Statistics

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ECONOMETRIC INSTITUTE

A QUADRATIC ENGEL CURVE DEMAND MODEL

(squaring with the representative consumer)

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Erasmus

REPORT 7911/E

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A Quadratic Engel Curve Demand Model
(squaring with the representative consumer)

by

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This paper starts with some reflections on the notion of the representative consumer. These reflections result in the conclusion that the "average consumer" is - in general - not a representative consumer in the sense that he may be considered as a (fictitious) consumer acting in the same way as the individual consumers are supposed to do, i.e. maximizing their utility subject to a budget constraint.

Nevertheless, relations between per capita consumption on the one hand and income and prices on the other hand, may be such that, though being non-linear in income, they are "aggregation consistent". This is at the cost of more information on incomes. Here we used a time series of the coefficients of variation of the income distribution in the Netherlands, leading to a Quadratic Engel Curve (Q.E.C.) demand system.

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* We gratefully acknowledge the positive criticisms made by Professors W.H. Somermeyer and T. Kloek and Dr. A.C.F. Vorst on the first version of this paper.

1. Introduction

The notion of "the" representative consumer has been extensively discussed in literature. One of the reasons for the multiplicity of these discussions is the vagueness of the concept; this means that there may be as many definitions of "the representative consumer" as there are discussants dealing with the subject, especially when they do not have to bother about practical applications.

A very appealing definition is the following: the representative consumer[1] is the consumer with the mean of all individual incomes as his income[2] and consuming per budget item precisely the mean of all amounts consumed by the individual consumers on that item. This definition has been adopted (whether or not implicitly) widely in empirical work; frequent lack of data other than totals and/or per capita figures often forced researchers to adopt it.

Everyone who needs a concept like the representative consumer is apparently convinced that relations between per capita consumption of the various budget items on the one hand, and prices and per capita income on the other hand, are <u>derived relations</u>, the basic relations existing only between consumption by individuals, their income and prices.

Here we assume that each individual consumer allocates his income to the various budget items according to preference relations between all possible baskets of consumer goods given that income and prices. Then the question arises: is it possible to construct a micro demand system such that a corresponding macro system (expressing per capita demand in per capita income and prices) exists that is consistent with the micro demand relations, and, if so, how are the micro and the macro systems related? In other words: we ask whether a representative consumer as defined above "exists".

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[1] Be it a household or an individual.

[2] For short, the word income will mean here the total amount spent on consumption.

In dealing with this question we have to take account of two famous (im)possibility theorems: that of Nataf[3] and that of Arrow[4]. With respect to our problem the first theorem implies that all equations have to be linear in incomes unless we take some other features into account; the second theorem simply states that <u>in general</u> there is no "collective preference scheme" based on the individuals' schemes. Consequently, we have to impose less general conditions than those of Arrow and to use more information than per capita figures only.

This can be done in several ways. Here, as a first approach, we circumvent Arrow's theorem in an "over-sufficient" way by assuming that all consumers have identical tastes, thus replacing Arrow's condition of the unrestricted domain by nearly its opposite. We do this for convenience' sake; besides, it is a generally accepted hypothesis in econometric research. Nataf's theorem is given its due by also taking into account characteristics of the income distribution other than mere means.

It will appear that our representative consumer is less dull than his "averageness" might suggest. In particular, his Slutsky matrix turns out to be generally biased compared to the average of the individual Slutsky matrices. Though symmetric for the individual demand relations resulting from utility maximization if the income distribution is invariant (meaning that each individual always has the "same part of the cake"), it might be indefinite or even positive semidefinite; in these cases the representative consumer does not maximize utility but chooses a "saddle point position" or even minimizes utility!

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[3] See Nataf (1948), Green (1964) and Somermeyer and Van Daal (1978).

[4] See Arrow (1963).

The only case in which these troubles do not arise is that in which the (utility maximizing) demand functions are linear in income. According to Gorman (1961) (see also Somermeyer (1974)) the demand functions are then:

$$(1.1)\quad q_{kj} = \frac{\partial \varphi}{\partial p_k} + \frac{1}{\psi}\frac{\partial \psi}{\partial p_k}(C_j - \varphi),$$

where:

q_{kj} = consumption of good k (= 1, ..., K) by individual j (= 1, ..., J),

p_k = the price of good k,

C_j = income of individual j,

and where φ and ψ are linear homogeneous functions of $p = (p_1, \ldots, p_K)'$ such that the corresponding cost of utility functions are concave in prices. Individual j's cost of utility function has then the form:

$$(1.2)\quad C_j = \varphi + \psi \cdot u_j,$$

where u_j is j's utility; (1.2) is known in literature as the "Gorman Polar Form".

In literature (see, for example, Blackorby, Boyce and Russell (1978)) one can find the opinion that the system (1.1) is most suited for empirical work on per capita data because of its sufficiency for "preference aggregation". The considerations below might reinforce this feeling.

One might object that assuming that all individuals have identical tastes and defining the representative consumer as we did above is somewhat confusing. The next section shows that this assumption and this definition do not mean the same thing nor contradict each other. In section 3 we substantiate our assertions about the (fictituous) representative consumer as a utility maximizer. Section 4 is devoted to an obvious specification of the demand system: the quadratic Engel curve system (Q.E.C.). In section 5 we present estimation results of a Q.E.C. model regarding Dutch data on consumption in the period 1948-1975. These estimates relate to the micro demand equations and can, therefore, be used for specifying a "true" cost of living index function.

