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On the Observation closest to the Origin

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ABSTRACT

Out of n i.i.d. random vectors in \mathbb{R}^d let \tilde{X}_n^* be the one closest to the origin. We show that \tilde{X}_n^* has a nondegenerate limit distribution if and only if the common probability distribution satisfies a condition of multidimensional regular variation.

We apply the result to a problem of density estimation.

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1. Introduction.

Suppose X_1, \dots, X_n are i.i.d. observations in a Euclidean space which for convenience we take to be the plane R^2 . The polar coordinates of X_1 are denoted by $(|X_1|, \theta(X_1))$. Call the observation with minimum modulus X_n^* ; if more than one observation has modulus equal to $\bigwedge_{i=1}^n |X_i|$, choose X_n^* from the tied observations by randomization.

As $n \rightarrow \infty$ X_n^* will converge to 0 but it may be possible to normalize X_n^* so that a non-trivial limiting distribution exists.

So we ask: when do there exist constants c_n , $0 < c_n \rightarrow 0$ such that

$$(1) \quad c_n^{-1} X_n^* \Rightarrow Y$$

where the limit vector Y is finite and $|Y|$ non-degenerate. Here the arrow " \Rightarrow " indicates weak convergence.

In section 2 we show that the necessary and sufficient condition for (1) is multivariate regular variation at 0 of the distribution of X_1 . Also the polar coordinates of Y are independent random variables.

In the latter section of the paper we examine a related density estimation problem.

2. Multivariate Regular Variation and the Observation closest to 0 .

The condition that the distribution of X_1 be regularly varying at 0 can conveniently be expressed either in terms of vague convergence of measures or, a regular variation property in terms of the distribution of $(|X_1|, \theta(X_1))$ and the following are equivalent:

(i) There exist constants c_n , $0 < c_n \rightarrow 0$ and a limit measure ν on R^2 which is finite on bounded neighbourhoods of 0 such that

$$n P[c_n^{-1} X_1 \in \cdot] \xrightarrow{v} \nu$$

(" \xrightarrow{v} " denotes vague convergence). This sequential form of the condition is the same as saying there exists a (necessarily regularly varying) function $U: R_+ \rightarrow R_+$ such that

$$P[X_1 \in t \cdot (\cdot)] / U(t) \xrightarrow{v} \nu \quad \text{as } t \downarrow 0.$$

(ii) There exist constants c_n , $0 < c_n \rightarrow 0$, $\alpha > 0$, and a probability measure S on $B([0, 2\pi])$ such that

$$n P[c_n^{-1} |\tilde{X}_1| \leq r, \theta(X_1) \leq \theta] \xrightarrow{y} r^\alpha S([0, \theta]).$$

This is the same as

$$\frac{P[|\tilde{X}_1| \leq tr, \theta(X_1) \leq \theta]}{P[|\tilde{X}_1| \leq t]} \xrightarrow{y} r^\alpha S([0, \theta])$$

as $t \downarrow 0$. Cf. Stam (1977), de Haan and Resnick (1977). We remark that the constants $\{c_n\}$ in (i) and (ii) need not be the same; their quotient tends to a positive constant.

Theorem 1.

There exists a sequence of positive constants $\{c_n\}$ such that $c_n^{-1} \tilde{X}_n^*$ converges in distribution to a finite limit vector \tilde{Y} with $|\tilde{Y}|$ non-degenerate iff the distribution of \tilde{X}_1 is regularly varying at 0 as described in (i) or (ii) above. In this case $P[|\tilde{Y}| > r, \theta(\tilde{Y}) \leq \theta] = S([0, \theta])e^{-r^\alpha}$, with $0 < \alpha < \infty$ so that $|\tilde{Y}|$ and $\theta(\tilde{Y})$ are independent.

