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ON THE OBSERVATION CLOSEST TO THE ORIGIN

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On the Observation closest to the Origin

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ABSTRACT

Out of n i.i.d. random vectors in $\mathbb{R}^d$ let $\underset{\sim}{X}_n^*$ be the one closest to the origin. We show that $\underset{\sim}{X}_n^*$ has a nondegenerate limit distribution if and only if the common probability distribution satisfies a condition of multidimensional regular variation.
We apply the result to a problem of density estimation.

Contents.

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1. Introduction.

Suppose $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ are i.i.d. observations in a Euclidean space which for convenience we take to be the plane R^2. The polar coordinates of $\underset{\sim}{X}_i$ are denoted by $(|\underset{\sim}{X}_i|, \Theta(\underset{\sim}{X}_i))$. Call the observation with minimum modulus $\underset{\sim}{X}_n^*$; if more than one observation has modulus equal to $\bigwedge_{i=1}^{n} |\underset{\sim}{X}_i|$, choose $\underset{\sim}{X}_n^*$ from the tied observations by randomization.
As $n \to \infty$ $\underset{\sim}{X}_n^*$ will converge to $\underset{\sim}{0}$ but it may be possible to normalize $\underset{\sim}{X}_n^*$ so that a non-trivial limiting distribution exists.
So we ask: when do there exist constants c_n, $0 < c_n \to 0$ such that

$$(1) \qquad c_n^{-1} \underset{\sim}{X}_n^* \Rightarrow \underset{\sim}{Y}$$

where the limit vector $\underset{\sim}{Y}$ is finite and $|\underset{\sim}{Y}|$ non-degenerate. Here the arrow "$\Rightarrow$" indicates weak convergence.

In section 2 we show that the necessary and sufficient condition for (1) is multivariate regular variation at $\underset{\sim}{0}$ of the distribution of $\underset{\sim}{X}_1$. Also the polar coordinates of $\underset{\sim}{Y}$ are independent random variables.

In the latter section of the paper we examine a related density estimation problem.

2. Multivariate Regular Variation and the Observation closest to $\underset{\sim}{0}$.

The condition that the distribution of $\underset{\sim}{X}_1$ be regularly varying at $\underset{\sim}{0}$ can conveniently be expressed either in terms of vague convergence of measures or, a regular variation property in terms of the distribution of $(|X_1|, \Theta(X_1))$ and the following are equivalent:

(i) There exist constants c_n, $0 < c_n \to 0$ and a limit measure ν on R^2 which is finite on bounded neighbourhoods of $\underset{\sim}{0}$ such that

$$n\, P[c_n^{-1} \underset{\sim}{X}_1 \in \cdot\,] \overset{v}{\to} \nu$$

("$\overset{v}{\to}$" denotes vague convergence). This sequential form of the condition is the same as saying there exists a (necessarily regularly varying) function U: $R_+ \to R_+$ such that

$$P[\underset{\sim}{X}_1 \in t.(\cdot)]/U(t) \overset{v}{\to} \nu \qquad \text{as } t \downarrow 0.$$

(ii) There exist constants c_n, $0 < c_n \to 0$, $\alpha > 0$, and a probability measure S on $\mathcal{B}([0,2\pi])$ such that

$$n\, P[c_n^{-1}\, |\underset{\sim}{X}_1| \le r,\ \Theta(X_1) \le \theta] \overset{v}{\to} r^\alpha\, S([0,\theta]).$$

This is the same as

$$\frac{P[|X_1| \le tr,\ \Theta(X_1) \le \theta]}{P[|X_1| \le t]} \overset{v}{\to} r^\alpha\, S([0,\theta])$$

as $t \downarrow 0$. Cf. Stam (1977), de Haan and Resnick (1977). We remark that the constants $\{c_n\}$ in (i) and (ii) need not be the same; their quotient tends to a positive constant.

Theorem 1.

There exists a sequence of positive constants $\{c_n\}$ such that $c_n^{-1}\, \underset{\sim}{X}_n^*$ converges in distribution to a finite limit vector $\underset{\sim}{Y}$ with $|\underset{\sim}{Y}|$ non-degenerate iff the distribution of $\underset{\sim}{X}_1$ is regularly varying at $\underset{\sim}{0}$ as described in (i) or (ii) above. In this case $P[|\underset{\sim}{Y}| > r,\ \Theta(Y) \le \theta] = S([0,\theta])e^{-r^\alpha}$, with $0 < \alpha < \infty$ so that $|\underset{\sim}{Y}|$ and $\Theta(\underset{\sim}{Y})$ are independent.