2. The form of the demand functions if a representative consumer exists

Suppose that for each of J individuals demand equations:

$$(2.1)\quad q_{kj} = f_k(p, C_j)$$

exist for k = 1, ..., K. Note that the functions f_k do not depend on j; all individuals are supposed to be equally constructed "pleasure machines" or (less negatively) "homines economici" maximizing identical utility functions subject to possibly different budget constraints.

Next we define the per capita variables:

$$(2.2)\quad \bar{q}_k = \frac{1}{J} \sum_{j=1}^{J} q_{kj}$$

and:

$$(2.3)\quad \bar{C} = \frac{1}{J} \sum_{j=1}^{J} C_j$$

and we wonder whether (2.1) can give rise to relations between $\bar{q}_k$, $\bar{C}$, p and some other characteristic (yet to be specified) of the income distribution:

$$(2.4)\quad \bar{q}_k = g_k(p, \bar{C}, w),$$

for k = 1, ..., K, where w is some characteristic of the income distribution other than the mean.

Because of the specification of the aggregates $\bar{q}_k$ and $\bar{C}$ (the only realistic ones if one wishes to do empirical work), the only case in which we can do without w is the already mentioned case of the q_{kj} being linear in incomes; this follows immediately from Nataf's theorem. Hence q_{kj} is essentially non-linear in C_j if some w appears in $\bar{q}_k$. Theoretically w may have any form, but in practice it is inconceivable that w has another form than some "summation-structure". Consequently, leaving further refinements for the theorists[5], we define:

$$(2.5)\quad w = \frac{1}{J} \sum_j \eta(C_j)$$

where η is some function of C_j.

From Nataf's theorem it follows that formulae (2.1) through (2.5) imply:

$$(2.6)\quad q_{kj} = \varphi_k \cdot \eta(C_j) + \psi_k \cdot C_j,$$

and

$$(2.7)\quad \bar{q}_k = \varphi_k \cdot w + \psi_k \cdot \bar{C},$$

where φ_k and ψ_k are k-specific functions of prices.

Relation (2.7) can be seen, for k = 1, ..., K, as the demand equations of a (fictituous) "representative" consumer and, therefore, because of the budget restriction:

$$(2.8)\quad \sum_{k=1}^{K} p_k \varphi_k = 0$$

and

$$(2.9)\quad \sum_{k=1}^{K} p_k \psi_k = 1.$$

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[5] See Muellbauer (1975 and 1976) and Vorst and Van Daal (1979).

These simple results are not very much different from those of Muellbauer's for his case of "Price Independent Generalized Linearity" (PIGL) derived under much more general conditions. As will be set out in section 4, under the condition of absence of money illusion they coincide with PIGL under the same condition; see Muellbauer (1976 and 1977).

3. The representative consumer as a utility maximizer

In this section we ask ourselves whether the assumption that the individual consumers are utility maximizers lead us to the conclusion that the[6] representative consumer is a utility maximizer as well. We try to answer this question by means of the Slutsky matrices. Consequently, we need derivatives with respect to individual and per capita income. The latter derivation can only be performed on the basis of particular assumptions concerning the income distribution. Here we simply assume that the relative income distribution does not change when per capita income $\bar{C}$ changes; this implies that for all $j = 1, \ldots, J$:

$$(3.1)\quad \frac{\partial C_j}{\partial \bar{C}} = \frac{C_j}{\bar{C}}$$

because the ratios of C_j and $\bar{C}$ do not change when $\bar{C}$ changes.

About the macro Slutsky coefficient $\bar{s}_{kk'}$, we can say now:

$$(3.2)\quad \bar{s}_{kk'} = \frac{\partial \bar{q}_k}{\partial p_{k'}} + \bar{q}_{k'} \frac{\partial \bar{q}_k}{\partial \bar{C}} = \frac{1}{J} \sum_j s_{kk',j} - \frac{1}{J} \sum_j (q_{k'j} - \frac{C_j}{\bar{C}} \bar{q}_{k'}) \frac{\partial q_{kj}}{\partial C_j}$$

The last term of the third member is the representative consumer's Slutsky matrix' aggregation bias, to be called $\bar{b}_{kk'}$. Because of our assumption on individual behaviour the micro Slutsky coefficients $s_{kk',j}$ are all symmetric. Therefore, the macro Slutsky matrix is symmetric if and only if its bias matrix $[\bar{b}_{kk'}]$ is symmetric. By means of (2.6) and (3.1) this matrix becomes:

$$(3.3)\quad \bar{b}_{kk'} = \frac{1}{J} \sum_j (\varphi_{k'} \eta + \psi_{k'} C_j - \frac{C_j}{\bar{C}} \varphi_{k'} w - \frac{C_j}{\bar{C}} \psi_{k'} \bar{C}) \cdot (\varphi_k \frac{d\eta}{dC_j} + \psi_k)$$

$$= \frac{1}{J} \varphi_k \varphi_{k'} \sum_j (\eta - w \cdot \frac{C_j}{\bar{C}}) \frac{d\eta}{dC_j} = \bar{b}_{k'k}.$$

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6 The (mis)use of the definite article does not mean that we pretend that there is only one way to define the notion of the representative consumer.