Proof:

We first prove the theorem under the added assumption that $|\tilde{X}_1|$ has a continuous distribution. This assumption is removed at the end. With this added assumption, it is easy to derive the distribution of \tilde{X}_n^* : by symmetry \tilde{X}_n^* could be any one of the \tilde{X}_i 's with equal probability. If $\tilde{X}_n^* = \tilde{X}_1$ then $\bigwedge_{i=2}^n (\tilde{X}_i) > |\tilde{X}_1|$. Thus we get

$$\begin{aligned} (2) \quad & P[c_n^{-1} \tilde{X}_n^* \leq r, \theta(\tilde{X}_n^*) \leq \theta] = \\ & = \int_0^r P[\bigwedge_{i=2}^n c_n^{-1} |\tilde{X}_i| > u] n P[c_n^{-1} |\tilde{X}_1| \in du, \theta(\tilde{X}_1) \leq \theta] \end{aligned}$$

Supposing (ii) holds we easily get $P[\bigwedge_{i=2}^n c_n^{-1} |\tilde{X}_i| > u] \rightarrow e^{-u^\alpha}$ uniformly. Assume $S\{\theta\} = 0$ and write the integral in (2) as

$$\begin{aligned} & \int_0^r \{P[\bigwedge_{i=2}^n c_n^{-1} |\tilde{X}_i| > u] - e^{-u^\alpha}\} n P[c_n^{-1} |\tilde{X}_1| \in du, \theta(\tilde{X}_1) \leq \theta] \\ & + \int_0^r e^{-u^\alpha} n P[c_n^{-1} |\tilde{X}_1| \in du, \theta(\tilde{X}_1) \leq \theta] \end{aligned}$$

The first term goes to zero by uniform convergence. The second term is seen to converge vaguely in the usual way by appealing to the Helly-Bray lemma (Loève, 1963, p. 180) and so

$$P[c_n^{-1} |\tilde{X}_n^*| \leq r, \theta(\tilde{X}_n^*) \leq \theta] \xrightarrow{Y} \int_0^r e^{-u^\alpha} du^\alpha S([0, \theta])$$

$$= P[|\tilde{Y}| \leq r, \theta(\tilde{Y}) \leq \theta].$$

For the converse suppose there exists $c_n, 0 < c_n \rightarrow 0$, and \tilde{Y} with $P[\tilde{Y} = 0] < 1$ such that $c_n^{-1} \tilde{X}_n^* \Rightarrow \tilde{Y}$. Then $c_n^{-1} \bigwedge_{i=1}^n |\tilde{X}_i| = c_n^{-1} |\tilde{X}_n^*| \Rightarrow |\tilde{Y}|$. Since the only normalization necessary here for convergence of the minima is scaling, we can easily identify the distribution of $|\tilde{Y}|$ to be the extreme value distribution $P[|\tilde{Y}| > x] = e^{-kx^\alpha}$, $x > 0$ for some $0 < \alpha < \infty$, $k > 0$. Without loss of generality, suppose $k = 1$.

This distribution is continuous so $P[\bigwedge_{i=2}^n c_n^{-1} |\tilde{X}_i| > u] \rightarrow e^{-u^\alpha}$ uniformly. To deal with the rest of the integral in the right side of (2) we use the selection theorem: Since $n P[c_n^{-1} |\tilde{X}_1| \leq u, \theta(\tilde{X}_1) \leq \theta] \leq n P[c_n^{-1} |\tilde{X}_1| \leq u] \rightarrow -\log P[|\tilde{Y}| > u] = u^\alpha$ as $n \rightarrow \infty$ (an easy consequence of $c_n^{-1} \bigwedge_{i=1}^n |\tilde{X}_i| \Rightarrow |\tilde{Y}|$) we have for each u, θ that $n P[c_n^{-1} |\tilde{X}_1| \leq u, \theta(\tilde{X}_1) \leq \theta]$ is a bounded sequence.

