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AN ABEL-TAUBER THEOREM ON CONVOLUTIONS WITH THE MÖBIUS FUNCTIONS

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by

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ABSTRACT

Suppose $n: R^{+} \to R^{+}$ and $\frac{n(x)}{x}$ is integrable on $(0,\infty)$.

Then $\tilde{n}(s) = s \int_{0}^{\infty} \frac{e^{-us}}{1-e^{-us}}\, n(u)\,du$ exists for $s>0$.

In this paper an Abel-Tauber theorem is proved concerning this transform.

Moreover the relation between $\tilde{n}(s)$ and $\sum_{m \leq s} \frac{1}{m}\, n(\frac{s}{m})$ is studied.

Contents

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Abel-Tauber theorems, regular variation.

Introduction

Following earlier work by Landau [1] and others, Ingham proved (in [2]) the following theorem:

Theorem A

Suppose that

(i) $f(x)$ is positive and non-decreasing for $x \geqq 1$

(ii) $F(x) \equiv \sum_{n \leqq x} f(\frac{x}{n}) = ax\log x + bx + o(x) \qquad (x \to \infty)$

where a and b are constants. Then

(a) $f(x) \sim ax \qquad (x \to \infty)$

(b) $\int_1^\infty \frac{f(x)-ax}{x^2}\,dx = b - a\gamma$

where γ is Eulers constant.

Moreover, for any function $f(x)$, bounded and integrable over every finite interval $(1,X)$ hypothesis (ii) implies conclusion (b). More recently Jukes [14] extended results of Segal (see [3], [4]) and proved the following theorem which can be considered as a generalization of theorem A. For a proof of this theorem see also [5].

Theorem B

Let $f(x)$ be bounded and integrable on every finite subinterval of $[1,\infty)$ and suppose that

$$\sum_{n \leqq x} f(\frac{x}{n}) = xg(x) + o\,(x^2 g'(x)) \qquad (x \to \infty)$$

where $g(x) \in C^2[1,\infty)$ is positive and satisfies

(i) $g'(x) > 0$ for all $x \in [1,\infty)$

(ii) for some real r, $xg'(x)(\log x)^{-r}$ is non-increasing for $x > x_0$

(iii) for some real s, $xg'(x)(\log x)^s \equiv q(x)$ is non-decreasing for $x > x_1$ and

$q(x) \to \infty \quad (x \to \infty)$

Then $\displaystyle\int_1^x \frac{f(t)}{t^2}\,dt = g(x) - \gamma\, xg'(x) + o\,(xg'(x)) \qquad (x\to\infty)$

If we suppose f(x) to be positive and non-decreasing we also have $f(x) \sim x^2 g'(x)$ $(x\to\infty)$

In this note we consider the situation in theorem B where g(x) is allowed to grow faster than in (ii).

We use the concept of regular variation as introduced by Karamata to formulate natural conditions on g(x) and prove a theorem similar to theorem B. Moreover, an application is given which gives a second order condition in a relation considered by Parameswaran in [6]. See our corollary 1 and the remark after theorem 4.

Parameswaran proves the following theorem ([6], theorem II).

<u>Theorem C</u>

If $\displaystyle\int_0^R \frac{n(u)}{u}\,du$ exists in the Lebesgue sense for every positive R

$$f(s) \equiv \exp\left\{ s\int_0^\infty \frac{e^{-su}}{1-e^{-su}}\, n(u)du \right\}$$

for all positive s and n(u) non-decreasing, then the relation

$$\log f(s) \sim \int_a^{1/s} \frac{L(u)}{u}\,du \text{ as } s\to 0+ \text{ implies the relation}$$

$n(u) \sim L(u)$ as $u\to\infty$, provided L(x) is a slowly oscillating function defined for $x \geq a$ such that

$$L(x) \sim K\exp\left\{ \int_a^x \frac{\delta(u)}{u}\,du \right\} \quad \text{as } x\to\infty \text{ where } \delta \text{ is a non-increasing function and}$$

K is a positive real number.

We use our theorem 3 to get a theorem similar to theorem C.

Results

First we define the subclass of slowly varying functions we want to consider.

Definition: The non-decreasing function $q : R^+ \to R$ belongs to the class Π if there exists a slowly varying function L such that $q(x) = \int_1^x \frac{L(t)}{t}\,dt + L(x)$

It can be shown that the class Π consists of all non-decreasing functions $q : R^+ \to R$ for which there exist functions $a : R^+ \to R^+$ and $b : R^+ \to R$ such that for all positive x

$$\lim_{t\to\infty} \frac{q(tx) - b(t)}{a(t)} = \log x$$

If the last relation holds, then it is true with $b(t) = q(t)$ and $a(t) = q(te) - q(t)$. The function $a(t)$ is of course determined up to asymptotic equivalence and is called the auxiliary function. As a consequence of these facts we mention the following result: if $q(x) \in \Pi$ with auxiliary function $L(x)$, $q_1(x)$ is non-decreasing and

$$\frac{q(x) - q_1(x)}{L(x)} \to C \ (x\to\infty), \text{ where}$$

$C \in R$ is a constant, then $q_1(x) \in \Pi$ with auxiliary function $L(x)$.

