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AN ABEL-TAUBER THEOREM FOR PARTITIONS

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by

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Summary.

Suppose $\Lambda = \{\lambda_1,\lambda_2,\ldots\}$ is a given set of real numbers such that $0 < \lambda_1 < \lambda_2 < \ldots$

Let $n(u) = \sum_{\lambda_k \leq u} 1$ and $P(u)$ the number of solutions of $n_1\lambda_1+n_2\lambda_2+\ldots \leq u$ in integers $n_i \geq 0$.

An Abel-Tauber theorem concerning $n(u)$ and $\log P(u)$ is proved for the case where $\frac{n(tx)}{n(t)} \to 1$ $(t\to\infty)$ for $x > 0$.

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Introduction

Suppose $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ is a given set of real numbers such that $0 < \lambda_1 < \lambda_2 < \ldots$

Let $0 = \nu_o < \nu_1 < \ldots$ be the elements of the additive semigroup generated by Λ.

Consider the weighted partition function $p(\nu_m)$ defined by the generating function

$$\prod_{r=1}^{\infty} (1-e^{-\lambda_r s})^{-\psi_r} = \int_0^{\infty} e^{-us}\, dP(u) := \hat{P}(s)$$

where $P(u) = \sum_{\nu_i \leqq u} p(\nu_i)$ and ψ_r is a given sequence with non-negative terms

We assume the λ_r and ψ_r are such that the product for $\hat{P}(s)$ is (absolutely) convergent λ

for $s > 0$.

Letting $n(u) = \sum_{\{k;\lambda_k \leqq u\}} \psi_k$ we have the following theorem.

Main theorem

Let c be a constant, $0 < c \leqq \infty$.

Under the above assumptions the following statements are equivalent.

(i) $\lim_{t\to\infty} n(t) = c$ and $\frac{n(tx)}{n(t)} \to 1$ $(t \to \infty)$ for all $x > 0$

(ii) there exists a non-decreasing function $a(t)$ on $(0,\infty)$ with $\lim_{t\to\infty} a(t) = c$ such that for all $x > 0$

$$\left(\frac{P(tx)}{P(t)}\right)^{1/a(t)} \to x \quad (t\to\infty)$$

If $c < \infty$ then

$$\int_0^x \frac{n(t)}{t}\, dt - \log P(x) \to \log \Gamma(1+c) \quad (x\to\infty)$$

If $c = \infty$ then

$$\log P(y) = \int_0^x \frac{n(t)}{t}\, dt + n(x) + o(n(x))$$

where $x,y \to \infty$ and $y \sim xn(x) \sim xa(y)$

If $c=\infty$ and the function n satisfies the relation $n(xn(x)) \sim n(x)$ then $n(x) \sim a(x)$

and $\int_0^x \frac{n(s)}{s}\, ds - \log P(x) \sim n(x) \log n(x) \quad (x\to\infty)$

In order to prove the main theorem we use the Lambert transform $\tilde{n}$ of n which is defined by

$$\tilde{n}(s) = \int_0^\infty \frac{s}{e^{us}-1} n(u)du, \text{ supposed the integral exists.}$$

This transform arises naturally since

$$\log \hat{P}(s) = -\sum_{r=1}^\infty \psi(r)\log (1-e^{-\lambda_r s}) = -\int_0^\infty \log (1-e^{-us})dn(u) = \tilde{n}(s)$$

by integration by parts.

We use Abelian and Tauberian theorems connected with the theory of regular variation. Earlier results in this direction were given by Kohlbecker for regularly varying functions and by Parameswaran for slowly varying functions. Our main theorem can be seen as a refinement of Parameswaran's work.

The main part of the proof is to show the equivalence of the statements (i) n is slowly varying,

(ii) $\tilde{n}(1/x) = \log \hat{P}(1/x)$ is an element of the class Π,

(iii) log (P-1) is an element of the class Π

(iv) log P is in Π. Precise conditions are given in the lemmas.