Thus we are assured of the existence of a vaguely convergent subsequence n' and a vague limit $\Psi(du, d\theta)$ such that

$$n' P[c_{n'}^{-1} |\tilde{X}_{1'}| \in du, \theta(\tilde{X}_{1'}) \in d\theta] \xrightarrow{Y} \Psi(du, d\theta).$$

As in the first half of the proof the uniform convergence of $P[\bigwedge_{i=2}^n |\tilde{X}_i| > c_n u]$ gives as $n' \rightarrow \infty$

$$P[c_{n'}^{-1} |\tilde{X}_{1'}^*| \leq r, \theta(\tilde{X}_{1'}^*) \leq \theta] \xrightarrow{Y} \int_0^r e^{-u^\alpha} \Psi(du \times [0, \theta])$$

If along another subsequence n''

$$n'' P[c_{n''}^{-1} |\tilde{X}_{1''}| \in du, \theta(\tilde{X}_{1''}) \in d\theta] \xrightarrow{Y} \Psi'(du, d\theta)$$

then the measures determined by the distribution functions

$$\int_0^r e^{-u^\alpha} \Psi(du \times [0, \theta]), \int_0^r e^{-u^\alpha} \Psi'(du \times [0, \theta]) \text{ and } P[|\tilde{Y}| \leq r, \theta(\tilde{Y}) \leq \theta]$$

are all equal.

Since e^{-u^α} is non-vanishing, it is easy to see $\Psi = \Psi'$. Therefore the vague limit of $n P[c_n^{-1} |\tilde{X}_1| \leq r, \theta(\tilde{X}_1) \leq \theta]$ exists.

This is equivalent to our first definition (i) of multivariate regular variation.

We now remove the assumption that $|X_1|$ has a continuous distribution thus allowing more than one observation to achieve minimum modulus. Formula (2) must be modified. As before, by symmetry \tilde{X}_n^* could be any one of the \tilde{X}_i 's with equal probability so that

$$(3) \quad \begin{aligned} & P[c_n^{-1} |\tilde{X}_n^*| \leq r, \theta(\tilde{X}_n^*) \leq \theta] \\ &= n P[c_n^{-1} |\tilde{X}_1| \leq r, \theta(\tilde{X}_1) \leq \theta, \tilde{X}_1 = \tilde{X}_n^*]. \end{aligned}$$

Write $[\tilde{X}_n^* = \tilde{X}_1] = \bigcup_{k=0}^{n-1} A_k$ where A_k is the event that k observations from $\tilde{X}_2, \dots, \tilde{X}_n$ have moduli equal to $|\tilde{X}_1|$, $n-k-1$ observations have moduli $> |\tilde{X}_1|$ and the randomization procedure selects \tilde{X}_1 from among those $k+1$ observations with minimum modulus.

Then (3) becomes

$$\begin{aligned} & n \int_{\{x | |x| \leq r, \theta(x) \leq \theta\}} P[c_n^{-1} \tilde{X}_n^* = x | c_n^{-1} \tilde{X}_1 = x] P[c_n^{-1} \tilde{X}_1 \in dx] \\ &= n \int_{\{x\}} \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} P[c_n^{-1} |\tilde{X}_1| = |x|]^k P[c_n^{-1} |\tilde{X}_1| > |x|]^{n-k-1} \\ & \quad P[c_n^{-1} \tilde{X}_1 \in dx] \\ &= \int_0^r \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} P[c_n^{-1} |\tilde{X}_1| = u]^k P[c_n^{-1} |\tilde{X}_1| > u]^{n-k-1} \\ & \quad n P[c_n^{-1} |\tilde{X}_1| \in du, \theta(\tilde{X}_1) \leq \theta]. \end{aligned}$$

Note the $k=0$ term (no ties) corresponds to the rightside of (2). For proving either necessity or sufficiency in this theorem, we have at our disposal the condition

$$(4) \quad P[\bigwedge_1^n |\tilde{X}_i| > c_n u] = P[|\tilde{X}_1| > c_n u]^n \rightarrow e^{-u^\alpha}$$

for some α . We now show (4) implies

$$I_n(u) := \sum_{k=1}^{n-1} \frac{1}{k+1} \binom{n-1}{k} p^k g^{n-k-1} \rightarrow 0$$

as $n \rightarrow \infty$ uniformly in $u \in [0, r]$, where $p = p_n(u) = P[c_n^{-1} |\tilde{X}_1| = u]$, $g = g_n(u) = P[c_n^{-1} |\tilde{X}_1| > u]$.