Proof:

We first prove the theorem under the added assumption that $|\underset{\sim}{X}_1|$ has a continuous distribution. This assumption is removed at the end. With this added assumption, it is easy to derive the distribution of $\underset{\sim}{X}_n^*$: by symmetry $\underset{\sim}{X}_n^*$ could be any one of the $\underset{\sim}{X}_i$'s with equal probability. If $\underset{\sim}{X}_n^* = \underset{\sim}{X}_1$ then $\bigwedge_{i=2}^{n} (\underset{\sim}{X}_i) > |\underset{\sim}{X}_1|$. Thus we get

(2) $$P[|c_n^{-1}\, \underset{\sim}{X}_n^*| \le r,\ \Theta(\underset{\sim}{X}_n^*) \le \theta] =$$

$$= \int_0^r P[\bigwedge_{i=2}^{n} c_n^{-1}\, |\underset{\sim}{X}_i| > u]\ n\, P[c_n^{-1}\, |\underset{\sim}{X}_1| \in du,\ \Theta(\underset{\sim}{X}_1) \le \theta]$$

Supposing (ii) holds we easily get $P[\bigwedge_2^n c_n^{-1}\, |\underset{\sim}{X}_i| > u] \to e^{-u^\alpha}$ uniformly. Assume $S\{\theta\} = 0$ and write the integral in (2) as

$$\int_0^r \{P[\bigwedge_{i=2}^{n} c_n^{-1}\, |\underset{\sim}{X}_i| > u] - e^{-u^\alpha}\}\ n\, P[c_n^{-1}|\underset{\sim}{X}_1| \in du,\ \Theta(\underset{\sim}{X}_1) \le \theta]$$

$$+ \int_0^r e^{-u^\alpha}\ n\, P[c_n^{-1}\, |\underset{\sim}{X}_1| \in du,\ \Theta(\underset{\sim}{X}_1) \le \theta]$$

The first term goes to zero by uniform convergence. The second term is seen to converge vaguely in the usual way by appealing to the Helly-Bray lemma (Loève, 1963, p. 180) and so

$$P[c_n^{-1} |\underset{\sim}{X}_n^*| \leq r, \Theta(\underset{\sim}{X}_n^*) \leq \theta] \overset{v}{\to} \int_0^r e^{-u^\alpha} du^\alpha S([0,\theta])$$

$$= P[|\underset{\sim}{Y}| \leq r, \Theta(\underset{\sim}{Y}) \leq \theta].$$

For the converse suppose there exists $c_n, 0 < c_n \to 0$, and $\underset{\sim}{Y}$ with $P[\underset{\sim}{Y} = \underset{\sim}{0}] < 1$ such that $c_n^{-1} \underset{\sim}{X}_n^* \Rightarrow \underset{\sim}{Y}$. Then $c_n^{-1} \bigwedge_{i=1}^{n} |\underset{\sim}{X}_i| = c_n^{-1} |\underset{\sim}{X}_n^*| \Rightarrow |\underset{\sim}{Y}|$. Since the only normalization necessary here for convergence of the minima is scaling, we can easily identify the distribution of $|\underset{\sim}{Y}|$ to be the extreme value distribution $P[|\underset{\sim}{Y}| > x] = e^{-kx^\alpha}$, $x > 0$ for some $0 < \alpha < \infty$, $k > 0$. Without loss of generality, suppose $k = 1$.

This distribution is continuous so $P[\bigwedge_{i=2}^{n} c_n^{-1} |\underset{\sim}{X}_i| > u] \to e^{-u^\alpha}$ uniformly. To deal with the rest of the integral in the right side of (2) we use the selection theorem: Since $n P[c_n^{-1} |\underset{\sim}{X}_i| \leq u, \Theta(\underset{\sim}{X}_1) \leq \theta] \leq n P[c_n^{-1} |\underset{\sim}{X}_1| \leq u] \to - \log P[|\underset{\sim}{Y}| > u] = u^\alpha$ as $n\to\infty$ (an easy consequence of $c_n^{-1} \bigwedge_{i=1}^{n} |X_i| \Rightarrow |\underset{\sim}{Y}|$) we have for each u,θ that $n P[c_n^{-1} |\underset{\sim}{X}_i| \leq u, \Theta(\underset{\sim}{X}_1) \leq \theta]$ is a bounded sequence.

