Π-REGULAR VARIATION

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by

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ABSTRACT

A function $U : R^+ \to R^+$ is said to be Π-regularly varying with exponent α if there exists a positive function L such that

$$\frac{\dfrac{U(\lambda x)}{\lambda^\alpha} - U(x)}{x^\alpha L(x)} \to \log \lambda \qquad (x \to \infty) \text{ for } \lambda > 0$$

Suppose $\tilde{U}(s) := s \int_0^\infty \frac{e^{-xs}}{1-e^{-xs}} U(x)dx, \quad U_\rho^*(s) := \int_0^\infty \frac{dU(x)}{(s+x)^\rho}$

and $\widehat{U}(t) := \int_0^\infty e^{-tx} dU(x)$ exist for $s, t > 0$ and some $\rho > 0$.

Furthermore we define $\chi(x) = \sum_{1 \leq m \leq x} \frac{1}{m} U(\frac{x}{m})$. We prove that $U, \tilde{U}, U_\rho^*, \widehat{U}$ and χ are Π-regularly varying if one of them is Π-regularly varying, supposed some extra assumptions are satisfied.

1. Introduction

First we give the definition of regular variation.

Definition: A function U is said to be regularly varying with exponent ρ at infinity if it is real-valued, positive and measurable on $(0,\infty)$ and if for each $\lambda > 0$

$$\lim_{x\to\infty} \frac{U(\lambda x)}{U(x)} = \lambda^\rho \quad \text{where } \rho\varepsilon\ R, \text{ (notation } U\varepsilon RV_\beta^{(\infty)})$$

Remark. Regular variation of U at zero can be defined analogously. Regularly varying functions with exponent zero are called slowly varying. The theory of regularly varying functions has been developed by Karamata. For some basic facts see (1), (10), (11).

A recent treatment of regular variation is also given in Seneta's book (13). Karamata proved the following theorems on regular variation which are basic in this theory:

Theorem A: Suppose $U : R^+ \to R^+$ is Lebesgue summable on finite intervals.

(i) If U varies regularly at infinity with exponent $\beta > -1$

then

$$\lim_{x\to\infty} \frac{xU(x)}{\int_0^x U(t)\ dt} = \beta + 1$$

(ii) If $\lim_{x\to\infty} \dfrac{xU(x)}{\int_0^x U(t)\ dt} = \beta + 1$ with $\beta > -1$ then $U\varepsilon RV_\beta^{(\infty)}$

The second theorem concerns the Laplace-Stieltjes transform

$$\hat{U}(t) = \int_0^\infty e^{-ts}\ dU(s) \text{ of } U.$$

Theorem B: Suppose $U : R^+ \to R^+$ is non-decreasing, right-continuous $U(0+) = 0$, $\hat{U}(t)$ is finite for $t > 0$.

For $\beta \geq 0$ the following assertions are equivalent:

(i) $U \varepsilon RV_\beta^{(\infty)}$

(ii) $\hat{U} \varepsilon RV_{-\beta}^{(0)}$

Both imply

(iii) $\lim_{x\to\infty} \dfrac{U(x)}{\hat{U}(1/x)} = \dfrac{1}{\Gamma(\beta+1)}$

The converse statement that (iii) implies regular variation of U is well-known. See e.g. Drasin (3).

For non-decreasing functions U we can combine the theorems A and B using the notion of a fractional integral:

Definition: $_\alpha U(x) = \dfrac{1}{\Gamma(\alpha+1)} \int\limits_0^x (x-t)^\alpha \, dU(t)$ where $\alpha > 0$.

Theorem C: Suppose $U : R^+ \to R^+$ is non-decreasing and right-continuous, $U(0+) = 0$ and $\hat{U}(t)$ is finite for $t > 0$. For $\alpha > 0$ and $\beta \geq 0$ the following assertions are equivalent:

(i) $U \in RV_\beta^{(\infty)}$

(ii) $_\alpha U \in RV_{\alpha+\beta}^{(\infty)}$

(iii) $\dfrac{_\alpha U(x)}{x^\alpha U(x)} \to \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}$ $\qquad (x \to \infty)$

(iv) $\hat{U} \in RV_{-\beta}^{(0)}$

(v) $\dfrac{U(x)}{\hat{U}(1/x)} \to \dfrac{1}{\Gamma(\beta+1)}$ $\quad (x \to \infty)$

Remark that the case $\alpha = 1$ yields theorem A with $\beta \geq 0$. For arbitrary $\alpha > 0$ theorem C can be proved by using theorems A and B and the relation

$$_\alpha \hat{U}(1/x) = x^\alpha \, \hat{U}(1/x)$$

since $_\alpha U(x)$ is non-decreasing.

It only remains to prove that (iii) implies (i).

This can be done using a result of Drasin and Shea.

We write

$$\Gamma(\alpha) \; \frac{\alpha \, U(x)}{x^\alpha} \; = \int_0^x U(t) \; \left(1 - \frac{t}{x}\right)^{\alpha-1} \; \frac{t}{x} \; \frac{dt}{t} \; \sim \; \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \; U(x) \qquad (x \to \infty)$$

Or $\displaystyle \int_0^\infty U(t) \; k\!\left(\frac{x}{t}\right) \frac{dt}{t} \sim c \, U(x)$ where the Kernel k is positive.

Application of theorem 6.2 of (4) then yields (i).

In 1963 Bojanic and Karamata (2) studied the class of functions U for which

$\displaystyle \lim_{x \to \infty} \frac{U(\lambda x) - U(x)}{x^\sigma L(x)}$ exists for some function L(x) and showed that σ can be

chosen such that L(x) is slowly varying.

