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ON THE SNAPPER, LIEBLER - VITALE,
LAM THEOREM ON PERMUTATION
REPRESENTATIONS OF THE SYMMETRIC GROUP

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ON THE SNAPPER, LIEBLER-VITALE, LAM THEOREM ON PERMUTATION REPRESENTATIONS OF THE SYMMETRIC GROUP.

Michiel Hazewinkel and Ton Vorst

ABSTRACT. Let $\kappa = (\kappa_1, \dots, \kappa_m)$ be a descending partition of n and $S_\kappa = S_{\kappa_1} \times S_{\kappa_2} \times \dots \times S_{\kappa_m}$ be the corresponding Young subgroup of S_n . Denote by $\rho(\kappa)$ the representation of S_n which one gets by inducing the trivial representation of S_κ . If $\lambda = (\lambda_1, \dots, \lambda_m)$ is another partition of n with $\sum_{i=1}^r \kappa_i \geq \sum_{i=1}^r \lambda_i$ ($\forall 1 \leq r \leq m$) then $\rho(\kappa)$ is a subrepresentation of $\rho(\lambda)$. In this note we give an elementary complete direct proof of this fact.

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1. INTRODUCTION.

1. Let $\kappa = (\kappa_1, \dots, \kappa_m)$, $\kappa_i \in \mathbb{N} \cup \{0\}$, $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m \geq 0$ be a descending partition of n . We identify partitions which differ only by the addition of some additional zero's. An ordering, which we call the specialization order, is defined on the set of all partitions by

$$(1.1) \quad \kappa > \lambda \iff \sum_{i=1}^r \kappa_i < \sum_{i=1}^r \lambda_i, \quad r = 1, 2, \dots$$

The reverse order has been called the dominance order. It occurs in many, seemingly unrelated parts of mathematics [1,2,3], and one of the central occurrences is in the representation theory of the symmetric groups in characteristic zero.

Let $S_\kappa = S_{\kappa_1} \times \dots \times S_{\kappa_m}$ be the Young subgroup of S_n (S_{κ_i} is viewed as the permutation subgroup of S_n permuting the letters $\kappa_1 + \dots + \kappa_{i-1} + 1, \dots, \kappa_1 + \dots + \kappa_i$) corresponding to the partition κ and let $\rho(\kappa)$ be the representation of S_n obtained by inducing the trivial representation of S_κ up to S_n . Also let $[\kappa]$ be the irreducible representation of S_n (in characteristic zero) associated to the partition κ . Snapper [5] proved that $[\kappa]$ occurs in $\rho(\lambda)$ implies $\kappa < \lambda$ (this also follows readily from Young's rule) and conjectured the reverse, which he proved for $m = 2$. Proofs of the conjecture were given by Liebler-Vitale [4] and Lam [3]. Liebler and Vitale proved more precisely that $\kappa < \lambda$ implies that $\rho(\kappa)$ is a subrepresentation of $\rho(\lambda)$ (which obviously implies, the conjecture because $[\kappa]$ occurs in $\rho(\kappa)$).

In this note we give a completely elementary direct proof of the Liebler-Vitale result which requires no representation theory at all (beyond the definition of the permutation representations $\rho(\kappa)$) by constructing explicit homomorphisms of representations.

2. THE SNAPPER, LIEBLER-VITALE, LAM THEOREM.

2.1. Description of the permutation representation $\rho(\kappa)$.

Let $W(\kappa)$ be the set of all words of length n in the symbols a_1, \dots, a_m such that each a_i occurs exactly κ_i times. The group S_n acts in the obvious way on $W(\kappa)$ ($\sigma(b_1 \dots b_n) = b_{\sigma(1)} \dots b_{\sigma(n)}$, $\sigma \in S_n$) and the vector-space $V(\kappa)$ with the elements of $W(\kappa)$ as basis and the action extended linearly is the representation $\rho(\kappa)$. We shall denote the elements of $W(\kappa)$ and the corresponding basis elements of $V(\kappa)$ with the same symbols.

2.2. "Reduction to the case $m = 2$ ". It obviously suffices to prove the statement " $\kappa < \lambda \rightarrow \rho(\kappa)$ is a subrepresentation of $\rho(\lambda)$ " in the case that $\kappa < \lambda$ and $\kappa < \mu < \lambda \Rightarrow \kappa = \mu$ or $\lambda = \mu$. In this case one easily shows that there exist i and j , $i > j$ such that $\lambda_i = \kappa_i + 1$, $\lambda_j = \kappa_j - 1$ and $\lambda_r = \kappa_r$ for $r \neq i, j$. In this case we define a linear map

$$(2.3) \quad \beta_{\lambda, \kappa}: V(\lambda) \rightarrow V(\kappa)$$

by the formula

$$(2.4) \quad \beta_{\lambda, \kappa}(b_1 \dots b_n) = \sum b'_1 \dots b'_n$$

where the sum extends over all words $b'_1 \dots b'_n$ such that $b'_t = b_t$ for all but one t . And for that one t we have $b_t = a_i$ and $b'_t = a_j$. I.e. the words in the sum on the right are obtained by replacing precisely one occurrence of a_i by a_j . This is obviously an S_n -equivariant map.

