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OPTIMAL HOUSEHOLD BEHAVIOUR UNDER TAXATION - A TWO-STAGE DYNAMIC PROGRAMMING APPROACH

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Abstract

This paper presents a method of solving a practical optimization problem which arose in the context of private households' reactions to income tax levied from couples.

Since the problem of constrained maximization of the household's life-time utility function in n prospective periods is complicated and sizeable, a two-stage dynamic programming approach is proposed.

Splitting the problem into two-sub problems: finding the optimum policy and approximation in the policy space, is based on the dynamic nature of problems of this type, which can be seen as a multi-period decision process.

Some provisional results obtained by experimental application appear to be sufficiently promising to justify elaboration.

Contents

1. Introduction

The optimization problems dealt with below arose in the context of private households' reactions to income tax levied from couples, with respect to their basic "instruments". The latter variables consist of working hours of both husband and wife, as well as their joint consumption for each year of their prospective life-time; they represent the arguments of the household's utility function to be maximized subject to constraints (cf. W.H. Somermeyer and R. Bannink, 1973).

The utility function is assumed to be separable with respect to time, such that it can be written as the sum of yearly sub-functions weighted by time discount factors. The latter relationships are specified as linear-homogeneous CES (constant elasticity of substitution) functions.

The constraints consist of both equalities and inequalities. Essentially, the equalities are of three kinds, viz.:

1. budget constraints, equating intertemporal changes in personal wealth = (dis)savings to differences between disposable income and consumption, with both beginning- and end-of-lifetime personal wealth set equal to zero;
2. income formation equations, proportionally relating labour and capital income to personal wealth, through supposedly fixed wage and interest rates, respectively;
3. relationships between taxes and their bases - mainly labour incomes of husband and wife, separately, and their capital income jointly.

The inequality constraints are both upper and lower bounds. The upper bounds apply to working hours (including do-it-yourself) of husband and wife. Lower bounds are imposed on personal wealth (maximum debt, proportional to income), and on consumption. Moreover, zeroes represent the logical lower bounds of essentially non-negative variables, such as working hours.

2. Formulation of the problem

We consider the problem of constrained maximization of the objective function:

$$F(x) = \sum_{\ell=\ell_o}^{\ell_n} U_\ell (1-\delta)^{\ell_o - \ell}, \tag{2.1}$$

in prospective periods $\ell = \ell_0, \ell_1, \ldots, \ell_n$, where U_ℓ are yearly utilities for the family and δ is a time-preference parameter. The maximization is subject to the equality budget constraint:

$$\sum_{\ell=\ell_o}^{\ell_n} [CKL_\ell - CKL_{\ell-1} - I_\ell + T_\ell + p_\ell(C_\ell - D_\ell)] = 0, \tag{2.2}$$

where:

CKL_ℓ - family wealth at the end of the period ℓ;
I_ℓ - taxable family income over the period ℓ;
T_ℓ - income tax paid over the period ℓ;
p_ℓ - price index for the period ℓ;
C_ℓ - family consumption during the period ℓ;
D_ℓ - value of "do-it-yourself" work during the period ℓ.

Economic considerations led to the choice of U_ℓ as Constant Elasticity of Substitution (CES) utility function (cf. W.J. Keller and A. Langhout, 1973):

$$U_\ell = (Y_{1\ell}^\rho + Y_{3\ell}^\rho + Y_{5\ell}^\rho)^{1/\rho}, \tag{2.3}$$

with:

$$Y_{1\ell} = \frac{F\ max - F_\ell}{\alpha_v}, \quad Y_{3\ell} = \frac{V\ max - V_\ell}{\alpha_v}, \quad Y_{5\ell} = \frac{C_\ell - C\ min}{\alpha_c} \tag{2.4}$$

and with:

$$\text{power } \rho = \frac{\sigma - 1}{\sigma}, \quad 0 < \sigma \leq 0.5, \tag{2.5}$$

where σ denotes the elasticity of substitution, and

$$\text{weights } \alpha_v, \alpha_c, \ 0 < \alpha_v, \alpha_c < 1, \text{ such that } \alpha_v + \alpha_c = 1 \tag{2.6}$$

Furthermore:

$F_\ell = F_{w\ell} + F_{d\ell}$ and $V_\ell = V_{w\ell} + V_{d\ell}$ number of working- and "do-it-yourself"-hours per week for men and women, respectively;

F max and V max: maximum available number of hours per week for activities and leisure for men and women, respectively;

C min: minimum yearly consumption for the family.