It is easy to see that if $L(x)$ slowly varying, then $\int_1^x \frac{L(t)}{t}\,dt$ is an element of Π with auxiliary function $L(x)$.

The class Π is a subclass of the slowly varying functions at infinity. Moreover, for each function $q \in \Pi$ it is possible to find a function L_* such that $q(x) =$

$$\int_1^x \frac{L_*(t)}{t}\,dt + o\,(L(x)) \ (x\to\infty)$$

where $L_* \sim L$ is slowly varying.

For the proofs of these statements and further properties see [10], [11] and [12].

We start with two Abelian results.

Theorem 1

Suppose $n : R^+ \to R^+$ is slowly varying, $\frac{n(x)}{x}$ is (Lebesgue) integrable on finite subintervals of $(0,\infty)$

Then $\tilde{n}\,(\frac{1}{s}) \in \Pi$ with auxiliary function $n(s)$, where $\tilde{n}$ is defined by

$$\tilde{n}(s) = s \int_0^\infty \frac{e^{-us}}{1-e^{-us}}\, n(u)du$$

Moreover,

$$\frac{\tilde{n}\,(\frac{1}{s}) \; - \int_0^s \frac{n(t)}{t}\, dt}{n(s)} \to 0 \quad (s\to\infty)$$

Proof:

$$\frac{\int_0^s \frac{n(t)}{t}dt - \tilde{n}(\frac{1}{s})}{n(s)} = \int_0^1 (\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}}) \frac{n(su)}{n(s)}\, du - \int_1^\infty \frac{e^{-u}}{1-e^{-u}} \; \frac{n(su)}{n(s)}\, du$$

We have $\int_0^1 \frac{n(su)}{n(s)}\, du = \frac{\int_0^s n(v)dv}{sn(s)} \to 1 \; (s\to\infty)$ by Karamata's theorem on regularly varying functions and since $\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}}$ is bounded on $(0,1)$ we can apply Pratt's lemma.

For the second part we have $\frac{(su)^{-\varepsilon} n(su)}{s^{-\varepsilon} n(s)} \to \; u^{-\varepsilon}$ $(s\to\infty)$ uniform on $(1,\infty)$ by co rollary 1.2.1.4 of [11].

So we have

$$\frac{\int_0^s \frac{n(t)}{t}dt - \tilde{n}(\frac{1}{s})}{n(s)} \to \int_0^1 (\frac{1}{u} - \frac{e^{-u}}{1-e^{-u}})\, du - \int_1^\infty \frac{e^{-u}}{1-e^{-u}}\, du \qquad (s\to\infty)$$

The right side is zero, as is shown by elementary integration. This implies that $\tilde{n}\,(1/s) \in \Pi$ with auxiliary function $n(s)$, since

$\int_0^s \frac{n(t)}{t}\, dt \in \Pi$ with auxiliary function $n(s)$ and $\tilde{n}\,(1/s)$ is non-decreasing.

Remarks: 1. The case $n \in RV_\alpha^{(\infty)}$ ($\alpha>0$) (regularly varying at infinity with exponent α) gives analogously $\frac{\tilde{n}(1/s)}{n(s)} \to \zeta(\alpha+1)\,\Gamma(\alpha+1)$ ($s\to\infty$). See [13], theorem 1.

2. The statement of the theorem implies $\tilde{n}(1/s) \sim \int_0^s \frac{n(t)}{t}\,dt$ since $n(s) = o(\int_0^s \frac{n(t)}{t}\,dt)$ by Karamata's theorem.

See theorem I of [6].

Theorem 2. If we define $\Psi(s) = \sum_{m\leq s} \frac{1}{m} n(\frac{s}{m})$ where n satisfies the conditions of theorem 1 and $xn(x)$ is of bounded variation on intervals of the form $(1, x_0)$, then

$$\frac{\Psi(s) - \int_0^s \frac{n(t)}{t}\,dt}{n(s)} \to \gamma \qquad (s\to\infty)$$