The terms with $\psi_{k'}$ in the second member of (3.3) cancel out (and, therefore, $\bar{b}_{kk'} = \bar{b}_{k'k}$) because of our assumption of invariance of the income distribution. Hence, if the income distribution changes with changing incomes the representative consumer need not be a utility maximizer. This does not mean that $\bar{b}_{kk'} = \bar{b}_{k'k}$ implies utility maximization by the representative consumer. The bias matrix of our representative consumer might be such that his Slutsky matrix may become indefinite or even positive semi-definite.

Consequently, in general there is no utility maximizing representative consumer as defined above.

Instead of looking for more "pathological" representatives one can use (2.6) and (2.7) for estimating the parameters of φ_k and ψ_k, given η, yielding "unbiased" estimates of micro parameters using aggregate data. This will be the subject of the next sections for some specifications of η and of the φ_k and ψ_k.

4. Further specification of the demand equations

The shape of the demand equations (2.6) can be further specified by using our assumption of utility maximization. Because of the implied absence of money illusion we can apply Euler's theorem on homogeneous functions:

$$(4.1)\quad \sum_{k'} p_{k'}\left\{\frac{\partial\varphi_k}{\partial p_{k'}}\cdot\eta(C_j)+\frac{\partial\psi_k}{\partial p_{k'}}\cdot C_j\right\}+\varphi_k\eta'(C_j)C_j+\psi_k C_j = 0.$$

Assuming $C_j \neq 0$ this identity can be written as:

$$(4.2)\quad \frac{\eta(C_j)}{C_j}\sum_{k'} p_{k'}\frac{\partial\varphi_k}{\partial p_{k'}}+\eta'(C_j)\varphi_k = -\sum_{k'} p_{k'}\frac{\partial\psi_k}{\partial p_{k'}}-\psi_k.$$

Consequently, the left hand side of (4.2) has to be independent of C_j; this means that identically has to hold:

$$(4.3)\quad \frac{\partial}{\partial C_j}\left\{\frac{\eta(C_j)}{C_j}\sum_{k'} p_{k'}\frac{\partial\varphi_k}{\partial p_{k'}}+\eta'(C_j)\varphi_k\right\} = 0,$$

yielding:

$$(4.4)\quad \frac{\eta''(C_j)}{\frac{d}{dC_j}\left(\frac{\eta(C_j)}{C_j}\right)} = -\frac{\sum_{k'} p_{k'}\frac{\partial\varphi_k}{\partial p_{k'}}}{\varphi_k}$$

Since (4.4) has to hold identically for all conceivable values of prices and income both members of this identity have to be independent of income <u>and</u> prices and, therefore, are constant, say h; this implies that φ_k has to be homogeneous of degree $-h$. Furthermore, η is the solution of the linear first order differential equation:

(4.5) $\eta'(C_j) = h \cdot \frac{\eta(C_j)}{C_j} + B,$

where B is a constant of integration. Hence η is the general solution of $\eta'(C_j) = h\eta(C_j)/C_j$ plus a particular solution of (4.5), i.e.:

(4.6) $\eta(C_j) = D \cdot C_j^h + \frac{B}{1-h} \cdot C_j$

for $h \neq 1$, and

(4.7) $\eta(C_j) = D \cdot C_j + B \cdot C_j \log C_j,$

for h = 1, where D, just like B, is a constant of integration.

Absorbing B and D in φ_k and ψ_k we can say that the micro demand functions become:

(4.8) $q_{kj} = \varphi_k \cdot C_j^h + \psi_k C_j$

for $h \neq 1$, and

(4.9) $q_{kj} = \varphi_k C_j \log C_j + \psi_k C_j$

for h = 1.

The corresponding macro equations are:

(4.10) $q_k = \varphi_k \cdot \varkappa_o \bar{C}^h + \psi_k \bar{C},$

with

(4.11) $\varkappa_o = \frac{1}{J} \sum_j \left(\frac{C_j}{\bar{C}}\right)^h,$

and, respectively:

$$(4.12)\quad q_k = \varphi_k \bar{C} \log \bar{C} + (\varkappa_1 \varphi_k + \psi_k)\bar{C},$$

with

$$(4.13)\quad \varkappa_1 = \frac{1}{J} \sum_j \frac{C_j}{\bar{C}} \log \frac{C_j}{\bar{C}}.$$

The income distribution characteristics $\varkappa_0$ and $\varkappa_1$ are indispensable to retain consistency between micro and macro relations. For h = 2 (the case to be considered below) $\varkappa_0 = 1 + v^2$, where v is the coefficient of variation of the income distribution; $\varkappa_1$ resembles Theil's entropy measure (see Theil (1967)). Relations (4.9) are dealt with by Deaton and Muellbauer (1978). The case h = 2 has already been suggested by Pearce (1964). In this case Engel curves become parabolae. The flexibility of parabolae and their relative simplicity are in our opinion, enough justification for further examination of this case[7].

Therefore, we have to ask ourselves what conditions have to be imposed on the φ_k and ψ_k in order that these functions are consistent with utility maximization at the micro level. In Muellbauer (1975, p. 533) the following useful theorem has been proved: if for each k = 1, ..., K the demand functions are defined as in (4.8) and individuals maximize their utility (subject to a budget constraint, of course) then the individuals' cost of utility functions have the form[8]:

$$(4.14)\quad C_j = \left(a^{-\varepsilon} - \frac{\varepsilon}{|\varepsilon|} u_j b^{-\varepsilon}\right)^{-1/\varepsilon}$$

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[7] Of course we are aware of possible objections that can be raised against this specification; especially the logical requirement of non-negativity need not be fulfilled here; see Somermeyer (1967). Apparently it is impossible to impose conditions of consistent aggregation of the simple (additive) kinds as expressed by (2.2), (2.3) and (2.5), and of logical consistency at the same time.