In this paper we shall see that the theorems A and B can be sharpened for functions U which satisfy the relation

$$\lim_{x \to \infty} \frac{\dfrac{U(\lambda x)}{\lambda^\sigma} - U(x)}{x^\sigma \, L(x)} = H(\lambda) \text{ for some function } L(x)$$

For $\lambda=0$ this relation is also studied by de Haan who gives a refinement of theorem B.

We need the following definitions and results which can also be found in (7) ,

<u>Definition</u> A function $U : R^+ \to R^+$ belongs to the class Π if U is measurable and if there exists an auxiliary function L(x) which is strictly positive such that for all $\lambda > 0$

$$\lim_{x \to \infty} \frac{U(\lambda x) - U(x)}{L(x)} = \log \lambda$$

The class Π is a subclass of the slowly varying functions.

Note that the auxiliary function L(x) is determined up to asymptotic equality: a positive function L(x) is an auxiliary function for $U(x) \in \Pi$ if and only if $L(x) \sim U(xe) - U(x)$ for $x \to \infty$. Let L(x) be slowly varying and in $L^1(dt/t)$ on finite intervals.

Then $U(x) = \displaystyle \int_0^x \frac{L(t)}{t} \, dt$ is an element of Π with auxiliary function L(x).

This follows since $\dfrac{L(tx)/tx}{L(x)/x} \to \dfrac{1}{t}$ for $x \to \infty$ uniformly on compact subsets of (o,∞)

Hence for $\lambda > 0$ $\dfrac{U(\lambda x) - U(x)}{L(x)} = \displaystyle \int_x^{\lambda x} \frac{L(t)}{L(x)} \frac{dt}{t} = \int_1^\lambda \frac{L(tx)/tx}{L(x)/x} \, dt \to \int_1^\lambda \frac{dt}{t} = \log \lambda$

It can be shown that each function $U \in \Pi$ for which $\int_0^a U(t)/t \, dt$ exists for all $a > 0$ can be written as $U(x) = L(x)+ \int_0^x \frac{L(t)}{t} \, dt$ where L is slowly varying.

Here also $L(x) \sim \frac{1}{x} \int_0^x s \, dU(s) \sim U(Xe) - U(x)$ $(x \to \infty)$ is the auxiliary function. From the definition of Π we can see that if $U(x) \in \Pi$ with auxiliary function $L(x)$, $U_1(x)$ is measurable and $\frac{U(x) - U_1(x)}{L(x)} \to c$ $(x \to \infty)$ where $c \in R$ is a

constant, then $U_1(x) \in \Pi$ with auxiliary function $L(x)$.

For an extensive treatment of the class Π the reader is referred to (7).

Combining the results of de Haan (8) and Embrechts (5) we get

Theorem D : Suppose $U : R^+ \to R^+$ is non-decreasing, $U(0+) = 0$ and $\hat{U}(s)$ is finite for $s > 0$

Then the following statements are equivalent:

(i) $U(x) \in \Pi$

(ii) $\hat{U}(1/x) \in \Pi$

(iii) $\dfrac{U(x) - \hat{U}(1/x)}{\frac{1}{x} \int_0^x s \, dU(s)} \to \gamma$ $(x \to \infty)$

We give a second order version of Karamata's theorems A and B for non-decreasing functions U. A necessary and sufficient condition for a function to obey the second order relation is formulated in the following definition.

Definition $U \in \Pi RV_\alpha$ iff $\frac{U(x)}{x^\alpha} \in \Pi$ where $\alpha \in R$

If $U \in \Pi RV_\alpha$ then we say that L is the auxiliary function of U if L is the auxiliary function of $\frac{U(x)}{x} \in \Pi$.

We call the function U Π-regularly varying with exponent α. The Π-varying functions with exponent α form a subclass of $RV_\alpha^{(\infty)}$.

2. Results

Our main result is the following theorem

Theorem 1. Suppose $\alpha > 0$, $\beta \geqq 0$, $U : R^+ \to R^+$, $U(x)/x^\beta$ non-decreasing

$$\lim_{x \downarrow 0} \frac{U(x)}{x^\beta} = 0, \text{ and } \hat{U}(t) \text{ exists for } t > 0$$

Then the following statements are equivalent:

(i) $\quad U \epsilon \Pi RV_\beta$

(ii) $\quad _\alpha U \epsilon \Pi RV_{\alpha+\beta}$

(iii) $\quad \hat{U}(1/x) \, \epsilon \Pi RV_\beta$

They imply

(iv) $\quad \dfrac{\dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} U(x) - \dfrac{_\alpha U(x)}{x^\alpha}}{x^{\beta-1} \int_0^x sd\, U(s)/s^\beta} \to -\dfrac{\partial}{\partial\beta} \left(\dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \right) \quad (x \to \infty)$

and

(v) $\quad \dfrac{U(x) - \dfrac{1}{\Gamma(\beta+1)} \hat{U}(1/x)}{x^{\beta-1} \int_0^x sd\, \dfrac{U(s)}{s^\beta}} \to -\psi(\beta+1) \quad (x \to \infty)$

where $\psi(x) = \dfrac{d}{dx} \log \Gamma(x)$

Conversely if (iv) with $\alpha\epsilon(0,1]$, $\beta \geq 0$ then (i)
and if (v) with $\beta \geqq 1$ then (i)