Thus we are assured of the existence of a vaguely convergent subsequence n' and a vague limit $\Psi(du, d\theta)$ such that

$n'P[c_{n'}^{-1} |\underset{\sim}{X}_1| \in du, \Theta(\underset{\sim}{X}_1) \in d\theta] \overset{v}{\to} \Psi(du, d\theta)$.

As in the first half of the proof the uniform convergence of $P[\bigwedge_2^n |\underset{\sim}{X}_i| > c_n \cdot u]$ gives as $n'\to\infty$

$$P[c_{n'}^{-1} |\underset{\sim}{X}_{n'}^*| \leq r, \Theta(\underset{\sim}{X}_{n'}^*) \leq \theta] \overset{v}{\to} \int_0^r e^{-u^\alpha} \Psi(du \times [0,\theta])$$

If along another subsequence n''

$$n''P[c_{n''}^{-1} |\underset{\sim}{X}_1| \in du, \Theta(\underset{\sim}{X}_1) \in d\theta] \overset{v}{\to} \Psi'(du,d\theta)$$

then the measures determined by the distribution functions $\int_0^r e^{-u^\alpha} \Psi(du \times [0,\theta])$, $\int_0^r e^{-u^\alpha} \Psi^1(du \times [0,\theta])$ and $P[|\underset{\sim}{Y}| \leq r, \Theta(\underset{\sim}{Y}) \leq \theta]$ are all equal.

Since e^{-u^α} is non-vanishing, it is easy to see $\Psi = \Psi'$. Therefore the vague limit of $n P[c_n^{-1} |\underset{\sim}{X}_1| \leq r, \Theta(\underset{\sim}{X}_1) \leq \theta]$ exists.

This is equivalent to our first definition (i) of multivariate regular variation.

We now remove the assumption that $|X_1|$ has a continuous distribution thus allowing more than one observation to achieve minimum modulus. Formula (2) must be modified. As before, by symmetry $\underset{\sim}{X}_n^*$ could be any one of the $\underset{\sim}{X}_i$'s with equal probability so that

$$\begin{aligned}(3)\quad & P[c_n^{-1}\,|\underset{\sim}{X}_n^*| \le r,\ \Theta(X_n^*) \le \theta] \\ & = n\,P[c_n^{-1}\,|\underset{\sim}{X}_1| \le r,\ \Theta(\underset{\sim}{X}_1) \le \theta,\ \underset{\sim}{X}_1 = \underset{\sim}{X}_n^*].\end{aligned}$$

Write $[\underset{\sim}{X}_n^* = \underset{\sim}{X}_1] = \bigcup_{k=0}^{n-1} A_k$ where A_k is the event that k observations from $\underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ have moduli equal to $|\underset{\sim}{X}_1|$, n-k-1 observations have moduli > $|\underset{\sim}{X}_1|$ and the randomization procedure selects $\underset{\sim}{X}_1$ from among those k+1 observations with minimum modulus.
Then (3) becomes

$$n \int_{\{\underset{\sim}{x}|\,|\underset{\sim}{x}| \le r,\ \Theta(\underset{\sim}{x}) \le \theta\}} P[c_n^{-1}\,\underset{\sim}{X}_n^* = \underset{\sim}{x} \mid c_n^{-1}\,\underset{\sim}{X}_1 = \underset{\sim}{x}]\ P[c_n^{-1}\,\underset{\sim}{X}_1 \in d\underset{\sim}{x}]$$

$$= n\int_{\{\ \}} \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} P[c_n^{-1}\,|\underset{\sim}{X}_1| = |\underset{\sim}{x}|\,]^k\ P[c_n^{-1}\,|\underset{\sim}{X}_1| > |\underset{\sim}{x}|\,]^{n-k-1}.$$

$$P[c_n^{-1}\,\underset{\sim}{X}_1 \in d\underset{\sim}{x}]$$

$$= \int_0^r \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{n-1}{k} P[c_n^{-1}\,|\underset{\sim}{X}_1| = u]^k\ P[c_n^{-1}\,|\underset{\sim}{X}| > u]^{n-k-1}.$$

$$n\,P[c_n^{-1}\,|\underset{\sim}{X}_1| \in du,\ \Theta(\underset{\sim}{X}_1) \le \theta].$$