[8] Incorrectly Muellbauer places a plus (+) instead of $-\varepsilon/|\varepsilon|$; see Vorst and Van Daal (1979).

for each $j = 1, \ldots, J$, where $\varepsilon = h-1$, a and b (both > 0) are linear homogeneous functions of prices such that the cost functions are concave in prices and where u_j is individual j's utility. For the case of (4.9) the (implicit) cost functions turn out to be:

$$(4.15)\quad \log C_j = \log a + u_j \log b,$$

where a is homogeneous of degree one in prices and b of degree zero.

Using Roy's theorem one can derive the following demand functions from (4.14) and (4.15):

$$(4.16)\quad q_{kj} = \frac{1}{a^{\varepsilon}}\left(\frac{1}{a}\cdot\frac{\partial a}{\partial p_k} - \frac{1}{b}\cdot\frac{\partial b}{\partial p_k}\right)C_j^{\varepsilon+1} + \frac{1}{b}\cdot\frac{\partial b}{\partial p_k}C_j,$$

and, respectively:

$$(4.17)\quad q_{kj} = \frac{1}{b\log b}\frac{\partial b}{\partial p_k}C_j\log C_j - \left(\frac{1}{a}\frac{\partial a}{\partial p_k} - \frac{1}{b}\frac{\log a}{\log b}\frac{\partial b}{\partial p_k}\right)C_j.$$

From (4.14) one can derive that, because $b > 0$, it always holds that $\partial C_j/\partial u_j > 0$. The derivatives of C_j with respect to any p_k are positive if $\partial a/\partial p_k$ and $\partial b/\partial p_k$ are positive and if $u_j > 0$ for $\varepsilon < 0$ and $u_j < 0$ for $\varepsilon > 0$; of course, these conditions are over-sufficient, but for practical purposes we do not need more sophisticated conditions (see the next section). For negative (positive) ε and positive (negative) u_j the cost function (4.14) is concave if a and b are concave; this, too, is over-sufficient for concavity. For $\varepsilon < 0$ and $u_j > 0$ C_j has to be greater than a; in that case a may be considered as a minimum subsistence budget.

The economically meaningful parts of the Engel curves corresponding to the system (4.17) are shaped as illustrated in figure 1. For $\varepsilon > 0$ the system (4.16) yields Engel curves that have the same shapes. For $\varepsilon < 0$ and $\neq -1$ the Engel

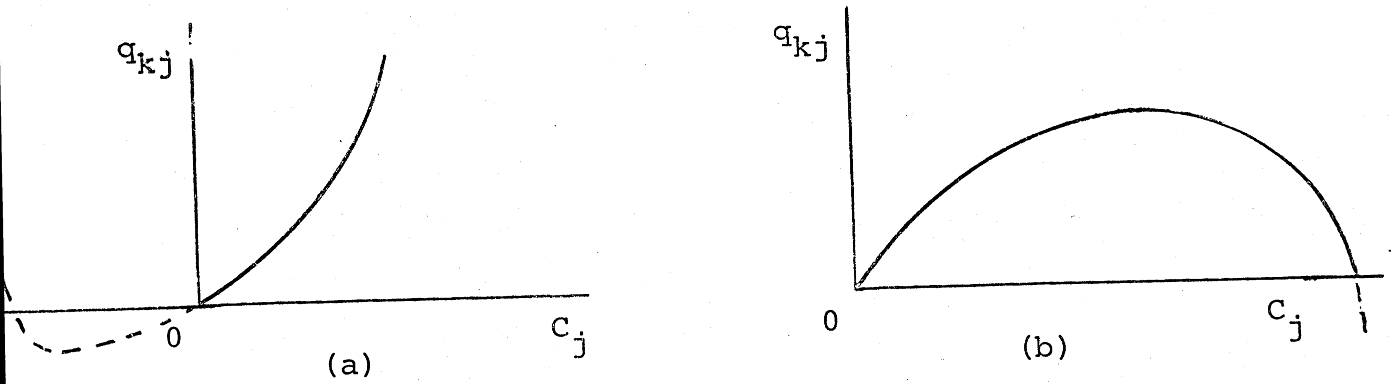

Figure 1. POSSIBLE SHAPES OF ENGEL CURVES CORRESPONDING WITH (4.16) FOR $\varepsilon > 0$ AND WITH (4.17).

curves resulting from (4.16) touch the vertical axis in the origin, a less realistic trait. Consequently, we only consider demand systems for which $\varepsilon > 0$.

Here we shall especially study the case $\varepsilon = 1$ $(h = 2)$, the quadratic Engel curve system. The individuals' demand equations are then:

$$(4.18)\quad q_{kj} = \frac{1}{a}\left(\frac{1}{a}\frac{\partial a}{\partial p_k} - \frac{1}{b}\frac{\partial b}{\partial p_k}\right)C_j^2 + \frac{1}{b}\frac{\partial b}{\partial p_k}C_j.$$

Simple specifications of a and b are:

$$(4.19)\quad a = \sum_{k'=1}^{k} \alpha_{k'} p_{k'},$$

and, respectively:

$$(4.20)\quad b = \prod_{k'=1}^{K} p_{k'}^{\beta_{k'}},$$

with $\sum_{k'} \beta_{k'} = 1$.