This choice of the form of the objective function requires the following supplementary inequality constraints and simple bounds:

$$\left.\begin{array}{l} F_\ell < F\ max \\ V_\ell < V\ max \\ C_\ell > C\ min \\ 0 \leq F_{w\ell},\ F_{d\ell},\ V_{w\ell},\ V_{d\ell} \end{array}\right\} \qquad (2.7)$$

3. Description of the model

Income functions $I_\ell = I_{s\ell} + I_{m\ell} + I_{v\ell}$ are assumed to be linear.

a. Income from savings:

$$I_{s\ell} = Rb\ CKL_{\ell-1}, \tag{3.1}$$

where bank rate $Rb = Sbr$, if $CKL_{\ell-1} \geq 0$, i.e. in the case of credits
$= Abr$, if $CKL_{\ell-1} < 0$. i.e. in the case of debits

b. Income from working:

$$I_{m\ell} = Ckh\ W_m\ F_{w\ell}, \text{ for men, and} \tag{3.2}$$

$$I_{v\ell} = Ckh\ W_v\ V_{w\ell}, \text{ for women,}$$

where Ckh: - holiday allowance fraction; and W_m, W_v - gross wage for every weekly working hour per year for men and women, respectively.

"Do-it-yourself" functions $D_\ell = D_{m\ell} + D_{v\ell}$ are assumed to be non-linear:

$$D_{m\ell} = d\ (1 - \frac{1}{Ckd\ F_{d\ell}}), \text{ for men, and} \tag{3.4}$$

$$D_{v\ell} = d\ (1 - \frac{1}{Ckd\ V_{d\ell}}), \text{ for women} \tag{3.5}$$

where d and Ckd are positive constants.

Income tax functions for the Netherlands are piece-wise linear $T_\ell = T_{m\ell} + T_{v\ell}$.

$$T_{m\ell} = (I_{s\ell} + I_{m\ell} - CIM(1) - CIM(I))\ Tk(I) + CIM(2)\ Tk(2) +$$
$$+ (CIM(3) - CIM(2))\ Tk(3) + \ldots + (CIM(I) - \tag{3.6}$$
$$- CIM(I-1))\ TK(I),$$

$$T_{v\ell} = (I_{v\ell} - CIV(1) - CIV(J))\ Tk(J) + CIV(2)\ Tk(2) + \\ + (CIV(3) - CIV(2))\ Tk(3) + \ldots + (CIV(J) - \\ - CIV(J-1))\ Tk(J), \qquad (3.7)$$

for men, if their taxable income is $I_{s\ell} + I_{m\ell}$ in the I-th tax region, and for women, if their taxable income is $I_{v\ell}$ in the J-th tax region, respectively, where:
CIM(1) and CIV(2): taxable minimum, CIM(I) or CIV(J), for I,J = 2,10: lower boundary for the I,J-th tax regions for men and for women, respectively;
Tk(i), i = 1, 11: tax coefficient for i-th tax region.

Maximum-savings functions are linear:
a. Maximum amount of advances (debts):

$$Fmaa_{\ell} = Abc\ (I_{m\ell} + I_{v\ell}), \qquad (3.8)$$

where Abc = bank advancing coefficient.

b. Maximum savings amount:

$$Fmsa_{\ell} = \sum_{m=\ell_o}^{\ell-1} Ysa(m), \qquad (3.9)$$

with:

$$Ysa(m) = I_m - T_m - p_m\ C\ min \qquad (3.10)$$

maximum possible yearly savings.

4. Dynamic analysis

A problem of this type can be seen as a multi-period decision process. Continually in its economic life each family engages in such a process in connection with distribution of activities and leisure and/or consumption and savings. The decision at any period consists of choice of I_ℓ, D_ℓ and C_ℓ, taking into account $CKL_{\ell-1}$, T_ℓ and p_ℓ according to (2.2). The purpose of the process is to maximize the value of the objective utility function subject to these constraints.

In the context of dynamic programming a solution can be given in terms of a sequence of yearly utility functions $\{UT_\ell\}$:

$$UT_\ell = (Y_{1\ell}^\rho + Y_{3\ell}^\rho + Y_{5\ell}^\rho)^{1/\rho} (1-\delta)^{\ell_o - \ell} = U_\ell (1-\delta)^{\ell_o - \ell} \qquad (4.1)$$

or, of a sequence of policy budget functions {CKL} :

$$CKL_\ell = CKL_{\ell-1} + I_\ell - T_\ell - p_\ell (C_\ell - D_\ell). \qquad (4.2)$$

The duality that arises from the interconnection between the sequences (4.1) and (4.2) makes it possible to approximate the optimum in the space of policies (R. Bellman, 1957). This is a simpler form of approximation and has the advantage that analytically it always leads to monotone approximations. For our problem it is by far the most natural approach since the budget constraint is the part of the problem about which a certain amount is known as a result of experience.