Note the k=0 term (no ties) corresponds to the rightside of (2).
For proving either necessity or sufficiency in this theorem, we have at our disposal the condition

$$(4)\quad P[\bigwedge_1^n |\underset{\sim}{X}_i| > c_n u] = P[|\underset{\sim}{X}_1| > c_n u]^n \to e^{-u^\alpha}$$

for some α. We now show (4) implies

$$I_n(u) := \sum_{k=1}^{n-1} \frac{1}{k+1} \binom{n-1}{k} p^k g^{n-k-1} \to 0$$

as $n\to\infty$ uniformly in $u \in [0,r]$, where $p = p_n(u) = P[c_n^{-1}\,|\underset{\sim}{X}_1| = u]$, $g = g_n(u) = P[c_n^{-1}\,|\underset{\sim}{X}_1| > u]$.

This then shows that $P[c_n^{-1} |\underset{\sim}{X}_n^*| \leq r, \; \Theta(\underset{\sim}{X}_n^*) \leq \theta]$ equals the righthandside of (2) plus $o(1)$; the case where the df of $|\underset{\sim}{X}_1|$ is not continuous can then be reduced to the continuous case.

Rewrite $I_n(u)$ as

$$\sum_{k=1}^{n-1} \frac{1}{k+1} \frac{(n-1)!}{k!(n-k-1)!} p^k g^{n-k-1} = \sum_{k=2}^{n} \frac{1}{n} \frac{n!}{k!(n-k)!} p^k g^{n-k}$$

$$= n^{-1} \sum_{k=2}^{n} \binom{n}{k} p^k g^{n-k}/p = [(p+g)^n - g^n - npg^{n-1}]/np$$

If $p=0$, $I_n(u) = 0$ so

$$I_n(u) = [(p+g)^n - g^n - npg^{n-1}]/np$$

where the right side is interpreted as 0 if $p=0$.

To help us check $I_n(u) \to 0$ uniformly for $u \in [0,r]$ we write down the following facts:

a) $g \to 1$ as $n\to\infty$ uniformly on $[0,r]$.

This is simply because $g = P[c_n^{-1} |\underset{\sim}{X}_1| > u] \geq P[c_n^{-1} |\underset{\sim}{X}_1| > r]$.

b) $n\, p_n(u) \to 0$ uniformly since for any given $\varepsilon > 0$

$$n\, p_n \leq n\, P[c_n^{-1} |\underset{\sim}{X}_1| \in (v, v+\varepsilon)] \to e^{-u^\alpha} - e^{-(v+\varepsilon)^\alpha}$$

and the last convergence is uniform.

c) As a consequence of (a) and (b) we have $np/g \to 0$ uniformly on $[0,r]$.

Finally we have

$$0 \leq I_n(u) = g^n [(1+p/g)^n - 1]/np - g^{n-1}$$

$$= g^n[\exp\{n \log(1+p/g)\} - 1]/np - g^{n-1}.$$

Since $p/g \to 0$, $n \log(1+p/g) \sim np/g \to 0$ by (c) and therefore $\exp\{n \log(1+p/g)\} - 1 \sim n \log(1+p/g) \sim np/g$ as $n\to\infty$. Thus given $\varepsilon > 0$, for n sufficiently large we have uniformly

$$0 \leq I_n(u) \leq g^n(1+\varepsilon)\,(np/g)/np - g^{n-1}$$

$$= (1+\varepsilon)\, g^{n-1} - g^{n-1} = \varepsilon\, g^{n-1} \leq \varepsilon$$

and this suffices for the desired result.

Remark

Under the regular variation conditions, the point process $\{c_n^{-1} \underset{\sim}{X}_x\}_{k=1}^{n}$ converges weakly to a Poisson point process with mean measure ν. Applying the functional which maps point processes to the point closest to $\underset{\sim}{0}$ gives via the continuous mapping theorem the sufficiency half of the theorem.