<u>Proof</u>

<u>(i) → (iv) and (i) → (ii)</u>

We write $U(x) = x^\beta \left(L(x) + \int_0^x \dfrac{L(t)}{t} dt \right)$ with $L(x) = \dfrac{1}{x} \int_0^x sd\dfrac{U(s)}{s^\beta} \epsilon RV_0^{(\infty)}$

Then

$$\frac{\dfrac{_\alpha U(x)}{x^\alpha} - \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} U(x)}{x^\beta L(x)} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^\beta \frac{\dfrac{U(tx)}{t^\beta x^\beta} - \dfrac{U(x)}{x^\beta}}{L(x)} dt \to$$

$$\to \frac{1}{\Gamma(\alpha)} \int_0^1 \log t \, (1-t)^{\alpha-1} t^\beta dt \quad (x \to \infty)$$

The last step is justified since by substituting the expression for U(x) we find

$$\frac{1}{\Gamma(\alpha)} \left[\int_0^1 (1-t)^{\alpha-1} t^\beta \left\{ \frac{L(tx)}{L(x)} - 1 \right\} dt - \int_0^1 (1-t)^{\alpha-1} t^\beta \int_t^1 \frac{L(sx)}{L(x)} \frac{ds}{s} dt \right]$$

and $\dfrac{s^\epsilon x^\epsilon L(sx)}{x^\epsilon L(x)} \to s^\epsilon$ $(x \to \infty)$ uniformly on $(0,1)$ where $\epsilon > 0$

(see de Haan (7)).

Now (iv) and (i) imply (ii) as mentioned in the introduction.

$\underline{(i) \to (v) \text{ and } (i) \to (iii)}$

We write $U(x) = x^\beta L(x) + K(x)$ where $K(x) = x^\beta \int_0^x \dfrac{L(t)}{t} dt$.

By Karamata's theorem we have

$$x^\beta L(x) - \frac{1}{\Gamma(\beta+1)} \int_0^\infty e^{-t/x} dt^\beta L(t) = o(x^\beta L(x)) \quad (x \to \infty).$$

Substituting the expression for K(x) we find

$$\frac{K(x) - \hat{K}(1/x)/\Gamma(\beta+1)}{x^\beta L(x)} = \int_0^1 \frac{L(tx)}{L(x)} \frac{dt}{t} - \frac{1}{\Gamma(\beta+1)} \int_0^\infty e^{-t} t^\beta \int_0^t \frac{L(ux)}{L(x)} \frac{du}{u} dt = (*)$$

Now we write $\dfrac{1}{\Gamma(\beta+1)} \int_t^\infty e^{-u} u^\beta du = G_\beta(t)$

Then $(*) = \int_0^1 \dfrac{L(tx)}{L(x)} \dfrac{dt}{t} + \int_0^\infty \int_0^t \dfrac{L(ux)}{L(x)} \dfrac{du}{u} dG_\beta(t) =$

$$= \int_0^1 \left(1 - G_\beta(t)\right) \frac{L(tx)}{L(x)} \frac{dt}{t} - \int_1^\infty G_\beta(t) \frac{L(tx)}{L(x)} \frac{dt}{t}$$

Since $\dfrac{t^\epsilon x^\epsilon L(tx)}{x^\epsilon L(x)} \to t^\epsilon$ uniformly on $(0,1)$ and

$$\frac{t^{-\epsilon} x^{-\epsilon} L(tx)}{x^{-\epsilon} L(x)} \to t^{-\epsilon} \text{ uniformly on } (1,\infty) \quad \text{(see (7) cor 1.2.1.4)}$$

we find $(*) \to \int_0^1 \{1 - G_\beta(t)\} \dfrac{dt}{t} - \int_1^\infty G_\beta(t) \dfrac{dt}{t} \quad (x \to \infty)$

By partial integration the last expression equals

$$-\int_0^\infty \frac{\log t}{\Gamma(\beta+1)} e^{-t} t^\beta dt = -\psi(\beta+1)$$

Now we have analogously that (v) and (i) imply (iii)

<u>(ii) → (iii)</u> follows immediately since $_\alpha\hat{U}(1/x) = x^\alpha \hat{U}(1/x)$

and we can use (i) → (iii)

<u>(iv) → (i)</u> We define $L(x) = \frac{1}{x}\int_0^x sd\frac{U(s)}{s^\beta}$ as in the proof of (i) → (v)

Now we can reformulate (v) as follows:

$$\frac{1}{\Gamma(\alpha)}\int_0^x \left(1 - \frac{t}{x}\right)^{\alpha-1} \left(\frac{t}{x}\right)^{\beta+1} L(t) \frac{dt}{t} + \frac{1}{\Gamma(\alpha)}\int_0^x \int_{t/x}^1 (1-u)^{\alpha-1} u^\beta du \, L(t) \frac{dt}{t}$$

$$- \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\int_0^x L(t) \frac{dt}{t} \sim \xi L(x) \qquad (x \to \infty)$$

where $\xi = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} + \frac{d}{d\beta}\left\{\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\right\}$

Or $\int_0^\infty L(t) k\left(\frac{x}{t}\right) \frac{dt}{t} \sim \xi L(x)$ $(x \to \infty)$ where the kernel k is defined by

$$k\left(\frac{1}{x}\right) = \frac{x}{\Gamma(\alpha)}\left\{(1-x)^{\alpha-1} x^\beta - \frac{1}{x}\int_0^x (1-u)^{\alpha-1} u^\beta du\right\} \text{ for } x < 1 \text{ and } 0 \text{ for } x \geq 1$$

For $\alpha\epsilon(0,1]$ and $\beta \geq 0$ the kernel is non-negative since $(1-x)^{\alpha-1} x^\beta$ is

increasing on $(0,1)$

Moreover we have $\lim_{\substack{x\to\infty \\ t\to 1+}} \inf \frac{L(tx)}{L(x)} \geq 1$ since $xL(x)$ is non-decreasing.