In our case the criterion function for choosing this or another policy appears to be yearly savings or dissavings.

$$R_\ell = CKL_\ell - CKL_{\ell-1}. \qquad (4.3)$$

The structure of this function R provides the possibility of making an optimal decision by using only information about the present condition of the system. This function has several important properties:

a. the dimension is not high;
b. the functional relations are comparatively simple;
c. it is separable in the sense of separating the past from the present.

The recurrence relation (4.2) can be rewritten as:

$$CKL_L - CKL_{\ell_o - 1} = \sum_{\ell=\ell_o}^{L} (I_\ell - T_\ell - p_\ell (C_\ell - D_\ell)) \text{ for } \ell_n = L. \qquad (4.4)$$

To find a unique solution we require initial values for the family wealth at the start and at the end of the life-time interval:

$$CKL_{\ell_o - 1} = SK, \qquad CKL_L = EK \qquad (4.5)$$

For the intermediate stages CKL_ℓ defines both family wealth at the start of the year $\ell+1$ and at the end of year ℓ. Such a recurrent function of CKL_ℓ can be used to organize approximation in the space of policies. For each ℓ determination of the decision region $\{FMA\}_\ell$ is required:

$$-Fmaa_\ell \leq CKL_{\ell-1} \leq Fmsa_\ell, \qquad (4.6)$$

where:

$Fmaa_\ell$: maximum possible advancing amount for the family with the income I_ℓ;

$Fmsa_\ell$: maximum possible saving amount over $\ell-1$ years.

By varying sequences of decision regions $\{FMA_\ell\}$ for $\ell = \ell_o, \ldots, L$, we can approximate the optimum iteratively.

5. Optimization procedure

The computational procedure consists of two stages: "Optimization two" - finding the optimum policy - and "Optimization one" - calculation of the approximation in the policy space (fig. 1).

a. "Optimization two"

Maximize

$$F(\overline{XS}) = \sum_{\ell=\ell_0}^{L} UT_\ell \tag{5.1}$$

subject to:

$$BL(m) \leq XS(m) \leq BU(m), \qquad m = 1, 2, \ldots, 2(L-1) \tag{5.2}$$

For each year ℓ the decision region is determined uniquely by the prospective family income and minimum consumption. Family income can be calculated according to (3.1), (3.2) and (3.3).

The set of instrumental variables is provisionally restricted to the 2(L-1):

$$\begin{aligned} XS(2k-1) &= Whm(k), \text{ and} \\ XS(2k) &= Whv(k) \qquad \text{for } k = 1, 2, \ldots, L-1, \end{aligned} \tag{5.3}$$

with Whm(k) and Whv(k): expected number of weekly working hours for men and for women, respectively.
The variables are initially fixed at their bounds, viz. upper bounds:
F max for XS(m) corresponding to Whm(k),
V max for XS(m) corresponding to Whv(k),
and viz. lower bounds equal to 0.

The optimal sequence of decision regions K is approximated according to (3.8) and (3.9) with:

$$Ysa(\ell) = Ysa(\ell-1)\ Rb + Ckh(W_m\ Whm(\ell) + W_v\ Whv(\ell)) - \\ - p_\ell\ C\ min\ .. \qquad (5.6)$$

b. "Optimization one"

It logically follows from the boundary conditions on the whole life-time interval to use backward dynamic maximization of yearly CES-utility function with time preferences (4.1) (see R. Bellman and E. Angel, 1972).

For each year ℓ, starting from $\ell = L$ and proceeding for $\ell = L-1$, L-2, etc., maximize:

$$F(\overline{x}) = UT_\ell = (Y^\rho_{1\ell} + Y^\rho_{3\ell} + Y^\rho_{5\ell})^{1/\rho}(1-\delta)^{\ell_o - \ell}, \qquad (5.7)$$

subject to:

$$\begin{aligned}
&CKL_\ell - CKL_{\ell-1} - I_\ell + T_\ell + p_\ell\ (C_\ell - D_\ell) = 0 \\
&-Fmaa_\ell \leq CKL_{\ell-1} \leq Fmsa_\ell \\
&F_{w\ell} + F_{d\ell} < F\ max \\
&V_{w\ell} + V_{d\ell} < V\ max \\
&C_\ell > C\ min \\
&0 \leq F_{w\ell},\ F_{d\ell},\ V_{w\ell},\ V_{d\ell}.
\end{aligned} \qquad (5.8)$$

The variables entering this stage are: $x(1) = F_{w\ell}$; $x(2) = F_{d\ell}$; $x(3) = V_{w\ell}$; $x(4) = V_{d\ell}$; $x(5) = C_\ell$ and for $\ell \neq \ell 0$ $x(6) = CKL_{\ell-1}$.