__Results__

Definition: A function $f : R^+ \to R^+$ is said to be slowly varying if it is measurable and $\lim_{t\to\infty} \frac{f(tx)}{f(t)} = 1$ for $x > 0$

In this paper we use properties of a subclass of the slowly varying functions. This subclass is defined as follows:

Definition: A non-decreasing function $U : R^+ \to R^+$ is said to belong to the class Π (notation: $U(.) \in \Pi$) if there exists an auxiliary function $a : R^+ \to R^+$ such that

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{a(t)} = \log x \text{ for } x > 0 \qquad (1)$$

The function a is (of course) determined up to asymptotic equivalence. It is possible to show that each function $U(x) \in \Pi$ with $\limsup U(x) = \infty$ can be represented as

$$U(x) = \int_1^x \frac{L(u)}{u} du + o(L(x)) \quad (x \to \infty) \qquad (2)$$

where L is slowly varying.

Conversely each function U of the form (2) satisfies (1). The functions a, L and U are related by $L(x) \sim a(x) \sim \frac{1}{x}\int_1^x t\,dU(t) \quad (x \to \infty)$.

For proofs and properties of regular variation and the class Π see (1), (6), (15).

<u>Lemma 1</u> Suppose $\sum_{k \le x} \frac{1}{k}\, n(\frac{x}{k}) = \int_1^x \frac{L(t)}{t}\,dt + o(L(x)) \quad (x \to \infty)$

where L is slowly varying, bounded on finite intervals and $\sup_{y \le x} L(y) = O(L(x)) \quad (x \to \infty)$.

If $xn(x)$ is non-decreasing for $x > 1$ then $n(x) \sim L(x) \quad (x \to \infty)$ and

$$\int_1^x \frac{n(t)-L(t)}{t}\,dt = -\gamma L(x) + o(L(x)) \quad (x \to \infty)$$

where γ is Euler's constant.

<u>Proof</u> We define $I(x) = \sum_{k \le x} \frac{1}{k}\, n(\frac{x}{k})$; $N(x) = \sum_{k \le x} \frac{\mu(k)}{k}$;

$$A(x) = n(x) + \int_1^x \frac{n(t)}{t}\,dt \quad \text{and} \quad a(x) = \frac{1}{x}\int_1^x A(t)\,dt$$

where μ is the Möbius function.

Möbius inversion in the first formula and partial integration in the third gives

$$n(x) = \sum_{k \le x} \frac{\mu(k)}{k}\, I(\frac{x}{k}) = \frac{1}{x}\int_1^x t\,dA(t)$$

Substituting this we get

$$\int_1^x \frac{I(x/u)N(u)}{u}\,du = \int_1^x \frac{I(x/u)}{u} \sum_{k \le u} \frac{\mu(k)}{k}\,du =$$

$$= \sum_{k \le x} \frac{\mu(k)}{k} \int_1^{x/k} I(\frac{x}{ku})\,\frac{du}{u} = \int_1^x \frac{1}{u}\, n(\frac{x}{u})\,du = \int_1^x \frac{n(u)}{u}\,du = a(x)$$

Now for $\lambda > 1$ fixed the function $\frac{I(\lambda x)-I(x)}{L(x)}$ is bounded for $x > 0$ and tends to $\log \lambda$ for $x \to \infty$.

Substituting the expression for $a(x)$ then gives

$$\frac{a(\lambda x)-a(x)}{L(x)} = \int_1^x \left(\frac{I(\lambda x/u)-I(x/u)}{L(x/u)} \cdot \frac{L(x/u)}{L(x)} \right) \frac{N(u)}{u}\,du + \int_1^\lambda \frac{I(\lambda/u)N(xu)}{u}\,du$$

and the term between brackets in the first integral is bounded for $u \in (1,x)$ by assumption.