We shall prove that $\beta_{\lambda, \kappa}$ is surjective if (and only if) $\lambda > \kappa$. This proves the theorem because the category of S_n -modules (in characteristic

zero) is semisimple. Alternatively observe that if $\alpha_{\kappa, \lambda}: V(\kappa) \rightarrow V(\lambda)$ is defined as $\beta_{\lambda, \kappa}$ with the letters a_j and a_i interchanged then $\alpha_{\kappa, \lambda}$ and $\beta_{\lambda, \kappa}$ are adjoint to each other in the sense that

$$(2.5) \quad \langle \alpha_{\kappa, \lambda} v, \omega \rangle = \langle v, \beta_{\lambda, \kappa} \omega \rangle, \quad v \in V(\kappa), \omega \in V(\lambda)$$

where the inner products on $V(\lambda)$ and $V(\kappa)$ are the ones for which $W(\lambda)$ and $W(\kappa)$ form orthonormal bases. This $\alpha_{\kappa, \lambda}$ is an S_n -equivariant injection iff $\beta_{\lambda, \kappa}$ is surjective and it remains to prove that $\beta_{\lambda, \kappa}$ is surjective if $\kappa < \lambda$.

To do this observe that as a vectorspace $V(\lambda)$ is the direct sum of $\binom{n}{\lambda} \binom{\lambda_i + \lambda_j}{\lambda_i}^{-1}$ copies of $V(\lambda_j, \lambda_i)$ indexed by all words in the symbols $a_1, \dots, \hat{a}_j, \dots, \hat{a}_i, \dots, a_m, c$ ($\hat{}$ denotes deletion) such that a_t occurs λ_t times and c occurs $\lambda_i + \lambda_j$ times. Similarly $V(\kappa)$ is the direct sum of $\binom{n}{\kappa} \binom{\kappa_i + \kappa_j}{\kappa_i}^{-1} = \binom{n}{\lambda} \binom{\lambda_i + \lambda_j}{\lambda_i}^{-1}$ copies of $V(\kappa_j, \kappa_i)$ and the homomorphism

(2.4) maps the copies of $V(\lambda_j, \lambda_i)$ and $V(\kappa_j, \kappa_i)$, labelled by the same word in $a_1, \dots, \hat{a}_j, \dots, \hat{a}_i, \dots, a_m, c$, into each other and is in fact the direct sum of these induced maps. Hence it is sufficient to prove the surjectivity of $\beta_{\lambda, \kappa}$ in the case $m = 2$.

2.6. Proof of the surjectivity of $\beta_{\lambda, \kappa}$ in the case $m = 2$. Let

$\lambda = (r-1, s+1)$, $\kappa = (r, s)$, $r + s = n$ and write x for a_j and y for a_i .

Then $W(r-1, s+1)$ consists of words of length n in $(r-1)$ x 's and $(s+1)$ y 's and $\beta = \beta_{\lambda, \kappa}$ changes such a word into the sum of all words which can be obtained from this word by changing precisely one y into an x . E.g.

$$(2.7) \quad \beta(xxxyyy) = xxxxyy + xxxxyx + xxxyyx$$

We shall now show that β is surjective if $r \geq s+1$ (We only need the case $r \geq s+2$). Let $W = W(r-1, s+1) \cup W(r, s)$. For each pair $\omega_1 = b_1 \dots b_n$,

$\omega_2 = b_1' \dots b_n'$ in W we define the distance $d(\omega_1, \omega_2)$ by

$$(2.8) \quad d(\omega_1, \omega_2) = \# \{t \mid b_t \neq b_t'\}.$$

(This distance is called Hamming distance in coding theory).

Now for $\omega_0 = x \dots xy \dots y \in W(r, s)$ let

$$(2.9) \quad E_t = \{\omega \in W \mid d(\omega_0, \omega) = t\}$$

Then $E_t \subset W(r, s)$ if t is even and $E_t \subset W(r-1, s+1)$ if t is odd.

Note that $\omega \in E_{2t}$ iff there are precisely t y 's among the first r letters and t x 's among the second s letters and similarly $\omega \in E_{2t+1}$ iff there are precisely $t+1$ y 's among the first r letters of ω and t x 's among the last s letters.

Now let

$$(2.10) \quad f = r^{-1} (c_0 \sum_{\omega \in E_1} \omega + c_1 \sum_{\omega \in E_3} \omega + \dots + c_s \sum_{\omega \in E_{2s+1}} \omega)$$

where

$$(2.11) \quad c_t = (-1)^t \binom{r-1}{t}^{-1}$$

We claim that $\beta(f) = \omega_0$. To see this observe that since $\omega \in W(r, s)$ and $r \geq s+1$ the maximum distance of a $\omega \in W$ to ω_0 is $2s+1$. Observe that if $\omega' \in E_{2t+1}$ then $\beta(\omega')$ is a sum of elements in E_{2t} and E_{2t+2} (except when $t = s$, then only elements of E_{2s} can occur by the maximum distance observation).

Now let $\omega'' = b_1 \dots b_n \in E_{2t}$ ($t \geq 1$) then the coefficient of $\omega'' \in \beta(f)$ is equal to

$$(2.12) \quad r^{-1} c_t (\# \{i \in \{1, \dots, r\} \mid b_i = x\}) + r^{-1} c_{t-1} (\# \{i \in \{r+1, \dots, r+s\} \mid b_i = x\}) =$$

$$r^{-1} c_t (r-t) + r^{-1} c_{t-1} t.$$

(The first contribution comes from the elements in E_{2t+1} whose i -th element was y and is transformed to x to decrease the distance to ω_0 ; the second contribution comes from elements of E_{2t-1} whose i -th element was y and is transformed to x to increase the distance).

By definition of c_t the right-hand side of (2.12) is zero.

The coefficient of $\omega_0 \in \beta(f)$ is equal to

$$(2.13) \quad r^{-1} c_0 (\#\{i \in \{1, \dots, r\} | b_i = x\}) = r^{-1} \cdot 1 \cdot r = 1.$$

This proves that $\omega_0 = \beta(f) \in \text{Im} \beta$ and hence $\omega \in \text{Im} \beta$ for all $\omega \in W(r, s)$ because β is S_n -equivariant and S_n acts transitively on $W(r, s)$. This concludes the proof.

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