The number of variables used:

$$N = \begin{cases} 6, & \text{if } \ell \neq \ell_o \\ 5, & \text{if } \ell = \ell_o \end{cases}$$

The "optimization two" stage provide bounds for x(6) according to (3.8) and (3.9).

The optimum vectos X^*_ℓ for the year ℓ are used to compute the "optimum" values of the objective sub-functions UT^*_ℓ and these in turn to compute:

UT* - to calculate the value of the objective function for "optimization two"

x* - to organize the backward dynamic procedure:

for i = ℓ - 1, $CKL_{i=\ell-1} = CKL_{i-1=\ell}$ = x*(6) for i = ℓ.

After L applications of "optimization one" we switch to "optimization two" with the value of the objective function (5.1), and, subsequently, repeat this cycle as often as required till a sufficient degree of approximation has been obtained.

Preferably the starting point should be an interior point of the feasible regions:

a. For "optimization one":

For variables in this stage feasible regions are determined with bounds and inequality constraints (5.8).

An attempt can be made to find a global maximum, using the standard routines for local maximization by varying the starting points, especially the x(1) and x(3). In our case, starting points for global "optimization one" are inferior for the tax regions:

$$CML(i) \leq x(1) < CMR(i) \text{ and } CVL(j) \leq x(3) < CVR(j), \qquad (5.9)$$

where:

$CML(i) = (CIM(1) + CIM(i-1))/CKH \cdot WM$, and

$CMR(i) = (CIM(1) + CIM(i))/CKH \cdot WM$ for i-th tax region (men)

$CVL(j) = (CIV(1) + CIV(j-1)) / CKH \cdot WV$, and

$CVR(j) = (CIV(1) + CIV(j))/CKH \cdot WV$ for j-th tax region (women).

b. For "optimization two"
The starting point was chosen as:

XS(m) = F max - 2k for XS(m) corresponding to Whm(k), and
XS(m) = V max - k for XS(m) corresponding to Whv(k).

This choice is based on the assumption that a person's prospective working-capacity is maximal at the start of his working-life and decreases monotonically to a minimum towards the end.

6. Computational procedure

The computational procedure for the "optimization two" is based on the standard program EØ4JAF, using the function value only (NAG, 1978). The number of iterations required by this routine depends on the NS (number of variables XS), the behaviour of the objective function (5.1) and the distance of the starting point from the solution. The run time will be dominated by the time spent on function evaluation.

The computational procedure for function evaluation, "optimization one", is based on the standard program EØ4HAF (sequential penalty function technique) or EØ4WAF (sequential augmented Lagrangean method), using analytically calculated first and second order derivatives (NAG, 1978).

The time taken by these routines depends on the N (number of variables x), the behaviour and number of the problem functions (5.7, 5.8) the accuracy demanded and the distance of the starting point from the solution. In such a situation minimization of time spent is clearly called for.

The acceleration scheme which we applied is based on increasing the absolute accuracy ATOL in "optimization one" by a factor 10 and changing the starting point XS to the optimum found in the previous run of "optimization two" after each 10 x NS iterations (Fig. 2).

7. Numerical results and conclusions

The results obtained for a minimum lifetime period of three years are presented and commented upon below.

For the sake of realism used in this model parameters were those estimated from statistical data for the Netherlands (cf. W.J. Keller and A. Langhout, 1973).

The optimal number of weekly working hours appear to decrease rapidly from the maximum amount in the first year of what looks like an asymptotic level (fig. 3). The opposite seems to hold for the optimal family consumption (fig. 4). After saving as much as possible at the beginning of the lifetime we expect a slow decrease of the optimal family savings to an asymptotic level, which depends on the tax-system (fig. 5).

We expect to retain this asymptotic behaviour when we extend the experiments to a larger number of periods.

Notwithstanding the simplifying assumptions mentioned above, the optimization problem is still both complicated and sizeable. The complications arise from:

a. its dynamic nature, implied by the budget constraint, and
b. the intricacies of the tax function, being broken-linear for as many as 10 successive tax regions.

The large size of the problem mainly results from its reference to an entire (remianing) lifetime, covering up to about 50 years.

For this reason, splitting the problem into two sub-problems appears to be in order. The first sub-problem - called "optimization two" - deals with the sequence of selecting decision regions, dealing with working hours only, with their upper bounds as initial values, to start the iterative process of solution. For tackling the remaining programming problem, called "optimization one", a dynamic backward solution is outlined; this includes all intrumental variables.

