the same properties as the functions f_i .

Definition. Let Ω be an open subset of \mathbb{R}^n such that $\overline{\Omega} \cap T \neq \mathscr{A}$.

 $(\bar{\Omega} \text{ denotes the closure of } \Omega)$. We call a permissible control u(t), $t \in [t_0, t_u]$, Ω -feasible if the corresponding trajectory x(t) is defined throughout the interval $[t_0, t_u]$ and satisfies the following conditions

- 1) $x(t_u) \in T$
- 2) $x(t) \in \Omega$ for all $t \in [t_0, t_u)$

If $\Omega = \mathbf{R}^n$ we say simply "feasible" instead of " Ω -feasible".

Fixing a set $\Omega \subset \mathbf{R}^n$ with Ω open and $\overline{\Omega} \land T \neq \emptyset$, the optimal control problem which we shall consider, may now be stated as follows:

Find among the Ω -feasible controls that control which minimizes the functional

$$J(u) = \int_{t_0}^{t_u} f_0(x(t), u(t)) dt$$
 (1.3)

(Of course, the function x(t) in (1.3) is the corresponding trajectory of u(t)).

An Ω -feasible control u^{*}(t), with corresponding trajectory x^{*}(t), which yields the solution of the above problem, is called an Ω -optimal control; x^{*}(t) is called the corresponding Ω -optimal trajectory. We shall refer to the pair (u^{*}(t),x^{*}(t)) as an Ω -optimal pair. In the case $\Omega = \mathbf{R}^{n}$ we speak simply about "(global)optimal" instead of " \mathbf{R}^{n} -optimal".

We conclude this introduction with some remarks on the notation which we shall use. A vector will be either a column or a row vector, as it will be clear from the context how the vector is to be considered. Thus we shall write the inner product of two nvectors x and y simply as xy. The partial differential operator

 $\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right]$

will be denoted by ∇ .

2. A sufficient condition for Ω -optimality.

The following theorem gives a sufficient condition for Ω -optimality.

<u>Theorem 1</u>. Let $u^{*}(t), t \in [t_{0}, t_{u^{*}}]$, be an Ω -feasible control with

corresponding trajectory $x^*(t)$. Suppose there exists a continuously differentiable function g: $\Omega \rightarrow \mathbf{R}$ which has the following properties:

(1) $\nabla g(x)f(x,u) + f_0(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^n \times U$

(2) $\nabla g(x^{*}(t))f(x^{*}(t),u^{*}(t))+f_{0}(x^{*}(t),u^{*}(t)) = 0$ for all $t \in [t_{0}, t_{u^{*}})$

(3) For every Ω -feasible control u(t), t \in [t₀,t_u], with corresponding trajectory x(t) we have

 $\lim_{t \to t} g(x(t)) \leq \lim_{t \to t} g(x^{*}(t)) = 0$

Then $(u^{*}(t), x^{*}(t))$ is an Ω -optimal pair.

Proof. It will be convenient to write t* for t *. Let

 $u(t), t \in [t_0, t_u]$, be an Ω -feasible control with corresponding trajectory x(t). Using equations (1.1) we see that

$$\nabla g(x(t))f(x(t),u(t)) = \nabla g(x(t))\dot{x}(t) = \frac{d}{dt}[g(x(t))]$$

Hence, for all $\tau \in [t_0, t_u)$ we have

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$$\int_{t_{o}} \nabla g(x(t)) f(x(t), u(t)) dt = g(x(\tau)) - g(x(t_{o})) = g(x(\tau)) - g(x_{o})$$
(2.1)

From this we conclude that

$$\lim_{t \to t} \int_{u}^{t} \nabla g(x(t)) f(x(t), u(t)) dt \qquad (2.2)$$

exists.

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Denoting the limit (2.2) by α and using (2.1) and (3) of the theorem, we see that

$$\alpha \leq -g(x_0) \qquad (2.3)$$

Using (2) and (3) of the theorem, we deduce analogously

$$\int_{t_{0}}^{t} f_{0}(x^{*}(t), u^{*}(t)) dt = g(x_{0})$$
 (2.4)

As a consequence of (1) of the theorem and (2.3), we obtain $t_{u} \int_{t_{0}}^{t} f_{0}(x(t), u(t)) dt \ge - \int_{t_{0}}^{t} \nabla g(x(t)) f(x(t), u(t)) dt = -\alpha \ge g(x_{0})$ (2.5)

Comparing (2.3) and (2.4) we obtain

$$t^{*}_{f_{o}}f_{o}(x^{*}(t),u^{*}(t))dt \leq \int_{t_{o}}^{t_{u}}f_{o}(x(t),u(t))dt$$

which is what we wanted to prove.