As $\int_1^\infty \frac{N(u)}{u}\,du$ is absolutely convergent by a classical result of Landau, we get by dominated convergence $\frac{a(\lambda x)-a(x)}{L(x)} \to \log \lambda$.

Since $xn(x) = \int_1^x t\,dA(t)$ is non-decreasing, A is non-decreasing.

This implies that $A\in\Pi$ with auxiliary function $L(x) \sim \frac{1}{x}\int_1^x t\,dA(t) \sim n(x)$ by the main theorem in (*).

Since n is slowly varying we can apply theorem 2 in (5) to get

$$I(x) - \int_1^x \frac{n(t)}{t}\,dt = (\gamma+o(1))n(x) \quad (x \to \infty)$$

Combination with $I(x) = \int_1^x \frac{L(t)}{t}\,dt + o(L(x))$ then gives the desired result.

Remark 1. This theorem is an extension of earlier results in (5), (7), (12), (13), (14). Remark that the conditions (see (7))

(i) for some real r, $xg'(x)(\log x)^{-r}$ is non-increasing for $x > x_0$

(ii) for some real s, $xg'(x)(\log x)^{s}$ is non-decreasing for $x > x_1$

imply that $xg'(x)$ is slowly varying and so $g(x)\in\Pi$ with auxiliary function $xg'(x)$.

2. If $L(x) \sim M(x) \quad (x \to \infty)$ where $M(x)$ is non-decreasing, then the condition

$\sup_{y\leq x} L(y) = O(L(x))$ holds.

This is trivial, since $\frac{L(y)}{L(x)} \leq \frac{\max(A,(1+\varepsilon)M(x))}{L(x)} < c$

for all $y \leq x$, where $L(x) \leq (1+\varepsilon)\, M(x) \quad (x \geq x_0)$,

$A = \sup_{x\leq x_0} L(x)$ and c is a constant.

This condition is sufficient to prove the main theorem.

For a slowly varying function n the following result gives the relation between n and $\tilde{n}$. For a proof see (5).

(*)(preprint) Π-regular variation

<u>Lemma 2</u> Suppose n is slowly varying, $\frac{n(x)}{x}$ is integrable on finite subintervals of R^+. Then $\tilde{n}(1/x)\ \varepsilon\Pi$

and $\tilde{n}(1/x) - \int_0^x n(t)/t\, dt = o(n(x)) \quad (x \to \infty)$

A converse statement is given in the next theorem.

<u>Lemma 3</u> Suppose $n : R^+ \to R^+$ and $n(x)/x$ is integrable on $(0,R)$ for $R > 0$, $n(x)$ is non-decreasing for $x > 0$ and

$$\tilde{n}(1/x) = \int_1^x \frac{L(u)}{u}\, du + o(L(x)) \quad (x \to \infty)$$

with L slowly varying and asymptotic to a non-decreasing function, then $n(x) \sim L(x)$ and $\int_0^x \frac{n(t)}{t}\, dt = \int_1^x \frac{L(u)}{u}\, du + o(L(x)) \quad (x \to \infty)$

<u>Proof</u> Since $\sum_{m \leq x} \frac{1}{m} n(\frac{x}{m})$ is non-decreasing, we have

$\tilde{n}(1/x)\varepsilon\Pi$ with auxiliary function L iff $\sum_{k \leq x} \frac{1}{k} n(\frac{x}{k})\varepsilon\Pi$ with auxiliary function L.

See corollary 1 in (5).

Moreover $\sum_{k \leq x} \frac{1}{k} n(\frac{x}{k}) = \tilde{n}(\frac{1}{x}) - \int_0^1 \frac{n(t)}{t}\, dt + \gamma L(x) + o(L(x))$

Substitution of $\tilde{n}(1/x) = \int_1^x \frac{L(u)}{u}\, du + o(L(x))$ in the expression for $\sum_{k \leq x} \frac{1}{k} n(\frac{x}{k})$ shows that there exists an $L_* \sim L$ such that

$$\sum_{k \leq x} \frac{1}{k} n(\frac{x}{k}) = \int_1^x \frac{L_*(t)}{t}\, dt + o(L_*(x)) \quad (x \to \infty)$$

since $\limsup \sum_{k \leq x} \frac{1}{k} n(x/k) = \infty$.