In that case the demand equations become:

$$(4.21)\quad p_k q_{kj} = \beta_k C_j + \frac{1}{\sum_{k'} \alpha_{k'} p_{k'}} \left(\frac{\alpha_k p_k}{\sum_{k'} \alpha_{k'} p_{k'}} - \beta_k \right) C_j^2,$$

for k = 1, ..., K. This system will be estimated on the basis of per capita data. Therefore, it has to be rewritten as:

$$(4.22)\quad p_k \bar{q}_k = \beta_k \bar{C} + \frac{1}{a}\left(\frac{\alpha_k p_k}{a} - \beta_k\right)\bar{C}^2 \cdot (1+v^2)$$

where the bars indicate per capita figures and where v is the coefficient of variation of the income distribution. Estimation results based on Dutch data with respect to $p_k \bar{q}_k$, p_k (k = 1, ..., K), $\bar{C}$ and v will be presented and commented below.

Note that (4.21) is a generalisation of the extremely simple elementary text-book model:

$$(4.23)\quad q_{kj} = \frac{\beta_k C_j}{p_k}$$

by adding a second term to the right-hand member of (4.23). In fact, all well-known models depart from (4.23). The addilog budget allocation model[9] is characterized by the dependence of the β_k on prices and income such that they "automatically" add up to 1. In the linear expenditure model[10] C_j has been replaced by "supernumerary income" i.e. income above subsistence level; the S-branch systems[11] combine this with endogenous β_k (i.e. depending on prices). Even the

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[9] See Somermeyer and Langhout (1972) and Houthakker (1960).

[10] See Stone (1954).

[11] See Brown and Heien (1972); see also Carlevaro (1975), who crosses (a.o.) the Addilog and the L.E.S.

Rotterdam model has a link with (4.23): assuming that the Rotterdam model describes utility maximizing behaviour it has to have the form (4.23)[12].

All these generalisations cannot be made haphazardly if one imposes consistency in some sense. Here, for instance, we saw that extending the naive model (4.23) with a quadratic term while maintaining the condition of utility maximization yields the model (4.18) with b specified according to (4.20).

An interesting feature of these models is that they give rise to realistic "true" price index functions. Let p_0 and p_1 be price levels in years 0 and 1, respectively, and let a_0, b_0, a_1 and b_1 be the values of a and b evaluated for prices p_0 and p_1, respectively. Because of (4.14) an individual with income C_0 in year 0 has the same utility in year 0 as well as in year 1 if he has an income C_1 in year 1 that, for the case $\varepsilon = 1$ (i.e.: $h = 2$) satisfies:

$$(4.24)\quad \frac{C_0^{-1} - a_0^{-1}}{b_0^{-1}} = \frac{C_1^{-1} - a_1^{-1}}{b_1^{-1}} .$$

From this we derive:

$$(4.25)\quad \frac{C_1}{C_0} = \frac{b_1}{b_0} \cdot \frac{1}{\left(1 - C_0 \cdot \frac{a_1 b_0 - a_0 b_1}{a_0 b_1 a_1}\right)} \approx \frac{b_1}{b_0} + \frac{a_1 b_0 - a_0 b_1}{a_0 b_0} \cdot \frac{b_1}{b_0} \cdot \frac{C_0}{a_1} .$$

See also Afriat (1977).

In the next section we shall exemplify the use of this formula.

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[12] See Theil (1975) and McFadden (1964, cited in Theil).

5. Some exploratory numerical exercises

For estimation of the model (4.22) we need, for a sufficiently large number of periods, the amounts spent per budget item and per capita, the prices of these items and the coefficients of variation of the income distributions. For the period 1948-1975 Dutch data on yearly consumption are presented in table 1; on the basis of this data estimation will be performed. The number of budget items is three: (1) food, beverages and tobacco, (2) durables, (3) other goods and services; the total amounts spent on these items and their prices (columns (1) through (6)) are taken from Keller (1977, tables 6 and 7). The numbers of private households (column (7)) are provided by the Netherlands Central Bureau of Statistics (unpublished series). We prefer dealing with amounts per household rather than "per head" because the household can be considered as the smallest decision unit with respect to spending its budget; all households have a budget, but not all persons have a budget. These numbers have to be considered with some caution: the figures of 1956, 1960 and 1971 are results of general censuses, those for other years are estimates; this may possibly explain the decrease from 1970 to 1971. Furthermore, people living in institutions, prisons, hospitals etc. do not belong to households; this gives rise to overestimation of income per household. The non-starred figures in column (8) are coefficients of variation of the distributions of income before taxation as presented by Hartog and Veenbergen (1978, table 3); the starred figures are "guestimated" coefficients of variation for which the authors are fully responsible. Of course, we are aware of the fact that the distribution of the households' budgets is better approximated by the distribution of income after taxation, but, unfortunately, this data is lacking for the time being.

The above mentioned shortcomings of the data and the very small number of budget items distinguished are reasons for us to consider the calculations presented below only as useful exercises that have to be followed by estimations on the basis of more disaggregated (with respect to goods and services) data of a better quality.

There is, however, still another reason to present our results with reserve: we do not take into account the distribution of size over the households. Here we simply assume that all households with the same income have the same spending pattern; further research on the effect of household composition on consumption and on how to build in this research into our model(s) is required; see Blokland (1975).