This then shows that $P[c_n^{-1} |\tilde{X}_n^*| \leq r, \theta(\tilde{X}_n^*) \leq \theta]$ equals the righthandside of (2) plus $o(1)$; the case where the df of $|\tilde{X}_1|$ is not continuous can then be reduced to the continuous case.

Rewrite $I_n(u)$ as

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k+1} \frac{(n-1)!}{k!(n-k-1)!} p^k g^{n-k-1} &= \sum_{k=2}^n \frac{1}{n} \frac{n!}{k!(n-k)!} p^k g^{n-k} \\ &= n^{-1} \sum_{k=2}^n \binom{n}{k} p^k g^{n-k} / p = [(p+g)^n - g^n - npg^{n-1}] / np \end{aligned}$$

If $p=0$, $I_n(u) = 0$ so

$$I_n(u) = [(p+g)^n - g^n - npg^{n-1}] / np$$

where the right side is interpreted as 0 if $p=0$.

To help us check $I_n(u) \rightarrow 0$ uniformly for $u \in [0, r]$ we write down the following facts:

a) $g \rightarrow 1$ as $n \rightarrow \infty$ uniformly on $[0, r]$.

This is simply because $g = P[c_n^{-1} |\tilde{X}_1| > u] \geq P[c_n^{-1} |\tilde{X}_1| > r]$.

b) $n p_n(u) \rightarrow 0$ uniformly since for any given $\epsilon > 0$

$$n p_n \leq n P[c_n^{-1} |\tilde{X}_1| \in (v, v+\epsilon)] \rightarrow e^{-u^\alpha} - e^{-(v+\epsilon)^\alpha}$$

and the last convergence is uniform.

c) As a consequence of (a) and (b) we have $np/g \rightarrow 0$ uniformly on $[0, r]$.

Finally we have

$$\begin{aligned} 0 \leq I_n(u) &= g^n [(1+p/g)^n - 1] / np - g^{n-1} \\ &= g^n [\exp \{n \log (1+p/g)\} - 1] / np - g^{n-1}. \end{aligned}$$

Since $p/g \rightarrow 0$, $n \log(1+p/g) \sim np/g \rightarrow 0$ by (c) and therefore $\exp \{n \log(1+p/g)\} - 1 \sim n \log(1+p/g) \sim np/g$ as $n \rightarrow \infty$. Thus given $\epsilon > 0$, for n sufficiently large we have uniformly

$$\begin{aligned} 0 \leq I_n(u) &\leq g^{n(1+\epsilon)} (np/g) / np - g^{n-1} \\ &= (1+\epsilon) g^{n-1} - g^{n-1} = \epsilon g^{n-1} \leq \epsilon \end{aligned}$$

and this suffices for the desired result.

Remark

Under the regular variation conditions, the point process $\{c_n^{-1} \tilde{X}_k\}_{k=1}^n$ converges weakly to a Poisson point process with mean measure ν . Applying the functional which maps point processes to the point closest to 0 gives via the continuous mapping theorem the sufficiency half of the theorem.