Application of theorem 6.2 in (4) then gives the result since

$\hat{k}(\rho) = \int_0^1 k\left(\frac{1}{t}\right) t^{\rho-1} dt$ is decreasing for $\rho > -\beta-1$ and so $\hat{k}(\rho) = \xi$ only if $\rho = 0$.

<u>(v) → (i)</u> We define $L(x)$ as in the proof of (iv) → (i)

Here we can reformulate (v) as follows:

$$\int_0^\infty k\left(\frac{x}{t}\right) L(t) \frac{dt}{t} \sim \xi L(x) \quad (x \to \infty)$$

where $\xi = 1 + \psi(\beta+1)$ and the kernel k is given by

$$k\left(\frac{1}{x}\right) = \frac{1}{\Gamma(\beta+1)} x^\beta e^{-x} - \frac{1}{x} \int_0^x u^\beta e^{-u} du + 1 - I_{(0,1)}(x)$$

If $\beta \geq 1$ this kernel is positive for all $x > 0$ since the term $x^\beta e^{-x}$ is increasing on $(0,\beta)$

Here we can also apply theorem 6.2 in (4).

<u>(i) $\leftrightarrow$ (iii)</u> Writing $V(x) = \frac{U(x)}{x^\beta}$ we have by proposition P4 in (9)

$$V \epsilon \Pi \text{ iff } \int_0^x t^\beta dV(t) \; \epsilon RV_\beta^{(\infty)} \text{ where } \beta > 0$$

Or : $U \epsilon \Pi RV_\beta$ iff $U(x) - \int_0^x \frac{U(t)}{t} dt \; \epsilon RV_\beta^{(\infty)}$

This is equivalent to

$$\hat{U}(1/x) - \beta x \hat{K}(1/x) \; \epsilon RV_\beta^{(\infty)} \text{ where } K(x) = \frac{U(x)}{x}$$

The last statement is equivalent to $\hat{U}(1/x) \; \epsilon \Pi RV_\beta$, since

$$x\hat{K}(1/x) = \int_0^x \frac{\hat{U}(1/t)}{t} dt \text{ by partial integration}$$

The case $\beta = 0$ is the result of de Haan (8).

Remarks

(a) (iv) and (v) imply (iii) and (iv) in theorem C since

$$\frac{U(x)}{x^{\beta-1} \int\limits_0^x sdU(s)/s^\rho} \to \infty \quad (x \to \infty) \quad \text{if} \quad \frac{U(s)}{s^\rho} \text{ is slowly varying}$$

(b) The proofs of (i) → (ii), (i) → (iv) remain valid for $\beta > -1$

That (ii) implies (i) for $-1 \le \beta < 0$ can be shown as follows:

$$\int\limits_x^\infty t^\beta d \frac{U(t)}{t^\beta} = -U(x) -\beta \int\limits_x^\infty \frac{U(t)}{t} dt \text{ is } \beta\text{-varying at infinity}$$

iff $U \epsilon \Pi RV_\beta$ by proposition P4 in (9)

Integrating from 1 to x gives

$$G(x) := -\beta \int\limits_1^x \int\limits_t^\infty \frac{U(s)}{s} ds \, dt - \int\limits_1^x U(t) dt \; \epsilon \; RV_{1+\beta}^{(\infty)}$$

iff $\hat{G}(1/x) = -\beta \, x \, \hat{K}(1/x) - x\hat{U}(1/x) \; \epsilon \; RV_{1+\beta}^{(\infty)}$

where $K(x) = \int\limits_x^\infty \frac{U(s)}{s} ds$. This proves the statement since

$$\hat{K}(1/x) = \int\limits_x^\infty \frac{\hat{U}(1/t)}{t} dt.$$

For $\beta < -1$ a similar method can be applied.

(c) For the Abelian theorems (i) → (ii) and (i) → (iii) it is not necessary that $U(x)/x^\beta$ is non-decreasing, the Tauberian part however cannot be derived without extra assumption on U.

(d) An alternative proof for the equivalence of (ii) and (iii) can be given as follows. We have $_\alpha U \epsilon \Pi RV_{\alpha+\beta}$ iff for $a > 1$ $\phi \epsilon RV_0^{(\infty)}$ where

$$\phi(x) = \frac{1}{x^{\alpha+\beta}} \int\limits_0^x (x-s)^{\alpha-1} \left\{ \frac{U(sa)}{a^\beta} - U(s) \right\} ds$$

by the definition of $_\alpha U$ and theorem 1.4.1 in (7).

Now $\phi \epsilon RV_0^{(\infty)}$ iff $x^{\alpha+\beta} \phi(x) \; \epsilon RV_{\alpha+\beta}^{(\infty)}$ iff $x^\alpha \hat{H}(\frac{1}{x}) \; \epsilon RV_{\alpha+\beta}^{(\infty)}$

where $H(x) = \frac{U(ax)}{a^\beta} - U(x)$ by theorem B.

This is equivalent to $x^{-\beta} \hat{H}(\frac{1}{x}) = \frac{1}{(ax)^{\beta}} \hat{U}(\frac{1}{ax}) - \frac{1}{x^{\beta}} \hat{U}(\frac{1}{x})$

$\epsilon RV_0^{(\infty)}$ and so to $\frac{1}{x^{\beta}} \hat{U}(\frac{1}{x}) \epsilon \Pi$.