Some provisional results obtained by experimental application of such a two-stage approach appear to be sufficiently promising to justify elaboration. Extension to a longer period, adding consumption to working hours as instrumental variables at "optimization two" and relaxation of some of the simplifying assumptions are being prepared.

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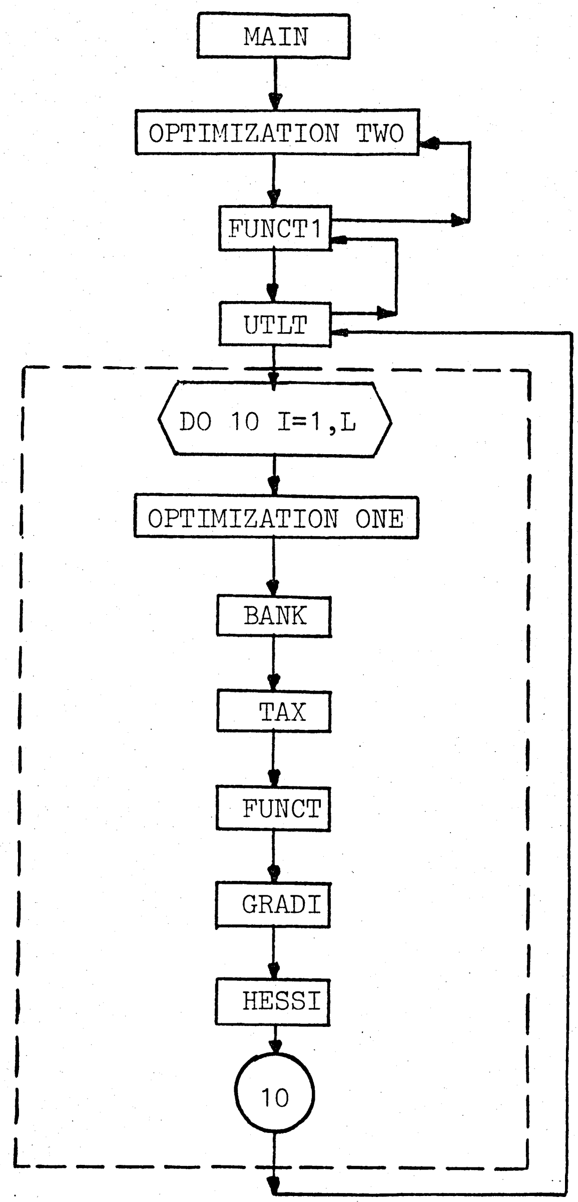
Fig. 1. Two-stage optimization procedure