3. A sufficient condition for global optimality.

Let G be the set of all starting points x_0 which can be steered to a point in T by a permissible control, i.e.

If the set Ω has the additional property that

$$\Gamma \cup \Omega \supset G$$
 (3.1)

then it is plain that Theorem 1 gives us a sufficient condition for global optimality. Thus we have

<u>Theorem 2</u>. Let Ω be an open subset of \mathbb{R}^n such that $\overline{\Omega} \cap T \neq \emptyset$ and $T \cup \Omega \supset G$. Let $u^*(t), x^*(t)$ and g be as in Theorem 1 and let g have the properties (1), (2), (3) of Theorem 1. Then $(u^*(t), x^*(t))$ is a global optimal pair.

4. A remark on the function g; an example.

In applying Theorem 1, it is of course necessary to determine the function g. Which choice to make for g is suggested by (2.4). Let y be an arbitrary point of Ω . Suppose that $(\widetilde{u}(t), \widetilde{x}(t))$, $t_0 \in [t_0, \widetilde{t}]$ is an Ω -optimal pair for the control problem which we considered, but with (1.2) replaced by $x(t_0) = y$. Now define

$$g(y) := \int_{t_0}^{t} f_0(\widetilde{x}(t), \widetilde{u}(t)) dt \qquad (4.1)$$

If the function g, defined by (4.1), happens to be continuously differentiable on Ω , then it is <u>the</u> candidate to use in the application of theorem 1. We end up with an example in which we apply the sufficient

conditions given by the two theorems.

Example.

$$t_1 \neq \min!$$

subject to
 $\dot{x}_1(t) = u_1(t)$
 $\dot{x}_2(t) = u_2(t)$
 $x_1(0) = x^0, x_2(0) = x_2^0, x_1^0 \text{ and } x_2^0 \text{ given}$

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 $x_1(t_1) = x_2(t_1) = 0$ $U = \{(u_1, u_2) \in \mathbb{R}^2 | u_1^2 + u_2^2 = 1\}$

In this example, $T = \{(0,0)\}$. Using Pontryagin's maximum

principle (see [1]), we find $u_1^{*}(t) = -1 - \frac{x_1^{0}}{t_1^{*}}$,

 $u_{2}^{*}(t) = \frac{-x_{2}^{o}}{t_{1}^{*}}$ with $t_{1}^{*} = -\frac{(x_{1}^{o})^{2} + (x_{2}^{o})^{2}}{2x_{1}^{o}}$ as extremal control; the

corresponding extremal trajectory is

$$x_1^*(t) = -t - \frac{x_1^{o}t}{t_1^*} + t_1^* + x_1^{o}, \ x_2^*(t) = \frac{-x_2^{o}t}{t_1^*} + x_2^{o}.$$

The corresponding value of the performance functional t₁ is

$$t_{1}^{*} = -\frac{(x_{1}^{\circ})^{2} + (x_{2}^{\circ})^{2}}{2x_{1}^{\circ}}$$
(4.2)

The set G for this problem turns out to be

$$G = \{(x_1, x_2) \in R^2 | x_1 < 0\} \cup \{(0, 0)\}$$

To apply Theorem 1, we take $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 | x_1 < 0\}$. In view of (4.2) we define g: $\Omega \rightarrow \mathbb{R}$ by

$$g(x_1, x_2) = -\frac{x_1^2 + x_2^2}{2x_1}$$

It is rather easy to verify that g meets all the conditions of Theorem 1. In fact, since $\Omega \cup T = G$, the extremal pair (u^{*}(t), x^{*}(t)) is global minimizing by Theorem 2.

Reference.

[1] Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V.,Mishenko, E.F.: The Mathematical Theory of Optimal ProcessesPergamon Press 1964.

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