Example

If \tilde{X}_1 is uniformly distributed on the unit disc then picking $c_n = (2n)^{-1/2}$ gives $c_n^{-1} \tilde{X}_n^* \Rightarrow \underline{Y} = (Y_1, Y_2)$ i.i.d. $N(0, 1)$. To check this one observes that for $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

$$P[|\tilde{X}_1| \leq r, \theta(\tilde{X}_1) \leq \theta] = \frac{r^2 \theta}{2\pi}$$

and hence $2n \pi P\{c_n^{-1} \tilde{X}_1 \in \cdot\}$ converges to Lebesgue measure. Note that the convergence to the normal distribution is no longer true in \mathbb{R}^k with $k \neq 2$.

Example

Suppose \tilde{X}_1 has a bivariate Cauchy density $(1+x_1^2 + x_2^2)^{-3/2} / 2\pi$. Then

$$P[|\tilde{X}_1| \leq r, \theta(\tilde{X}_1) \leq \theta] = (1 - (1+r^2)^{-1/2}) \theta / 2\pi.$$

Since $P[|\tilde{X}_1| \leq t] \sim t^2/2$ as $t \downarrow 0$ we may set $c_n = n^{-1/2}$ and

$$\lim_{n \rightarrow \infty} n P[|\tilde{X}_1| \leq c_n r, \theta(\tilde{X}_1) \leq \theta] = \frac{1}{2} r^2 \theta / 2\pi$$

so that $\alpha = 2$, $S([0, \theta]) = \theta / 2\pi$. As before $\underline{Y} = (Y_1, Y_2)$ is i.i.d. $N(0, 1)$ and $\nu(dx_1, dx_2) = dx_1 dx_2 / 2\pi$.

Remark

In both previous examples we had $\alpha = 2$. This is due to the fact that in both cases \tilde{X}_1 has a continuous density which is positive at 0 . Cf. section 3 below. An example where this is not the case is the following:

Example

Suppose \tilde{X}_1 has a bivariate gamma density concentrated on \mathbb{R}_+^2 :

$$f(x_1, x_2) = e^{-(x_1 + x_2)} (1 - e^{-x_1 \wedge x_2}); x_1, x_2 \geq 0$$

(Johnson and Kotz, 1972, p. 218). As $x_1 \rightarrow 0, x_2 \rightarrow 0$

$$f(x_1, x_2) \sim x_1 \wedge x_2.$$

Since $f(t, t) = t$, we have for $\underline{x} = (x_1, x_2) \in (0, \infty)^2$:

$$P[\underline{X}_1 \leq t \underline{x}] / t^3 = \int_0^{x_1} \int_0^{x_2} \frac{f(t \underline{y}) d\underline{y}}{f(t, t)} \rightarrow$$

$$\int_0^{x_1} \int_0^{x_2} y_1 \wedge y_2 dy_1 dy_2 =: v([0, x_1] \times [0, x_2])$$

as $t \rightarrow 0$ (cf. de Haan and Resnick, 1979, Theorem 1) and we have verified that the regular variation condition (i) holds.

Hence $\alpha = 3$. To find S observe for $\theta_0 \in [0, \pi/2]$:

$$v\{\underline{x} : |\underline{x}| \leq r_0, \theta(\underline{x}) \leq \theta_0\} = \int_{r \leq r_0} \int_{\theta \leq \theta_0} (r \cos \theta \wedge r \sin \theta) r dr d\theta$$

$$= \frac{1}{3} r^3 \int_0^{\theta_0} (\cos \theta \wedge \sin \theta) d\theta$$

so that (note that v is specified only up to a multiplicative constant)

$$(2 - \sqrt{2}) S([0, \theta]) = \begin{cases} 1 - \cos \theta & 0 \leq \theta \leq \pi/4 \\ 1 - \sqrt{2} + \sin \theta & \pi/4 \leq \theta \leq \pi/2. \end{cases}$$

3. Density Estimation

Suppose \underline{Y}_i are i.i.d. random vectors in \mathbb{R}^2 with distribution function F . Suppose F has a positive derivative F' at some point \underline{x}_0 in the sense that for every rectangle I with one endpoint in the origin

$$(5) \lim_{t \rightarrow 0} P[\underline{Y}_1 - \underline{x}_0 \in t \cdot I] / t^2 |I| = F'(\underline{x}_0)$$

where $|I|$ is Lebesgue measure of I (cf. Saks, 1964).