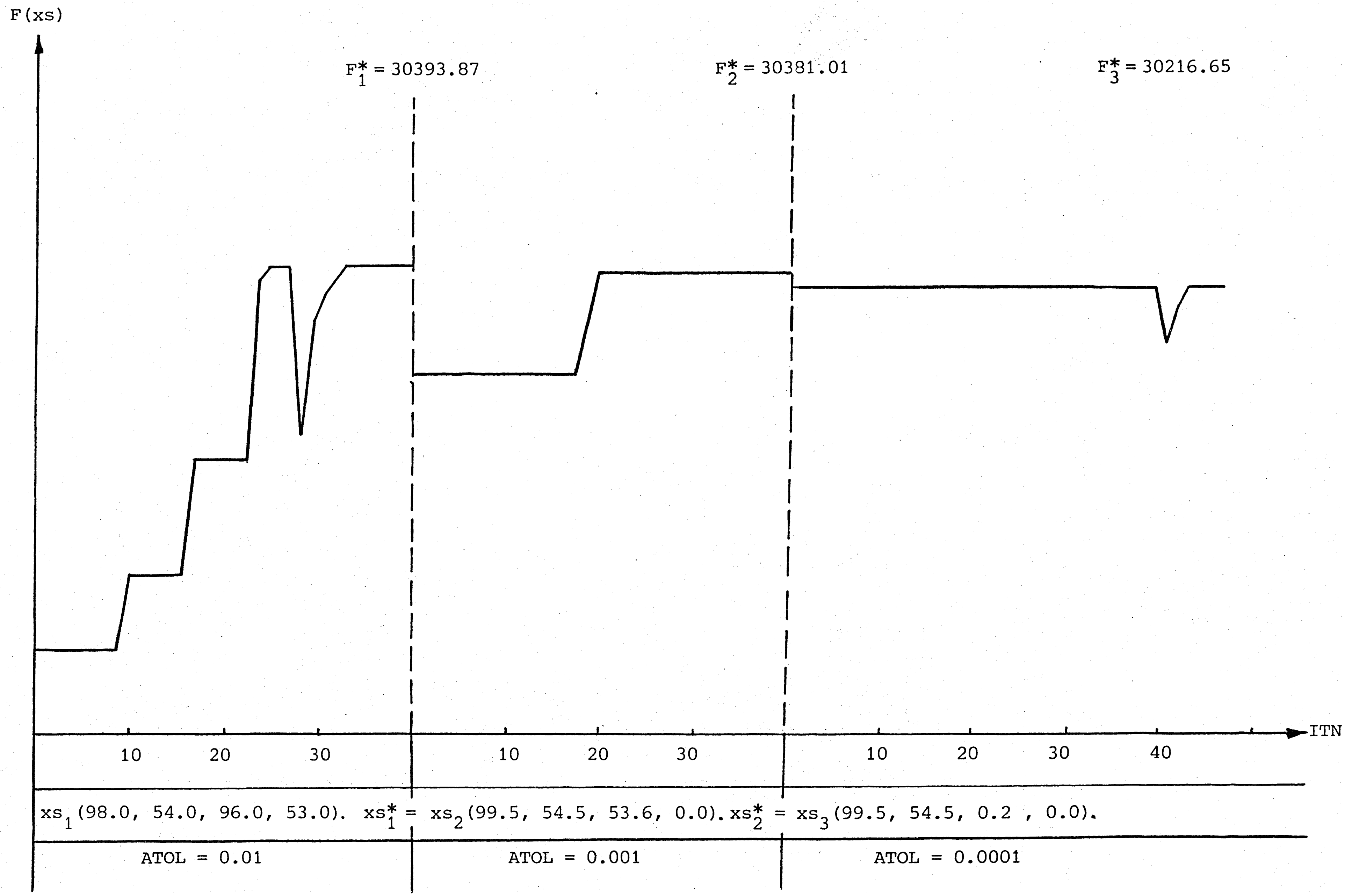


Fig. 2. "Optimization two" with acceleration scheme

Fig. 3. Optimal number of weekly working hours