Table 1. The data

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1948	3.905	2.664	3.901	.441	.610	.351	2.319	1.76*
1949	4.441	3.192	3.990	.481	.643	.367	2.385	1.69*
1950	5.007	3.623	4.282	.528	.716	.390	2.390	1.71
1951	5.555	3.821	4.723	.581	.825	.430	2.431	1.66*
1952	5.842	3.428	4.896	.596	.752	.445	2.469	1.71
1953	6.137	3.611	5.142	.589	.735	.452	2.501	1.55
1954	6.678	4.173	5.653	.606	.753	.479	2.533	1.53
1955	6.947	4.745	6.259	.618	.759	.490	2.568	1.44
1956	7.558	5.398	6.819	.636	.743	.514	2.606	1.40
1957	7.989	5.511	7.327	.673	.759	.552	2.651	1.35
1958	8.182	5.374	7.739	.672	.763	.569	2.702	1.55
1959	8.588	5.705	8.244	.688	.762	.584	2.760	1.36
1960	8.930	6.520	9.077	.690	.767	.607	2.804	1.31
1961	9.635	7.156	9.565	.694	.772	.632	2.847	1.29*
1962	10.268	7.881	10.535	.716	.774	.653	2.895	1.28
1963	11.361	8.834	11.714	.737	.791	.690	2.945	1.35
1964	12.302	10.261	13.328	.798	.828	.739	2.990	1.32
1965	13.667	11.507	14.863	.828	.837	.780	3.050	1.58
1966	14.629	11.817	16.990	.879	.867	.826	3.117	1.23
1967	15.498	12.504	18.963	.901	.893	.863	3.187	1.48
1968	16.087	13.798	21.264	.915	.912	.888	3.242	1.45*
1969	17.386	15.562	25.149	.969	.950	.95	3.307	1.47*
1970	18.500	18.671	28.171	1.	1.	1.	3.377	1.49
1971	19.940	20.770	32.561	1.046	1.091	1.101	3.356	1.53*
1972	22.260	22.872	37.443	1.104	1.156	1.219	3.423	1.56
1973	24.710	25.611	43.340	1.193	1.247	1.35	3.477	1.54*
1974	27.011	28.660	49.380	1.259	1.371	1.511	3.523	1.52*
1975	30.482	31.803	56.971	1.362	1.494	1.714	3.572	1.50

(1): total amount spent on food, beverages and tobacco,

(2): total amount spent on durables,

(3): total amount spent on other services,
all measured nominally in billions of guilders.

(4), (5), (6): prices corresponding to (1), (2) and (3), respectively,

(7) numbers of households x 10^{-6},

(8) coefficients of variation of the distribution of income.

As a stochastic specification of the model (4.22) we adopted for $t = 1, \ldots, 28$ and $k = 1, 2, 3$:

$$(5.1) \quad \frac{(p_k\bar{q}_k)_t}{\bar{c}_t} = \beta_k + \frac{1}{\sum_{k'} \alpha_{k'} p_{k't}} \left(\frac{\alpha_k p_{kt}}{\sum_{k'} \alpha_{k'} p_{k't}} - \beta_k \right) \bar{c}_t (1+v_t^2) + u_{kt}.$$

About the disturbances u_{kt} we assume that they are normally distributed with zero expectation and with covariances:

$$(5.2) \quad E u_{kt} u_{k't'} = \delta_{tt'} \omega_{kk'},$$

where $\delta_{tt'} = 1$ for $t = t'$ and $\delta_{tt'} = 0$ for $t \neq t'$; further we assume that the disturbances are independent of the explanatory variables.

According to Barten (1969) we compute the likelihood function after deleting one budget category; this likelihood function is next concentrated on the parameters α_k and β_k (see, e.g., Bard (1974, p. 92)). Finally, this concentrated likelihood function is maximized as a function of the α_k and β_k by means of a sequential augmented Lagrangian method, where the minimization subproblems involved are solved by a quasi-Newton method (see Murray (1976)). The estimation results are presented in table 2.

Table 2. Estimation results (asymptotic standard errors between brackets)

	α	β
food etc.	17.736	.501
	(3.490)	(.015)
durables	31.829	.183
	(8.108)	(.015)
other	48.052	.316
	(12.211)	(.013)

Note that these figures are estimates of the micro demand model (4.21) not deformed by aggregation. Therefore, we can "unbiasedly" discuss the behaviour of individual households that satisfy our assumptions culminating in (4.21). The fairly high estimate of β_1 is an indication that all households consider food etc. as necessary, whereas the low β_2 estimate suggests that durables are luxuries; this is in accordance with the figures of table 3.

Table 3. Some income elasticities for households with income $\bar{C}$ and for households with income $2\bar{C}$

	1948		1957		1966		1975	
	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$
$E(q_1, C)$	.929*	.847	.916	.816	.889	.741	.825	.576
$E(q_2, C)$	1.126	1.223	1.122	1.218	1.121	1.218	1.130	1.230
$E(q_3, C)$	1.021*	1.041*	1.041*	1.078	1.080*	1.147	1.130	1.230

* not significantly different from 1 (level of confidence: 95 per cent)

As noted before, the cost of utility function, for cost less than a, is concave if the functions $a = \sum_k \alpha_k p_k$ and $b = \prod_k p_k^{\beta_k}$ are concave. These second order conditions are fulfilled for all households with budgets of - roughly - less then five times average income as can be concluded from table 4. The dimension of a is the same as that of C. Therefore, a is measured in thousands of guilders. In this table we present also some other features of the data and the estimates. It appears that, according to our model, the individual budget quotes differ considerably from aggregated budget quotes as presented in table 4. Individual shares for two classes of households can be found in table 5; they are computed by using the estimates of table 2 and relation (4.21).