This condition is satisfied for instance if F has a density continuous and positive at \underline{x}_0 . We are going to estimate $F'(\underline{x}_0)$ using the \underline{Y} -observations.

Translating \tilde{x}_0 to the origin by setting $\tilde{X}_i = Y_i - \tilde{x}_0$ shows that (5) is the regular variation condition (i) of section 2 and $v(dx_1, dx_2) = F'(x_0)dx_1 dx_2$. Since S is the uniform distribution on $[0, 2\pi]$, the problem is really one-dimensional and we may as well work with $\{|\tilde{X}_i|, i \leq n\}$. Since

$$n P[c_n^{-1} |\tilde{X}_1| \leq r] \rightarrow r^2$$

with $c_n = (n\pi F'(x_0))^{-\frac{1}{2}}$, the point process on R_+ with points $\{|\tilde{X}_i|/c_n, i \leq n\}$ converges weakly to a Poisson process on R_+ with mean measures on $[0, x] = x^2$.

So the order statistics $\{c_n^{-1} |\tilde{X}_n^{(i)}|, i \leq n\}$ from $\{c_n^{-1} |\tilde{X}_i|, i \leq n\}$ form asymptotically a Poisson process and we base our estimate on the k^{th} smallest $\tilde{X}_n^{(k)}$ where $k = k(n) \rightarrow \infty$ and $k/n \rightarrow 0$. The law of large numbers then will give weak consistency.

It is convenient to use the Renyi representation for order statistics $(Z_n^{(1)}, Z_n^{(2)}, \dots, Z_n^{(n)})$ from the exponential distribution (Feller, 1971, chapter 1): If E_1, E_2, \dots are i.i.d., $P[E_i > x] = e^{-x}$, then the k^{th}

smallest $Z_n^{(k)} \stackrel{d}{=} \frac{E_1}{n} + \frac{E_2}{n-1} + \dots + \frac{E_k}{n-k+1}$. If $k = k(n) \rightarrow \infty$, $k/n \rightarrow 0$ then

$$\frac{n}{k} \left(\frac{Z_1}{n} + \frac{Z_2}{n} + \dots + \frac{Z_k}{n-k+1} \right) \xrightarrow{p} 1$$

We now use a probability integral transformation:

Let $\Psi(x) = -\log P[|\tilde{X}_1| > x]$. From the regular variation condition we have $\lim_{t \downarrow 0} \Psi(t)/t^2 = F'(x_0)\pi$ and so for the inverse function Ψ^{-1} we have

$$\lim_{t \downarrow 0} \Psi^{-1}(t)/t^{\frac{1}{2}} = (F'(x_0)\pi)^{-\frac{1}{2}}.$$

Since $|\tilde{X}_n^{(k)}| \stackrel{d}{=} \Psi^{-1}\left(\frac{Z_1}{n} + \dots + \frac{Z_k}{n-k+1}\right)$ it follows

$$\begin{aligned} \left(\frac{n}{k}\right)^{\frac{1}{2}} |\tilde{X}_n^{(k)}| &\stackrel{d}{=} \left(\frac{n}{k}\right)^{\frac{1}{2}} \Psi^{-1}\left(\frac{k}{n} \left(\frac{Z_1}{n} + \dots + \frac{Z_k}{n-k+1}\right)\right) \\ &\sim \left(\frac{n}{k}\right)^{\frac{1}{2}} \Psi^{-1}(k/n) \xrightarrow{p} (F'(x_0)\pi)^{-\frac{1}{2}}. \end{aligned}$$

So our estimate of $F'(x_0)$ is

$$k/(n\pi |\tilde{X}_n^{(k)}|^2).$$

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