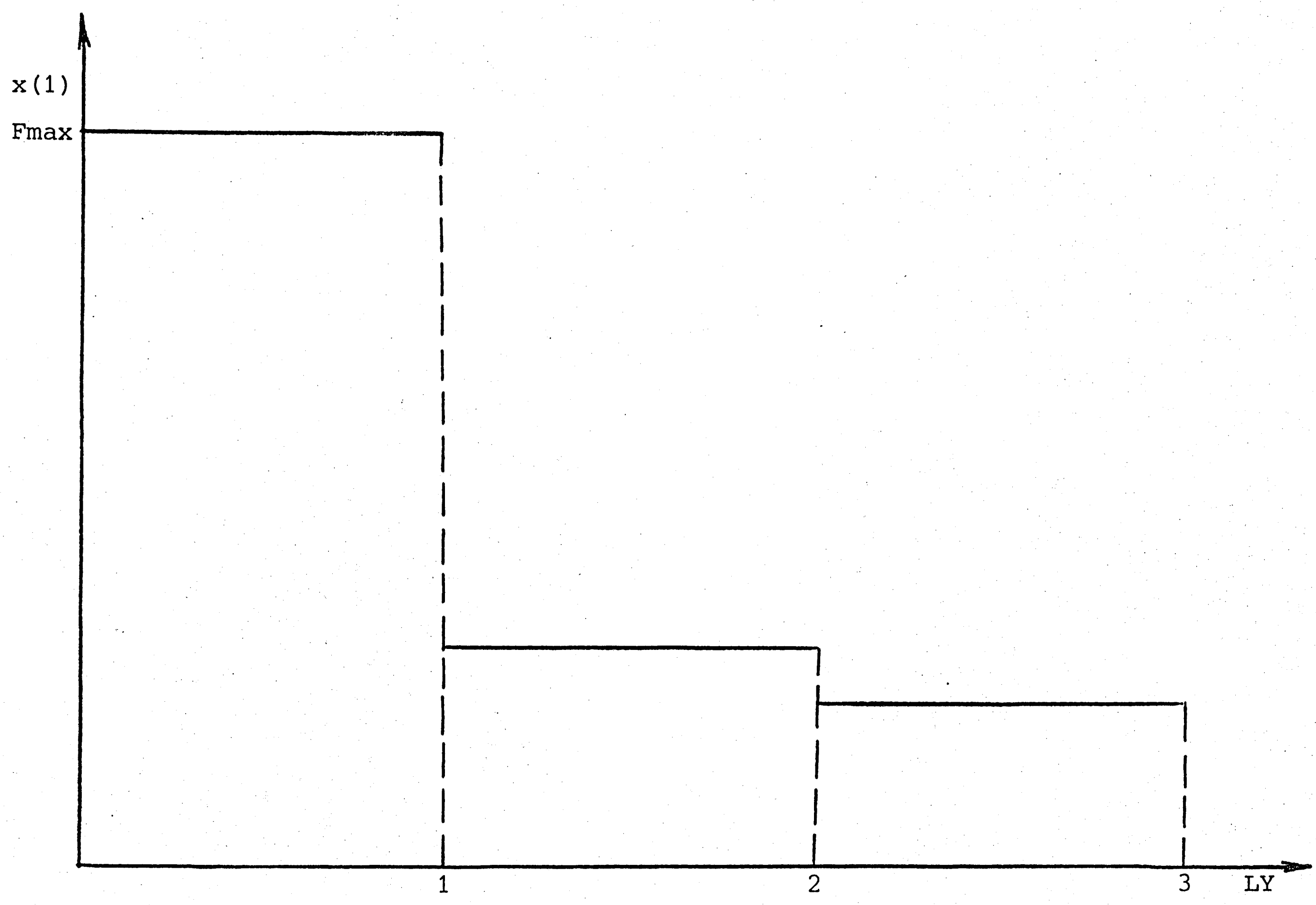


Fig. 4. Optimal family consumption

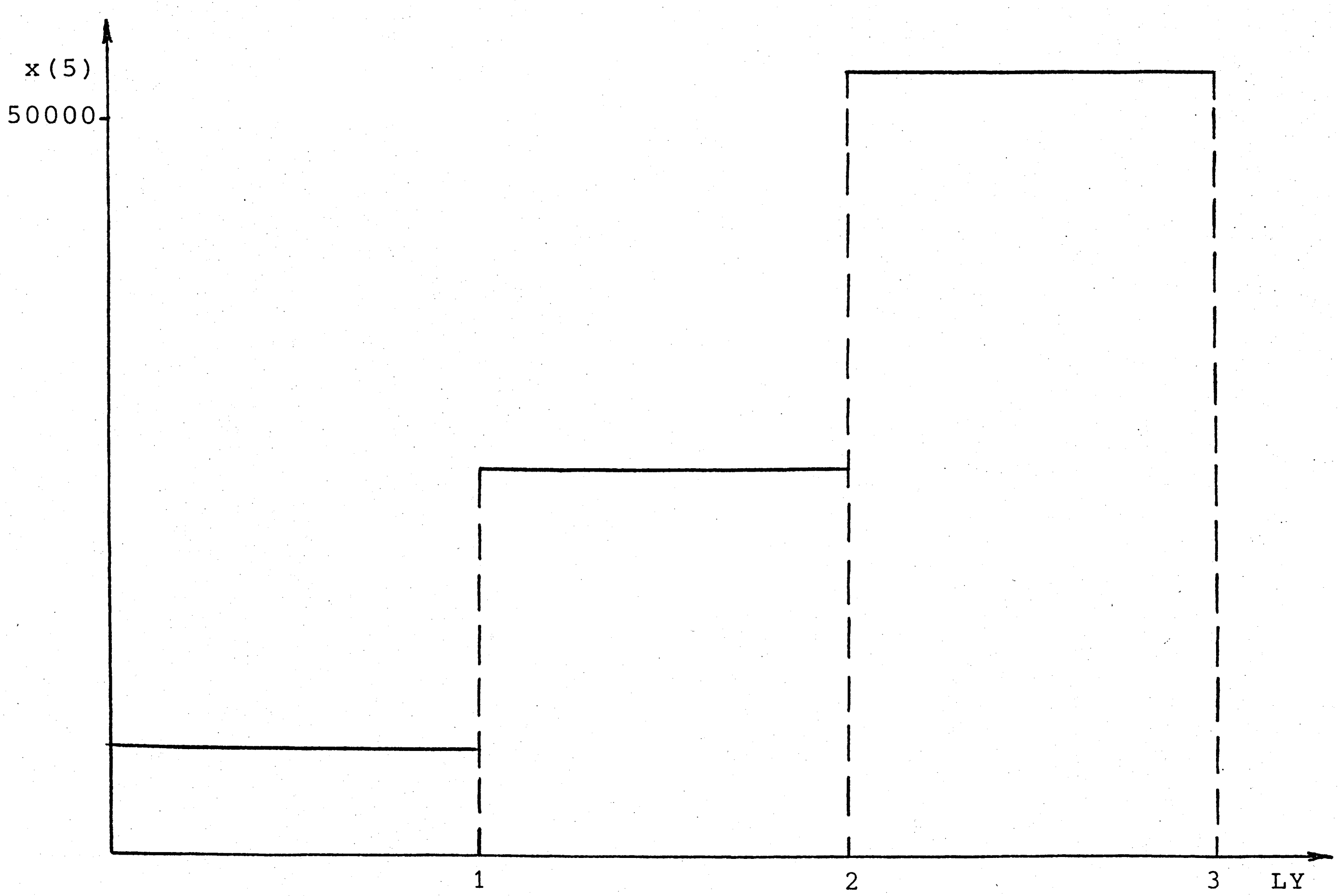