Table 4. Budget shares, (quasi-)parameters in relation to average income for selected years

	1948	1957	1966	1975
$\bar{C}$	4.515	7.856	13.935	33.385
a	44.10	62.62	82.88	154.07
b	.435	.646	.860	1.490
$p_1\bar{q}_1/\bar{C}$	.373	.384	.337	.256
$p_2\bar{q}_2/\bar{C}$	.254	.264	.272	.267
$p_3\bar{q}_3/\bar{C}$	.373	.352	.391	.477
$\alpha_1 p_1/a$	.177	.191	.188	.157
$\alpha_2 p_2/a$	.440	.386	.333	.309
$\alpha_3 p_3/a$	.383	.423	.479	.534

Table 5. Budget shares for households with income $\bar{C}$ and with income $2\bar{C}$, respectively.

	1948		1957		1966		1975	
	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$
p_1q_{1j}/C_j	.468	.435	.462	.423	.448	.398	.426	.352
p_2q_{2j}/C_j	.209	.236	.209	.234	.208	.233	.210	.238
p_3q_{3j}/C_j	.323	.329	.329	.343	.344	.379	.364	.410

Macro income elasticities cannot be computed unless we know how v depends on $\bar{C}$. If we assume, very unrealistically, that from 1975 onwards v does not change, macro income elasticities (i.e. $(\partial\bar{q}_k/\partial\bar{C})/(\bar{q}_k/\bar{C})$ evaluated by means of (4.22) and using table 2) turn out to be of the order of magnitude of 5.

In table 6 we present some own price elasticities at the micro level.

Table 6. Price elasticities for households with income $\bar{C}$ and $2\bar{C}$, respectively.

	1948		1957		1966		1975	
	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$	$\bar{C}$	$2\bar{C}$
$E(q_1,p_1)$	-.899	-.783	-.914	-.812	-.910	-.796	-.930	-.830
$E(q_2,p_2)$	-.893	-.810	-.874	-.775	-.840	-.715	-.880	-.788
$E(q_3,p_3)$	-.801	-.610	-.863	-.737	-.898	-.811	-.954	-.919

For further completion of the picture we computed the 1975 matrices of compensated price elasticities for households with income $\bar{C}$ and $2\bar{C}$, respectively. These matrices turned out to be:

$$\begin{bmatrix} -.553 & .649 & .412 \\ & -.653 & .046 \\ & & -.511 \end{bmatrix} \text{ and } \begin{bmatrix} -.566 & .288 & .308 \\ & -.205 & .047 \\ & & -.357 \end{bmatrix}, \text{respectively.}$$

As could be expected our three categories are all substitutes of each other.

Finally, table 7 shows some "true" index functions of the cost of living. These indices are computed by means of formula (4.25) for the case that in 1976 food prices increase by 100 k per cent, the other prices remaining at their 1975 level.

Table 7. Index functions of the cost of living: $C_1/C_0 \approx A+B \cdot C_0$ (see (4.25)).

Increase of food prices in per cents	A	B
2	1.009	-4.419×10^{-6}
4	1.019	-8.835×10^{-5}
6	1.029	-1.324×10^{-4}
8	1.039	-1.764×10^{-4}
10	1.048	-2.204×10^{-4}

According to our model, say, 6 per cent increase in the 1975 price of food has to be met by an income compensation of 2.5 per cent for average income $\bar{C}$ in 1975 in order to maintain the level of utility; if income is $2\bar{C}$ this compensation decreases to 2.0 per cent.

In Appendix II we deal with some obvious alternative specifications of the model. The pertinent estimates turned out to be less satisfactory from an economic and/or econometric point of view; this was a reason for confining ourselves - in the main text of this paper - to the calculations based on (4.21) and (4.22).

6. Conclusion

In the theoretical part of this paper we showed that simultaneously requiring consistent aggregation of a simple additive nature and utility maximization leads to a surprisingly limited class of demand models that fulfill these requirements. These models have to be examined further with respect to their empirical usefulness. Deaton and Muellbauer (1978) were among the first to do this. They, however, did not use other moments of the income distributions than mere means; hence they had to assume that income distribution is stable during the period in consideration.

Our data, however, though still poor, makes clear that (Dutch) income distribution does change over time and that this variability has to be used in estimating.

If one confines himself to use first and second order moments of the income distribution the only model is the quadratic Engel curve model (see Appendix I). Here we examined only the most simple version of the Q.E.C. system. At first sight the results did not look too unrealistic as may be concluded from tables 2 through 7. This does not, however, remove the fact that the model has serious drawbacks.

As a theoretical drawback we mention that (4.21) leaves room for negative amounts spent if prices and/or incomes exceed particular values; furthermore for extreme values of C the second order conditions for utility maximization are violated. A practical point is that the model is fairly "rigid". If a good is, say, a "necessary" for one household it is so for any household irrespective of its income.

The advantages of Q.E.C. are already fully emphasized: it is "aggregation proof", consistent with utility maximizing and, therefore, yields "unbiased" estimates of micro parameters in the sense that these estimates are not subject to aggregation errors. Furthermore, it does not entirely preclude the Giffen paradox and, last but not least, it is easy to estimate.