Proof We give the proof by an Euler-Maclaurin kind of argument

$$\int_{\nu-1}^{\nu} \frac{x}{t} n\left(\frac{x}{t}\right) dt = xn\left(\frac{x}{\nu}\right) - xn\left(\frac{x}{\nu-1}\right) - \int_{\nu-1}^{\nu} t\,d\frac{x}{t} n\left(\frac{x}{t}\right) =$$

$$\frac{x}{\nu}n\left(\frac{x}{\nu}\right) - \int_{\nu-1}^{\nu} \{t\}\,d\frac{x}{t} n\left(\frac{x}{t}\right)$$

where we use the notation $\{t\} = t-[t]$

Summing over ν gives

$$\sum_{1\leq\nu\leq x} \frac{1}{\nu} n\left(\frac{x}{\nu}\right) - \int_1^x \frac{1}{t} n\left(\frac{x}{t}\right) dt = n(x) - \int_{[x]}^{x} \frac{1}{t} n\left(\frac{x}{t}\right) dt + \int_1^{[x]} \{t\}\,d\frac{1}{t} n\left(\frac{x}{t}\right)$$

Now with $M=\sup_{x\,\varepsilon\,(1,2)} n(x)$ we have $\left|\frac{\int_{[x]}^{x} \frac{1}{t} n\left(\frac{x}{t}\right) dt}{n(x)}\right| \leq \frac{M \log \frac{x}{[x]}}{n(x)} \leq \frac{M}{n(x)(x-1)} \to 0 \; (x\to\infty)$

since $(x-1)\,n(x)$ is 1-varying at infinity.

Furthermore $\int_1^{[x]} \{t\}\, d\, \frac{1}{t}\, n\, (\frac{x}{t}) = -\frac{1}{x} \int_1^{[x]} \frac{x}{t}\, n\, (\frac{x}{t})\, d\, \{t\} \sim -n(x) \int_1^{[x]} \frac{1}{t}\, d\, \{t\} \sim (\gamma-1)\, n(x)$

since $\frac{\frac{x}{t}\, n\, (\frac{x}{t})}{xn(x)} \to t^{-1}$ uniform for $t \in (1,\infty)$ (see [11], cor. 1.2.1.4)

and $\int_1^{\infty} \frac{\{t\}}{t^2}\, dt = 1-\gamma$ (see e.g. [7])

This proves theorem 2.

Remarks: 1. The theorem implies $\frac{\Psi(tx) - \Psi(t)}{\Psi(te) - \Psi(t)} \to \log x \; (t\to\infty)$

where $x>0$ is fixed. If Ψ is non-decreasing we have $\Psi \in \Pi$.

2. With the additional supposition that Ψ is non-decreasing a simpler proof is possible:

$$\tilde{n}(s) - s \int_0^1 \frac{e^{-us}}{1-e^{-us}}\, n(u)\, du = s \int_1^{\infty} \sum_{k=1}^{\infty} e^{-kus}\, n(u)\, du = \int_0^{\infty} e^{-us}\, d\Psi(u)$$

Since $\lim_{s\to\infty} \frac{1}{s} \int_0^1 \frac{e^{-u/s}}{1-e^{-u/s}}\, n(u)\, du = \int_0^1 \frac{n(u)}{u}\, du$ we have

by theorem 1 $\int_0^{\infty} e^{-u/s}\, d\Psi(u) \in \Pi$ with auxiliary function n.

Hence $\frac{\tilde{n}(1/s) - \Psi(s)}{n(s)} \to -\gamma (s\to\infty)$ (see [10], theorem 1)

Combining this with theorem 1 gives the desired result.

3. The case $n \in RV_{\alpha}^{(\infty)}$ $(\alpha>0)$ in remark 2 gives

$$\frac{\Psi(s)}{\int_0^s \frac{n(t)}{t}\, dt} \to \alpha \quad \zeta(\alpha+1) \; (s\to\infty)$$

4. The statement of the theorem implies $\Psi(s) \sim \int_0^s \frac{n(t)}{t}\, dt$ since $n(s) =$

$o(\int_0^s \frac{n(t)}{t}\, dt) \; (s\to\infty)$

As a consequence of remark 2 we mention the following:

Corollary 1. Suppose $n : R^+ \to R^+$ satisfies the conditions $\int_0^R \frac{n(u)}{u}\, du < \infty$ for $R > 0$

and $\sum_{m \leqq s} \frac{1}{m} n\left(\frac{s}{m}\right)$ is non-decreasing for $s > 0$.

The assertions $\sum_{m \leqq s} \frac{1}{m} n\left(\frac{s}{m}\right) \in \Pi$ with auxiliary function $L(s) \to \infty$ $(s \to \infty)$

and $\tilde{n}(1/s) \in \Pi$ with auxiliary function $L(s) \to \infty$ $(s \to \infty)$ are equivalent.

Both imply: $$\frac{\sum_{m \leqq s} \frac{1}{m} n\left(\frac{s}{m}\right) - \tilde{n}(1/s)}{L(s)} \to \gamma \quad (s \to \infty)$$

For the Tauberian counterpart of the theorems 1 and 2 we need three lemmas.