Application of lemma 1 now gives the desired result.

<u>Proof</u> (of the main theorem).

Suppose n is slowly varying. Application of lemma 2 now gives $\log \int_0^\infty e^{-u/x}\, dP(u) = \tilde{n}(1/x)\varepsilon\Pi$ with auxiliary function $n(x)$. To apply theorem 1 in (2) we take $P_o(y) = -1 + P(y)$.

Then $P_o(0+) = 0$ and $\log P \in \Pi$ iff $\log P_o \in \Pi$

since $\log P(y) - \log P_o(y) \to 0 = o$ (auxiliary function)

and the auxiliary function tends to $c > 0$.

For $c < \infty$ we get $\log P \in \Pi$ with auxiliary function $\sim c$.

For $c = \infty$ the result is $\log P(y) \in \Pi$ with auxiliary function $\sim a(y) \sim n(x)$

where $x,y \to \infty$ are related by $y \sim xn(x) \sim xa(y)$.

For the converse statement we replace lemma 2 by lemma 3. All regularity conditions are satisfied since n and P are as mentioned in the introduction.

In case $c < \infty$ we combine $\tilde{n}(1/x) = \int_0^x \frac{n(t)}{t} dt + o(n(x))$ with formula (1.3) in (2).

The case $c = \infty$ is treated similarly with (1.5) in (2).

Now suppose $n(xn(x)) \sim n(x) \quad (x \to \infty)$

Then $n^*(y) \sim 1/n(y) \quad (y \to \infty)$ where n^* is the conjugate function in the sense of de Bruijn (see (3)).

Since $y \sim xn(x) \sim xa(y)$ we get $x \sim yn^*(y) \sim y/a(y)$ and so $n(x) \sim a(x)$.

This implies that $\log P(t) \in \Pi$ with auxiliary function $n(t)$ satisfying $n(tx) \sim n(t)$ for $t \to \infty$ uniformly in $x \in [1,n(t)]$ since n is non-decreasing.

Application of theorem 13 in (2) now gives

$\log \hat{P}(1/t) - \log P(t) \sim n(t) \log n(t)$ and the final result follows since

$$\log \hat{P}(1/t) = \tilde{n}(1/t) = \int_0^t \frac{n(s)}{s} ds + o(n(t)) \quad (t \to \infty)$$

This finishes the proof.

Remark 1. The function n satisfies the condition $n(xn(x)) \sim n(x)$ if

$$\frac{n(2t)}{n(t)} - 1 = o(1/\log n(t)) \text{ for } t \to \infty.$$

2. An application to Mahler's partition problem (see (4), (10)) where $\Lambda = \{1,r,r^2,\ldots\}$ and $\psi_r = 1$ gives

$$n(u) = \left[\frac{\log u}{\log r} + 1\right] \sim \frac{\log u}{\log r} \quad (u \to \infty)$$

Application of the main theorem to this case gives

$$\log P(u) = \frac{1}{2\log r}\log^2 u - (1+o(1))\,\frac{\log u.\log\log u}{\log r}$$

3. If in the case $c = \infty$ and $n(xn(x)) \sim n(x)$ there exists a slowly varying function L such that $L(x)/x$ is convex and $\log P(x) = \int_0^x \frac{L(t)}{t}\,dt + o(1)$, then we have

$$\tilde{n}\left(\frac{L(t)}{t}\right) + L(t) - \log P(t) - \frac{1}{2}\log 2\pi L(t) \to 0 \quad (t \to \infty)$$

by theorem 15 in (2) (in this case $L \sim n$).

This stronger result is not very useful however, since log P is expressed in terms of $\tilde{n}$ and not in terms of n.

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