Remark that $x^{-\beta} \hat{U}(\frac{1}{x})$ is non-decreasing, since $\frac{U(x)}{x^{\beta}}$ is non-decreasing.

(e) The existence of $\hat{U}$ is implied by the regular variation of U for non-decreasing U.

Indeed for β-varying U we have

$$\hat{U}(1/x) = \frac{1}{x} \int_0^{\infty} e^{-t/x} U(t)\, dt = \frac{1}{x} \int_0^{x} e^{-\frac{t}{x}} U(t)\, dt + \frac{1}{x} \sum_{n=1}^{\infty} \int_{2^{n-1}x}^{2^n x} e^{-\frac{t}{x}} U(t)\, dt$$

$$\leq U(x) + \sum_{n=1}^{\infty} U(2^n x)\, e^{-2n-1} \leq U(x) + U(x) \sum_{n=1}^{\infty} 2^{n(\beta+\epsilon)}\, e^{-n-1} < \infty$$

(f) It is evident that (i) implies regular variation of

$$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\ U(x) - \frac{\alpha U(x)}{x^{\alpha}}\ \text{with exponent } \beta.\ \text{It is an open question whether}$$

the converse statement holds true.

Example

$\log \Gamma(x+1)\ \epsilon \Pi RV_1$ with auxiliary function 1. This can be derived from

$$\psi(x) - \log x \to 0 \quad (x \to \infty)\ \text{and}\ \log \Gamma(x+1) = \int_0^x \psi(t+1)\, dt$$

3. Applications

Our first application concerns the Stieltjes-transform. Suppose U is non-decreasing and right-continuous. We define

$$U_\rho^*(x) = \int_0^\infty \frac{dU(t)}{(t+x)^\rho} \quad \text{for } \rho > 0, \text{ supposed the integral exists.}$$

In 1931 Karamata proved the following theorem about U_ρ^* (see [11]).

Theorem 2 For $\rho \geq \sigma > 0$ $U_\rho^*(x) \sim x^{-\sigma}L(x)$ $(x \to \infty)$ where $L(x)$ is slowly varying, U non-decreasing

implies $U(x) \sim \dfrac{\Gamma(\rho)}{\Gamma(\sigma)\ \Gamma(\rho-\sigma+1)}\ x^{\rho-\sigma}\ L(x)$ $(x \to \infty)$

It is obvious that the converse statement $U \epsilon RV_{\rho-\sigma} \Rightarrow U_\rho^* \epsilon RV_{-\sigma}$ also holds. To extend this theorem for Π-regularly varying functions we need a lemma.

Lemma 1 (a) Suppose $U_1 \epsilon RV_\alpha$ is increasing and continuous $(\alpha > 0)$, U is non-decreasing. Then $U \epsilon \Pi$ iff $\dfrac{1}{U_1(x)} \int_0^x U(t)dU_1(t) \epsilon \Pi$. In this case

$U(x) - \dfrac{1}{U_1(x)} \int_0^x U(t)dU_1(t) \sim \dfrac{1}{\alpha} L(x)$ $(x \to \infty)$ where L is the auxiliary function of U.

(b) Suppose $U_1 \epsilon RV_{-\alpha}$ is decreasing and continuous $(\alpha > 0)$, U is non-decreasing. Then $\dfrac{-1}{U_1(x)} \int_x^\infty U(t)dU_1(t) \epsilon \Pi$ iff $U \epsilon \Pi$. In this case

$\dfrac{-1}{U_1(x)} \int_x^\infty U(t)dU_1(t) - U(x) \sim \dfrac{1}{\alpha} L(x)$ where L is the auxiliary function of U.

Proof We only prove the first part; the second part is proved similarly. Now

$$U(x) - \frac{1}{U_1(x)} \int_0^x U(t)dU_1(t) = U(x) - \frac{1}{U_1(x)} \int_0^{U_1(x)} U(U_1^{-1}(s))ds =$$

$$= M(U_1(x)) \text{ where } M(x) = U(U_1^{-1}(x)) - \frac{1}{x}\int_0^x U(U_1^{-1}(s))ds$$

From $U_1 \epsilon RV_\alpha$ and the definition of the class Π we see that $U \epsilon \Pi$ with auxiliary function L iff $U\ U_1^{-1} \epsilon \Pi$ with auxiliary function M. The functions L and M satisfy the relation $L(x) \sim \alpha M(U_1(x))$. This finishes the proof of part a.

Theorem 3 Suppose $\rho \geq \sigma > 0$, $\dfrac{U(x)}{x^{\rho-\sigma}}$ non-decreasing on R^+, $\lim\limits_{x\downarrow 0} \dfrac{U(x)}{x^{\rho-\sigma}} = 0$.