Example

If $\underset{\sim}{X}_1$ is uniformly distributed on the unit disc then picking $c_n = (2n)^{-\frac{1}{2}}$ gives $c_n^{-1} \underset{\sim}{X}_n^* \Rightarrow \underset{\sim}{Y} = (Y_1, Y_2)$ i.i.d. $N(0, 1)$. To check this one observes that for $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

$$P[|X_1| \leq r, \Theta(\underset{\sim}{X}_1) \leq \theta] = \frac{r^2\theta}{2\pi}$$

and hence $2n\,\pi\, P\{c_n^{-1} \underset{\sim}{X}_1 \in \cdot\}$ converges to Lebesgue measure. Note that the convergence to the normal distribution is no longer true in $\mathbb{R}^k$ with $k \neq 2$.

Example

Suppose $\underset{\sim}{X}_1$ has a bivariate Cauchy density $(1+x_1^2 + x_2^2)^{-3/2}/\ 2\pi$. Then

$$P[|\underset{\sim}{X}_1| \leq r, \Theta(\underset{\sim}{X}_1) \leq \theta] = (1-(1+r^2)^{-\frac{1}{2}})\ \theta/\ 2\pi.$$

Since $P[|\underset{\sim}{X}_1| \leq t] \sim t^2/2$ as $t \downarrow 0$ we may set $c_n = n^{-\frac{1}{2}}$ and

$$\lim_{n\to\infty} n\, P[|\underset{\sim}{X}_1| \leq c_n r, \Theta(\underset{\sim}{X}_1) \leq \theta] = \tfrac{1}{2}r^2\ \theta/\ 2\pi$$

so that $\alpha = 2$, $S([0,\theta]) = \theta/\ 2\pi$. As before $\underset{\sim}{Y} = (Y_1, Y_2)$ is i.i.d. $N(0,1)$ and $\nu(dx_1, dx_2) = dx_1 dx_2/\ 2\pi$.

Remark

In both previous examples we had $\alpha = 2$. This is due to the fact that in both cases $\underset{\sim}{X}_1$ has a continuous density which is positive at $\underset{\sim}{0}$. Cf. section 3 below. An example where this is not the case is the following:

Example

Suppose $\underset{\sim}{X}_1$ has a bivariate gamma density concentrated on R_+^2:

$$f(x_1, x_2) = e^{-(x_1 + x_2)} (1 - e^{-x_1 \wedge x_2}); \; x_1, x_2 \geq 0$$

(Johnson and Kotz, 1972, p. 218). As $x_1 \to 0$, $x_2 \to 0$

$$f(x_1, x_2) \sim x_1 \wedge x_2.$$

Since $f(t, t) = t$, we have for $\underset{\sim}{x} = (x_1, x_2) \in (0,\infty)^2$:

$$P[\underset{\sim}{X}_1 \leq t \underset{\sim}{x}]/ t^3 = \int_0^{x_1} \int_0^{x_2} \frac{f(t \underset{\sim}{y}) \, d\underset{\sim}{y}}{f(t, t)} \longrightarrow$$

$$\int_0^{x_1} \int_0^{x_2} y_1 \wedge y_2 \, dy_1 \, dy_2 =: \nu([0,x_1] \times [0,x_2])$$

as $t \downarrow 0$ (cf. de Haan and Resnick, 1979, Theorem 1) and we have verified that the regular variation condition (i) holds.
Hence $\alpha = 3$. To find S observe for $\theta_0 \varepsilon [0,\pi/2]$:

$$\nu\{\underset{\sim}{x} : |\underset{\sim}{x}| \leq r_0, \, \Theta(\underset{\sim}{x}) \leq \theta_0\} = \int_{r \leq r_0} \int_{\theta \leq \theta_0} (r \cos \theta \wedge r \sin \theta) \, r \, dr \, d\theta$$

$$= \frac{1}{3} r^3 \int_0^{\theta_0} (\cos \theta \wedge \sin \theta) \, d\theta$$

so that (note that ν is specified only up to a multiplicative constant)

$$(2 - \sqrt{2}) \, S([0,\theta]) = \begin{cases} 1 - \cos \theta & 0 \leq \theta \leq \pi/4 \\ 1 - \sqrt{2} + \sin \theta & \pi/4 \leq \theta \leq \pi/2. \end{cases}$$

3. Density Estimation

Suppose $\underset{\sim}{Y}_i$ are i.i.d. random vectors in $\mathbb{R}^2$ with distribution function F. Suppose F has a positive derivative F' at some point $\underset{\sim}{x}_0$ in the sense that for every rectangle I with one endpoint in the origin

(5) $$\lim_{t \to 0} P[\underset{\sim}{Y}_1 - \underset{\sim}{x}_0 \in t \cdot I]/ t^2 \, |I| = F'(\underset{\sim}{x}_0)$$

where $|I|$ is Lebesgue measure of I (cf. Saks, 1964).
This condition is satisfied for instance if F has a density continuous and positive at $\underset{\sim}{x}_0$. We are going to estimate $F'(x_0)$ using the $\underset{\sim}{Y}$-observations.