As subjects of further research we mention:

- estimation of the model under the assumption of autocorrelation of the disturbances (see also Appendix II);
- estimation of the model for more disaggregated categories;
- estimation of the model for other specification of a and/or b; this can be done in two ways: first, one may replace b by, say, $(\sum_k \beta_k p_k^2)^{\frac{1}{2}}$ yielding:

$$(6.1) \quad p_k q_{kj} = \frac{\beta_k p_k^2}{\sum_{k'} \beta_{k'} p_{k'}^2} c_j + \frac{1}{\sum_{k'} \alpha_{k'} p_{k'}} \left(\frac{\alpha_k p_k}{\sum_{k'} \alpha_{k'} p_{k'}} - \frac{\beta_k p_k^2}{\sum_{k'} \beta_{k'} p_{k'}^2} \right) c_j^2$$

  (see also Appendix II);
  second, one may replace a and b by functions that are not homogeneous a priori as Deaton and Muellbauer did in their paper. In that case one can either impose homogeneity before estimating the parameters of the model, or one can test afterwards whether or not the homogeneity requirements are met.

Unlike many researchers we prefer the first procedure, because we do not try to refute the hypothesis of the utility maximizing consumer but, maintaining this hypothesis, try to find realistic consumer demand models.

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Appendix I An alternative derivation of the Q.E.C. model

Relations (2.1), (2.2) and (2.4) give rise to:

(A1) $g_k(p, \bar{C}, w) = \frac{1}{J} \sum_{j''} f_k(p, C_{j''})$.

Suppose that w is the variance σ^2 of the income distribution. Then, differentiation of both members of (A1) with respect to C_j and $C_{j'}$, for $j \neq j'$ results in the identity:

(A2) $\frac{1}{J^2} \cdot \frac{\partial^2 g}{\partial \bar{C}^2} + \frac{2}{J^2} \cdot \frac{\partial^2 g}{\partial \bar{C} \partial w} (C_j + C_{j'} - 2\bar{C}) - \frac{2}{J^2} \frac{\partial g}{\partial w} = 0,$

where we omitted the subscripts k. This yields:

(A3) $\frac{\partial^2 g}{\partial \bar{C}^2} - 2\frac{\partial g}{\partial w} = (C_j + C_{j'} - 2\bar{C}) \frac{\partial^2 g}{\partial \bar{C} \partial w}.$

This identity can only be fulfulled if:

(A4) $\frac{\partial^2 g}{\partial \bar{C} \partial w} = 0.$

Hence:

(A5) $\frac{\partial^2 g}{\partial \bar{C}^2} - 2\frac{\partial g}{\partial w} = 0.$

The system of differential equations (A4) and (A5) has the following <u>rational</u> solution:

(A6) $g = \varphi \cdot (\bar{C}^2 + w) + \chi \cdot \bar{C} + \psi,$

where φ, χ and ψ are constants of integration depending on prices only. This is the quadratic Engel curve system.

Appendix II. Some further empirical evidence

As an alternative stochastic specification we assumed that the disturbances of (5.1) are autocorrelated (normal) variables:

(A7) $u_t = Ru_{t-1} + \varepsilon_t$

where $u_t = (u_{1t}, \ldots, u_{Kt})'$ and where R is a $K \times K$ matrix of coefficients of autocorrelation, and where ε_t is a K dimensional vector of normal variates:

(A8) $\varepsilon_t \sim N(0, \Omega)$;

note that we assume R and Ω to be independent of t.

Concentration of the likelihood function of our sample (the data of table 1) on the parameters α and β results in the following concentrated likelihood function:

(A9) $L = C - \frac{1}{2}T \log \det A'A$,

where:

(A10) $A = U - U_{-1}U'U_{-1}(U_{-1}' U_{-1})^{-1}$

with U a $(T-1)\times(K-1)$ matrix of rows $u_t^{*\prime}$ $(t= 2, \ldots, T)$ resulting from u_t after deleting one budget category and U_{-1} a $(T-1)\times(K-1)$ matrix of rows $u_t^{*\prime}$ for $t = 1, \ldots, T-1$; C is a constant.

The estimates of the parameters of (4.21) according to this specification and based on the data of table 1 are presented in table A1.

<u>Table A1</u> Estimation results under the assumption of autocorrelated disturbances (third category deleted, standard errors between brackets).

	α	β
food, etc.	7.112 (17.801)	.166 (.234)
durables	2.046 (5.787)	.037 (.079)
other	37.500 (27.221)	.797 (-)

Although the value of the log-likelihood function increased by about 20 (highly significant for 4 degrees of freedom) we reject these results on economic grounds: the estimates of the β_k are unrealistic and those of the α_k are such that for a large part of the set of feasible incomes the second order conditions are violated; besides, the standard errors are unacceptably high. For completeness' sake we mention that the estimate of the matrix R in relation (A7) turned out to be nearly the identity matrix.

Furthermore, we estimated the model (6.1) with the stochastic specification of section 4. Estimation results can be found in table A2.

<u>Table A2</u> Estimation results of (6.1) (β_3 fixed at the value 1, standard errors between brackets)

	α	β
food, etc.	8.741 (.398)	2.005 (.128)
durables	4.717 (.183)	.844 (.077)
other	6.382 (.493)	1 (-)

Unfortunately, the estimates of the parameters α_k are such that for most incomes the second order conditions are violated.

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