Then the following statements are equivalent

(i) $U \varepsilon \Pi RV_{\rho-\sigma}$

(ii) $U_\rho^* \varepsilon \Pi RV_{-\sigma}$

Both imply

$$\frac{U(x) - \dfrac{\Gamma(\rho)}{\Gamma(\sigma)\,\Gamma(\rho-\sigma+1)}\, x^\rho U_\rho^*(x)}{x^{\rho-\sigma-1} \displaystyle\int_0^x sd\,\frac{U(s)}{s^{\rho-\sigma}}} \to \psi(\sigma) - \psi(\rho-\sigma+1) \quad (x \to \infty)$$

Proof Since for $\rho > 0$ $(t+x)^{-\rho} = \dfrac{1}{\Gamma(\rho)} \displaystyle\int_0^\infty e^{-t\tau} e^{-x\tau} \tau^{\rho-1}\, d\tau$

it follows that $U_\rho^*(x) = \displaystyle\int_0^\infty g(\tau) e^{-x\tau}\, d\tau$ where

$$g(\tau) = \frac{\tau^{\rho-1}}{\Gamma(\rho)} \int_0^\infty e^{-t\tau}\, dU(t) \qquad (\dot{x})$$

By theorem 1 $U \varepsilon \Pi RV_{\rho-\sigma}$ is equivalent to $\hat{U}(1/x) \varepsilon \Pi RV_{\rho-\sigma}$

and since $g(\frac{1}{x}) = \dfrac{1}{\Gamma(\rho)} x^{1-\rho} \hat{U}(1/x)$ we have $g(\frac{1}{x}) \varepsilon \Pi RV_{1-\sigma}$

Writing $H(x): = x^{\sigma-1} g(\frac{1}{x})$ we have by lemma 1 (b)

$H \varepsilon \Pi$ iff $x^\sigma \displaystyle\int_x^\infty \frac{H(u)}{u^{\sigma+1}}\, du = x^\sigma \int_0^{1/x} g(t)dt \varepsilon \Pi$ iff $\displaystyle\int_0^{1/x} g(t)dt \varepsilon \Pi RV_{-\sigma}$

if we can show that $H(x) = \dfrac{1}{\Gamma(\rho)} x^{-\rho+\sigma} \hat{U}(\frac{1}{x})$ is non-decreasing and this is so since $U(x)/x^{\rho-\sigma}$ is non-decreasing

Thus $g(1/x) \varepsilon \Pi RV_{1-\sigma}$ iff (i)

Equivalently $U_\rho^*(x) = \hat{g}(x)/x \varepsilon \Pi RV_{-\sigma}$ by interchanging the roles of zero and infinity in (i) $\to$ (iii) of theorem 1. This proves the first part of the theorem. By (*) and theorem 1 we have

$$\frac{U(x) - \dfrac{\Gamma(\rho)}{\Gamma(\rho-\sigma+1)}\, x^{\rho-1}\, g(1/x)}{x^{\rho-\sigma-1} \displaystyle\int_0^x sdU(s)/s^{\rho-\sigma}} \to -\psi(\rho-\sigma+1) \quad (x \to \infty).$$

From the last expression we can see that $g(\frac{1}{x}) \in \Pi RV_{1-\sigma}$

with auxiliary function $\dfrac{\Gamma(\rho-\sigma+1)}{\Gamma(\rho)} \; \dfrac{1}{x} \int_0^x sd \, \dfrac{U(s)}{s^{\rho-\sigma}}$

This implies that

$$\frac{g(\frac{1}{x}) - \dfrac{1}{\Gamma(\sigma)} \, \hat{g}(x)}{\dfrac{\Gamma(\rho-\sigma+1)}{\Gamma(\rho)} \; x^{-\sigma} \int_0^x sd \, \dfrac{U(s)}{s^{\rho-\sigma}}} \;\to\; \psi(\sigma) \qquad\qquad (x \to \infty)$$

Combining this with the last expression gives the desired result.

Remark   The existence of U_ρ^{*} is implied by the regular variation of U. To prove this we partition the domain of integration by the points $x, 3x, 7x, \ldots, (2^n-1)x, \ldots$

If U is $\rho-\sigma$ varying then $\dfrac{U(2x)}{U(x)} \leqq 2^{\rho-\sigma/2}$ for $x \geq x_0$

Repeated application of this inequality yields

$$U_\rho^{*}(x) = \rho \int_0^\infty \frac{U(t)}{(t+x)^{\rho+1}} \, dt \leqq c \, U(x) . x^{-\rho} \sum_{n=0}^\infty 2^{-\frac{n\sigma}{2}} < \infty.$$

The second application concerns the Lambert-transform
Suppose $n: \; R^+ \to R^+$ and $n \in L'(dt/t)$ on $(0,\infty)$

We define $\tilde{n}(s) = s \int_0^\infty \frac{e^{-us}}{1-e^{-us}} \, n(u) du$ for $s > 0$

An Abel-Tauber theorem on regular variation of n, $\tilde{n}$ is given by Kohlbecker (12), a similar result for the class Π is derived in (6).

We start with an Abelian result.

Theorem   Suppose $n \in \Pi RV_\beta$ $(\beta > 0)$ and $\dfrac{n(t)}{t}$ integrable on finite intervals of $(0,\infty)$

Then $\tilde{n}(1/x) \in \Pi RV_\beta$
Moreover

$$\frac{c_1 \, n(x) - \tilde{n}(1/x)}{x^\beta L(x)} \;\to\; c_2 \qquad (x \to \infty)$$

where $c_1 = \zeta(\beta+1) \, \Gamma(\beta+1)$

$$c_2 = - \frac{d}{d\beta} (\zeta(\beta+1) \, \Gamma(\beta+1))$$

and L is the auxiliary function of n.