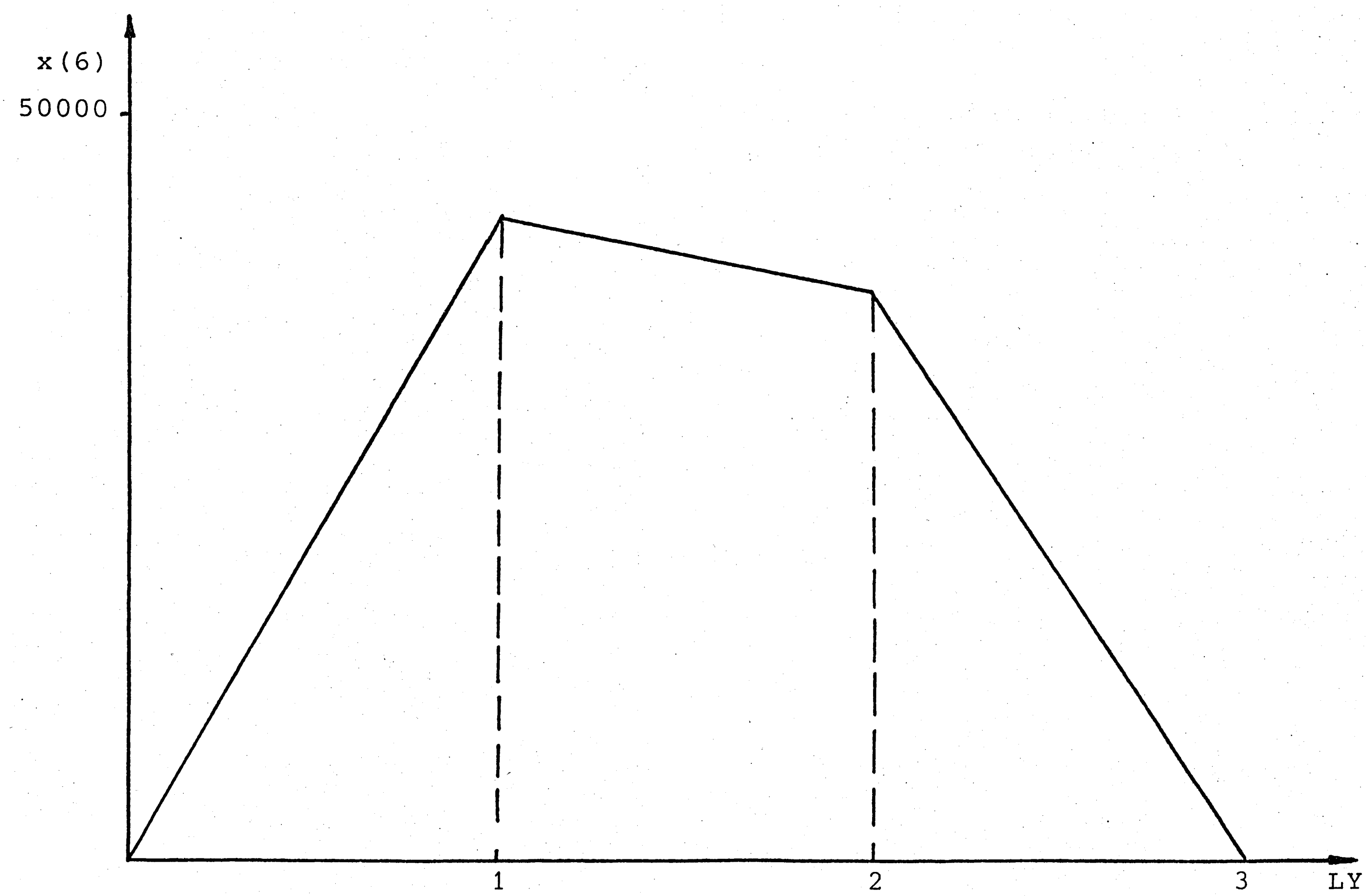


Fig. 5. Optimal family savings



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