Translating $\underset{\sim}{x}_0$ to the origin by setting $\underset{\sim}{X}_i = \underset{\sim}{Y}_i - \underset{\sim}{x}_0$ shows that (5) is the regular variation condition (i) of section 2 and $\nu(dx_1, dx_2) = F'(x_0)dx_1\, dx_2$. Since S is the uniform distribution on $[0, 2\pi]$, the problem is really one-dimensional and we may as well work with $\{|\underset{\sim}{X}_i|, i \le n\}$. Since

$$n\, P[c_n^{-1}\, |\underset{\sim}{X}_1| \le r] \to r^2$$

with $c_n = (n\pi F'(\underset{\sim}{x}_0))^{-\frac{1}{2}}$, the point process on R_+ with points $\{|\underset{\sim}{X}_i|/c_n, i \le n\}$ converges weakly to a Poisson process on R+ with mean measures on $[0,x] = x^2$.
So the order statistics $\{c_n^{-1}\, |\underset{\sim}{X}_n^{(i)}|, i \le n\}$ from $\{c_n^{-1}\, |\underset{\sim}{X}_i|, i \le n\}$ form asymptotically a Poisson process and we base our estimate on the k^{th} smallest $\underset{\sim}{X}_n^{(k)}$ where $k = k(n) \to \infty$ and $k/n \to 0$. The law of large numbers then will give weak consistency.

It is convenient to use the Renyi representation for order statistics $(Z_n^{(1)}, Z_n^{(2)}, \ldots, Z_n^{(n)})$ from the exponential distribution (Feller, 1971, chapter 1): If $E_1, E_2, \ldots$ are i.i.d., $P[E_i > x] = e^{-x}$, then the k^{th} smallest $Z_n^{(k)} \overset{d}{=} \frac{E_1}{n} + \frac{E_2}{n-1} + \ldots + \frac{E_k}{n-k+1}$. If $k = k(n) \to \infty$, $k/n \to 0$ then

$$\frac{n}{k}\left(\frac{Z_1}{n} + \frac{Z_2}{n} + \ldots + \frac{Z_k}{n-k+1}\right) \overset{p}{\to} 1$$

We now use a probability integral transformation:
Let $\Psi(x) = -\log P[|\underset{\sim}{X}_1| > x]$. From the regular variation condition we have $\lim_{t\downarrow 0} \Psi(t)/t^2 = F'(\underset{\sim}{x}_0)\pi$ and so for the inverse function Ψ^{-1} we have $\lim_{t\downarrow 0} \Psi^{-1}(t)/t^{\frac{1}{2}} = (F'(x_0)\pi)^{-\frac{1}{2}}$.
Since $|\underset{\sim}{X}_n^{(k)}| \overset{d}{=} \Psi^{-1}\left(\frac{Z_1}{n} + \ldots + \frac{Z_k}{n-k+1}\right)$ it follows

$$\left(\frac{n}{k}\right)^{\frac{1}{2}} |\underset{\sim}{X}_n^{(k)}| \overset{d}{=} \left(\frac{n}{k}\right)^{\frac{1}{2}} \Psi^{-1}\left(\frac{k}{n}\left[\left(\frac{n}{k}\right)\left(\frac{Z_1}{n} + \ldots + \frac{Z_k}{n-k+1}\right)\right]\right)$$

$$\sim \left(\frac{n}{k}\right)^{\frac{1}{2}} \Psi^{-1}(k/n) \overset{p}{\to} (F'(x_0)\pi)^{-\frac{1}{2}}.$$

So our estimate of $F'(\underset{\sim}{x}_0)$ is

$$k/(n\pi\, |\underset{\sim}{X}_n^{(k)}|^2).$$

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