<u>Proof</u> For $\beta > 0$ we define $c_\beta = \zeta(\beta+1)\,\Gamma(\beta+1) - \frac{1}{\beta} =$

$$= -\int_0^1 \left(\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}}\right) u^\beta\, du + \int_1^\infty \frac{u^\beta e^{-u}}{1-e^{-u}}\, du$$

Substituting this we can write

$$\frac{c_\beta n(x) + \int_0^x \frac{n(t)}{t}\, dt - \tilde{n}(1/x)}{x^\beta L(x)} =$$

$$= \int_0^1 u^\beta \left(\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}}\right) \frac{\frac{n(xu)}{x^\beta u^\beta} - \frac{n(x)}{x^\beta}}{L(x)}\, du - \int_1^\infty \frac{u^\beta e^{-u}}{1-e^{-u}} \frac{\frac{n(xu)}{x^\beta u^\beta} - \frac{n(x)}{x^\beta}}{L(x)}\, du$$

For $x \to \infty$ the last expression tends to

$$\int_0^1 u^\beta \left(\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}}\right) \log u\, du - \int_1^\infty \frac{u^\beta e^{-u}}{1-e^{-u}} \log u\, du = -\frac{d}{d\beta} c_\beta$$

as in the proof of the main theorem

The Abelian side of the main theorem gives now

$$\frac{\frac{1}{\beta} n(x) - \int_0^x \frac{n(t)}{t}\, dt}{x^\beta L(x)} \to -\frac{1}{\beta^2} \quad (x \to \infty)$$

Combination of the last results yields the theorem.

<u>Remark</u> This theorem is a refinement of Kohlbecker's result

$$\zeta(\beta+1)\,\Gamma(\beta+1)\, n(x) \sim \tilde{n}(1/x) \quad (x \to \infty) \text{ for } \beta\text{-varying } n.$$

<u>Theorem</u> Suppose $n\epsilon\Pi RV_\beta$ $(\beta > 0)$ and $n(x)/x$ integrable on finite intervals of $(0,\infty)$

If we define $\chi(x) = \sum_{1 \le k \le x} \frac{1}{k} n(\frac{x}{k})$ and suppose

$n(x)/x^\beta$ to be non-decreasing, then $\chi\epsilon\Pi RV_\beta$.

Moreover

$$\frac{\zeta(\beta+1)\, n(x) - \chi(x)}{x^\beta L(x)} \;\to\; -\,\zeta'(\beta+1) \qquad (x \to \infty)$$

where L is the auxiliary function of n.

Proof   We may suppose w.l.o.g. that $n(x) = 0$ on $(0,1)$

Then $\tilde{n}(s) = s \int\limits_{1}^{\infty} \sum\limits_{k=1}^{\infty} e^{-kus}\, n(u)\,du = \int\limits_{0}^{\infty} e^{-us}\, d\chi(u)$

By the main theorem we have

$$\frac{\chi(x) - \dfrac{1}{\Gamma(\beta+1)}\,\tilde{n}(1/x)}{\zeta(\beta+1)\, x^\beta L(x)} \;\to\; -\,\psi(\beta+1) \qquad (x \to \infty)$$

Combination with the last theorem then gives the desired result.

Corollary In this case we have $\zeta(\beta+1)\, n(x) \sim \chi(x)$ $(x \to \infty)$

For the Tauberian converse of the last theorem we need the following lemma.

Lemma If L is slowly varying and non-decreasing for $x > 0$

$$\frac{L(x)}{L(x-1)} \;\le\; 1 + x^{-\gamma} \quad \text{for some} \quad > 0,\; x \ge x_o(\gamma),\; \text{then}$$

$$\sum_{m \le x} \frac{\mu(m)}{m}\, U\!\left(\frac{x}{m}\right) = o\,(x^\beta L(x)) \qquad (x \to \infty)$$

where $\beta > 0$ and $U(x) = \int\limits_{1}^{x} t^\beta\, dL(t)$

Proof As in (6) we define $a_n = U(n) - U(n-1),\; n \ge 2,\; a_1 = 0.$

We divide the sum into three parts.

Then $\displaystyle\sum_{m \le x} \frac{\mu(m)}{m}\, U\!\left(\left[\frac{x}{m}\right]\right) = \sum_{m \le x} \frac{\mu(m)}{m} \sum_{\nu \le \frac{x}{m}} a_\nu = \sum_{m \le x} a_m\, N\!\left(\frac{x}{m}\right)$

where $N(x) = \displaystyle\sum_{m \le x} \frac{\mu(m)}{m}$

Since $N(x) \to 0$ $(x \to \infty)$ we have

$$\sum_{m \le x} a_m N(\tfrac{x}{m}) \le c \sum_{m \le x} a_m = c \ U([x]) = o(x^\beta L(x)) \quad (x \to \infty)$$

by Karamata's theorem A.

Now $U(x) - U(x-1) = x^\beta L(x) - (x-1)^\beta L(x-1) - \beta \int_{x-1}^{x} t^{\beta-1} L(t) \ dt$

$\le x^\beta (L(x) - L(x-1))$ since L is non-decreasing.

Substituting this we find for $x \ge x_0 = x_0(\gamma)$

$$\sum_{1 \le m < \frac{x}{x_0}} \frac{\mu(m)}{m} \left\{ U(\tfrac{x}{m}) - U([\tfrac{x}{m}]) \right\} \le$$

$$\le \sum_{1 \le m < \frac{x}{x_0}} \frac{1}{m} (\tfrac{x}{m})^\beta \left\{ L(\tfrac{x}{m}) - L(\tfrac{x}{m} - 1) \right\} \le$$

$$\le \sum_{1 \le m < \frac{x}{x_0}} \frac{1}{m} (\tfrac{x}{m})^{\beta-\gamma} L(\tfrac{x}{m}) \le x^{\beta-\gamma} \left\{ L(x) + \int_{1}^{x/x_0} u^{-1-\beta+\gamma} L(\tfrac{x}{u}) \ du \right\}$$

$$= x^{\beta-\gamma} L(x) + \int_{x_0}^{x} \frac{L(y)}{y^{1-\beta+\gamma}} \ dy = o(x^\beta L(x)) \quad (x \to \infty)$$

by theorem A if we choose $0 < \gamma < \beta+1$

The last sum is bounded since

$$\sum_{\frac{x}{x_0} < m \le x} \frac{\mu(m)}{m} \left\{ U(\tfrac{x}{m}) - U([\tfrac{x}{m}]) \right\} \le 2 \sum_{\frac{x}{x_0} < m \le x} \frac{1}{m} U(\tfrac{x}{m})$$

$$\le 2 \ U(x_0) \sum_{\frac{x}{x_0} < m \le x} \frac{1}{m} = O(1) = o(x^\beta L(x)) \quad (x \to \infty)$$

This proves the lemma.

