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OLS ESTIMATION IN A MODEL WHERE A MICROVARIABLE IS EXPLAINED BY AGGREGATES AND CONTEMPORANEOUS DISTURBANCES ARE EQUICORRELATED

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OLS ESTIMATION IN A MODEL WHERE A MICROVARIABLE IS EXPLAINED BY AGGREGATES AND CONTEMPORANEOUS DISTURBANCES ARE EQUICORRELATED *

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Abstract

In the model $y = X\beta + u$ with $Eu = 0$ and $Euu' = \sigma^2 G$ it is possible that the OLS and GLS estimators are identical, even if $G \neq I$. However, the conditions for this identity do not necessarily imply the second equality sign in $V(\hat{\beta}_{OLS}) = \sigma^2(X'X)^{-1}X'GX(X'X)^{-1} = \sigma^2(X'X)^{-1}$, the latter being the usual formula for the OLS covariance matrix. This problem is illustrated for a particular model which may be applicable when a microvariable is explained by aggregates and contemporaneous disturbances are equicorrelated.

Contents

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* An earlier version of this paper was written in the form of a comment on a paper by Riddell [2]. In its present form it is more self-contained so that reading [2] is no longer a prerequisite. The author gratefully acknowledges an extensive comment by Riddell. He is also indebted to R. Harkema, J. Kmenta, P. Kooiman, A. Kunstman, S. Schim van der Loeff, H.K. van Dijk and a referee for valuable comments and suggestions. None of these persons should be held responsible for remaining errors.

1. INTRODUCTION

In the general case of a linear model $y = X\beta + u$ with $E\,u = 0$, $E\,uu' = \sigma^2 G$ (where G is positive-definite symmetric and X is fixed of order $n\times k$ with full column rank) it can be shown (Rao [1]) that the OLS and GLS estimators are identical if and only if matrices C and D exist such that

$$(1.1)\qquad G = XCX' + ZDZ' + \sigma^2 I$$

where Z is an $n \times (n-k)$ matrix with full column rank, satisfying $Z'X = 0$. If both estimators are identical they have the same covariance matrix which may be written as

$$(1.2)\qquad V(\hat{\beta}_{OLS}) = \sigma^2(X'X)^{-1}X'GX(X'X)^{-1} = \sigma^2 C + \sigma^2(X'X)^{-1}$$

Hence, the usual formula for the OLS covariance matrix applies if and only if $C = 0$, which is true only in a subset of the cases where (1.1) holds. In the present paper we shall consider two particular models. In the first (1.1) does not hold exactly but perhaps approximately, while in the second (which is a special case of the first) (1.1) holds with $C \neq 0$.

In the models we consider we can partition y, X and u as

$$(1.3)\qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} \qquad X = \begin{bmatrix} e_1 x_1' \\ \vdots \\ e_T x_T' \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix}$$

y_t, e_t and u_t are column vectors each consisting of m_t components ($t = 1, \ldots, T$); $e_t' = [1 \ldots 1]$ and x_t is a column vector consisting of k components. So the model may be applicable in case we have m_t observations on micro-variables in time period t which are all explained by the same vector of aggregates x_t. An empirical example can be found in Riddell [2, equations (1) through (12)], where y_t is a vector of money wage changes in m_t wage contracts for individual decision making units in period t. The explanatory variables are (functions of) aggregates, such as national unemployment and the (expected) consumer price index. So, all elements of y_t are explained by identical rows in the X matrix.

The disturbances are assumed to be homoskedastic, equicorrelated within time periods and uncorrelated across time periods. So the covariance matrix $\sigma^2 G$ is block diagonal and the t-th diagonal block can be written as

$$E\, u_t u_t' = G_t = (1-\rho)I_t + \rho e_t e_t' \tag{1.4}$$

where I_t is the unit matrix of order m_t. This concludes the assumptions for the first model to be considered. The second model is a particular case of the first, the additional assumption being that $m_t = m$ $(t = 1, \ldots, T)$.

For these models we start to derive simple expressions for the OLS and GLS estimators and their covariance matrices. Then we show that these estimators are identical in the second model. Finally we show for the second model that the usual covariance formula for the OLS estimator underestimates the true covariance matrix. The error made in this way may be serious if ρm is greater than two or three.