<u>Theorem</u> If $\chi(x) = \sum\limits_{1 \leq m \leq x} \frac{1}{m} n(\frac{x}{m}) . \varepsilon \Pi RV_\beta$ $(\beta > 0)$,

$L(x) := \frac{1}{x} \int\limits_0^x s\, d\, \frac{\chi(s)}{s^\beta}$ is non-decreasing on R^+

$\frac{L(x)}{L(x-1)} \leq 1 + x^{-\gamma}$ for some $\gamma > 0$, $x \geq x_0(\gamma)$

then $n \in \Pi RV_\beta$ with auxiliary function $\frac{1}{\zeta(\beta+1)} L(x)$

<u>Proof</u> By the representation theorem in de Haan (7) we can write

$\chi(x) = x^\beta L(x) + x^\beta \int\limits_1^x \frac{L(t)}{t} dt$ where L is slowly varying.

Möbius inversion now gives

$n(x) = x^\beta \sum\limits_{m \leq x} \frac{\mu(m)}{m^{\beta+1}} \phi(\frac{x}{m}) + x^\beta V(x)$

where $\phi(x) = \int\limits_1^x \frac{L(u)}{u} du$ and $V(x) = \sum\limits_{m \leq x} \frac{\mu(m)}{m^{1+\beta}} L(\frac{x}{m})$

Substituting this gives

$n(x) - \beta \int\limits_1^x \frac{n(t)}{t} dt = \int\limits_1^x u^{\beta-1} L(u) N(\frac{x}{u}) du + R(x)$

where $R(x) = x^\beta V(x) - \beta \int\limits_1^x t^{\beta-1} V(t) dt$ and $N(x) = \sum\limits_{m \leq x} \frac{\mu(m)}{m}$

Now we have by dominated convergence

$\int\limits_1^x u^{\beta-1} L(u) N(\frac{x}{u}) du = x^\beta \int\limits_1^x \frac{N(v)}{v^{\beta+1}} L(\frac{x}{v}) dv \sim x^\beta L(x) \int\limits_1^\infty \frac{N(v)}{v^{\beta+1}} dv \sim$

$\frac{1}{\beta\zeta(\beta+1)} x^\beta L(x)$ $(x \to \infty)$ since

$\int\limits_1^x \frac{N(v)}{v^{\beta+1}} dv = \sum\limits_{k \leq x} \frac{\mu(k)}{k} \int\limits_k^x \frac{dv}{v^{\beta+1}} = \frac{1}{\beta} \sum\limits_{k \leq x} \frac{\mu(k)}{k^{1+\beta}} - \frac{N(x)}{\beta x^\beta} \to \frac{1}{\beta\zeta(1+\beta)}$ $(x \to \infty)$

and $N(x) \to 0$ $(x \to \infty)$.

Application of the lemma now gives

$$R(x) = x^\beta V(x) - \beta \int_1^x t^{\beta-1} V(t) \, dt = \sum_{m \le x} \frac{\mu(m)}{m} \int_1^{x/m} t^\beta dL(t) = \sigma(x^\beta L(x))$$

This proves that $\dfrac{n(x)}{x^\beta} - \dfrac{\beta}{x^\beta} \displaystyle\int_1^x \dfrac{n(t)}{t} \, dt \sim \dfrac{1}{\beta \zeta(\beta+1)} \, L(x) \qquad (x \to \infty)$

Application of proposition P4 in (9) then gives $n \in \Pi RV_\beta$.

Remark In this case we have $n(x) \sim \dfrac{1}{\zeta(\beta+1)} \, x^\beta \displaystyle\int_1^x \dfrac{L(u)}{u} \, du \qquad (x \to \infty)$

Theorem Suppose $R(x) := \displaystyle\sum_{m \le x} \dfrac{1}{m} \, n(\tfrac{x}{m}) = x^\beta \int_1^x \dfrac{L(u)}{u} \, du + o(x^\beta L(x)) \qquad (x \to \infty)$

with L slowly varying and $\beta > 0$.

If $L_*(x) := \dfrac{1}{x} \displaystyle\int_0^x s \, d\dfrac{R(s)}{s^\beta}$ is non-decreasing on R^+

and $\dfrac{L_*(x)}{L_*(x-1)} \le 1 + x^{-\gamma}$ for some $\gamma > 0$, $x \ge x_0(\gamma)$

then $n \in \Pi RV_\beta$ with auxiliary function $\dfrac{1}{\zeta(\beta+1)} \, L(x)$

Proof We have $R(x) = x^\beta L_*(x) + x^\beta \displaystyle\int_1^x \dfrac{L_*(t)}{t} \, dt \in \Pi RV_\beta$

where L_* satisfies the conditions of the last theorem.
This proves the theorem.

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