2. DERIVATION OF RESULTS

The following properties of G_t and G_t^{-1} are readily verified:

$$G_t^{-1} = [I_t - (\rho/\tau_t)e_t e_t']/(1-\rho) \tag{2.1}$$

where

$$\tau_t = \rho(m_t - 1) + 1 \tag{2.2}$$

Furthermore we have

$$G_t e_t = \tau_t e_t \qquad e_t' G_t e_t = m_t \tau_t \tag{2.3}$$

$$G_t^{-1} e_t = (1/\tau_t) e_t \qquad e_t' G_t^{-1} e_t = m_t/\tau_t \tag{2.4}$$

These results may be used to obtain simple expressions for the OLS and GLS estimators, as follows:

$$\hat{\beta}_{OLS} = [\Sigma m_t x_t x_t']^{-1} \Sigma m_t x_t \bar{y}_t \tag{2.5}$$

$$\hat{\beta}_{GLS} = [\Sigma(m_t/\tau_t)x_t x_t']^{-1}\Sigma(m_t/\tau_t)x_t\bar{y}_t \tag{2.6}$$

where all summations are over $t = 1, \ldots, T$ and where $\bar{y}_t = e_t'y_t/m_t$. It is seen that the individual observations on the dependent variable only enter into (2.5) and (2.6) <u>via</u> the averages $\bar{y}_t$. In order to obtain efficient estimates one need not use the individual observations, provided the means are weighted with the appropriate weights m_t/τ_t. The corresponding covariance matrices are given by

$$V_{OLS} = \sigma^2(\Sigma m_t x_t x_t')^{-1}\Sigma m_t \tau_t x_t x_t'(\Sigma m_t x_t x_t')^{-1} \tag{2.7}$$

$$V_{GLS} = \sigma^2[\Sigma(m_t/\tau_t)x_t x_t']^{-1} \tag{2.8}$$

The estimators (2.5) and (2.6) are identical if and only if

$$[\Sigma(m_t/\tau_t)x_t x_t']^{-1}x_s = (\Sigma m_t x_t x_t')^{-1}\tau_s x_s \tag{2.9}$$

for $s = 1, \ldots, T$. A special case where this occurs is $m_t = m$ (all t). One might conjecture that, if (2.9) is mildly violated, (2.5) will yield a good approximation to (2.6).

Suppose next that the OLS estimator (2.5) has been used and that the standard errors have not been estimated according to (2.7) but using the traditional formula $\sigma^2(X'X)^{-1}$ which in the present case amounts to[1]

$$V^*_{OLS} = \sigma^2(\Sigma m_t x_t x_t')^{-1} \tag{2.10}$$

We analyse the consequences of this for the case of the second model where $m_t = m$ (all t).

There are two effects: the traditional estimator

$$\hat{\sigma}^2 = \frac{1}{mT - k}\Sigma\hat{u}_t'\hat{u}_t \tag{2.11}$$

is biased downward, while the matrix $(\Sigma m x_t x_t')^{-1}$ underestimates the

[1] Upon comparing (1.2), (2.7) and (2.8) it follows that, if $m_t = m$ (all t), $C = (\tau - 1)(\Sigma m x_t x_t')^{-1}$. Given this result one may also construct Z and D matrices satisfying (1.1), but this is tedious and not very illuminating.

the correct matrix $[\Sigma(m/\tau)x_t x_t']^{-1}$. The former effect is usually is a minor one, the latter may be quite important, as we shall proceed to show now. We first consider the bias of the traditional estimator (2.11). If $Euu' = \sigma^2 G$ and $\hat{u}$ is a vector of least-squares residuals, it is easily seen that

$$E\hat{u}'\hat{u} = \sigma^2[\operatorname{tr} G - \operatorname{tr}(X'X)^{-1}X'GX] \tag{2.12}$$

In the present particular case this amounts to

$$E\hat{u}'\hat{u} = \sigma^2(mT - k\tau) \tag{2.13}$$

compare (2.7). So (2.11) has to be multiplied by $(mT - k)/(mT - k\tau)$ to get an unbiased estimator. If mT is not too small this effect will usually not be important. Therefore we shall ignore it in the next paragraph.

The underestimation of the standard errors may be far more serious because of the second effect, i.e. the omission of the factor τ, which was missing in the matrix $(\Sigma m x_t x_t')^{-1}$. In Table 1 we have tabulated $\sqrt{\tau}$, the factor by which the standard errors obtained from (2.10) have to be multiplied.

TABLE 1. $\sqrt{\tau}$ AS A FUNCTION OF ρ AND m

ρ =	0	.05	.10	.20	.30	.50
m=10	1	1.20	1.38	1.67	1.92	2.35
m=30	1	1.57	1.97	2.61	3.11	3.94
m=50	1	1.86	2.43	3.29	3.96	5.05

It is seen that ρ cannot be ignored without serious consequences for the conclusions.[2] In particular, if m is large, the effect is sizable even for small values of ρ such as .05.

[2] In Riddell's case the assumptions of the first model are applicable. Since the variation in the m_t is not too large, we conjecture that the properties of the second model hold approximately. As Riddell's m_t are of the order of 10, the first row of Table 1 has to be used.

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