Lemma 1: If

(i) $L(x)$ is non-decreasing for $x > 0$, slowly varying, $L(x) \to \infty$ $(x \to \infty)$

(ii) $\frac{L(x)}{L(x-1)} \leqq 1 + x^{-\alpha}$ for some $\alpha > 0$, $x \geqq x_0 = x_0(\alpha)$

then

$$\sum_{m \leqq x} \frac{\mu(m)}{m} L\left(\frac{x}{m}\right) = o\,(L(x)) \quad (x \to \infty)$$

Proof: We define $a_n = L(n) - L(n-1)$ for $n \geqq 2$, $a_1 = L(1)$

Then

$$\sum_{m \leqq x} \frac{\mu(m)}{m} L\left(\left[\frac{x}{m}\right]\right) = \sum_{m \leqq x} \frac{\mu(m)}{m} \sum_{\nu \leqq \frac{x}{m}} a_\nu = \sum_{m \leqq x} a_m N\left(\frac{x}{m}\right)$$

where

$N(x) = \sum_{m \leqq x} \frac{\mu(m)}{m}$. Since $|N(x)| < \varepsilon$ for $x \geqq x_\varepsilon$ (see [15]) we have:

$$\left| \sum_{m \leqq \frac{x}{x_\varepsilon}} a_m N\left(\frac{x}{m}\right) \right| < \varepsilon \sum_{m \leqq \frac{x}{x_\varepsilon}} a_m = \varepsilon\, L\left(\left[\frac{x}{x_\varepsilon}\right]\right) \leqq \varepsilon(1+\varepsilon)\, L(x) \quad (x > x_\varepsilon).$$

and $$\left|\sum_{\frac{x}{x_\varepsilon}<m\leqq x} a_m\, N\left(\frac{x}{m}\right)\right| \leqq c \sum_{\frac{x}{x_\varepsilon}<m\leqq x} a_m \leqq c\,\{L(x) - L\left(\frac{x}{x_\varepsilon}\right)\} = o(L(x))\quad (x\to\infty)$$

For $x \geqq x_0(\alpha)$ we have by (ii)

$L(x) - L([x]) \leqq L(x) - L(x-1) \leqq \frac{L(x)}{x^\alpha}$ since L is non-decreasing. Hence:

$$\left|\sum_{1\leqq m\leqq\frac{x}{x_0}} \frac{\mu(m)}{m}\{L\left(\frac{x}{m}\right) - L\left(\left[\frac{x}{m}\right]\right)\}\right| \leqq \sum_{m\leqq\frac{x}{x_0}} \frac{1}{m}\,\frac{L(x/m)}{x^\alpha/m^\alpha} \leqq$$

$$x^{-\alpha}\,\{\int_1^{x/x_0} \frac{1}{u^{1-\alpha}}\, L\left(\frac{x}{u}\right) du + L(x)\} = \int_{x_o}^{x} \frac{L(v)}{v^{1+\alpha}}\, dv + o(L(x)) = o(L(x))$$

if we choose $0<\alpha<1$.

We estimate the last sum as follows:

$$\left|\sum_{\frac{x}{x_o}<m\leqq x} \frac{\mu(m)}{m}\{L\left(\frac{x}{m}\right) - L\left(\left[\frac{x}{m}\right]\right)\}\right| \leqq 2 \sum_{\frac{x}{x_o}<m\leqq x} \frac{1}{m}\, L\left(\frac{x}{m}\right) \leqq 2\, L(x_o) \sum_{\frac{x}{x_o}<m\leqq x} \frac{1}{m}$$

$= O(1) = o(L(x))$ $(x\to\infty)$. This proves the lemma.

<u>Remark</u>: The conclusion of the lemma is incorrect for arbitrary slowly varying functions.
There exists a function $L(x) \to C\ (\neq 0)$ $(x\to\infty)$ such that:

$$\sum_{m\leqq x} \frac{\mu(m)}{m}\, L\left(\frac{x}{m}\right) \neq o(1)\quad (x\to\infty)$$

See e.g. [4] example 2.

<u>Lemma 2</u> Suppose $\sum_{m\leqq x} \frac{1}{m}\, n\left(\frac{x}{m}\right) = \int_1^x \frac{L(u)}{u}\, du + L(x)$ where L satisfies the conditions of lemma 1. Then $n(x) \sim L(x)$ $(x\to\infty)$.

Proof: Möbius inversion and lemma 1 gives:

$$n(x) = \sum_{m \leq x} \frac{\mu(m)}{m} \int_1^{x/m} \frac{L(u)}{u} du + \sum_{m \leq x} \frac{\mu(m)}{m} L\left(\frac{x}{m}\right)$$

$$= \int_1^x \frac{L(u)}{u} N\left(\frac{x}{u}\right) du + o(L(x)) = \int_1^x L\left(\frac{x}{v}\right) \frac{N(v)}{v} dv + o(L(x)) \sim L(x) \int_1^\infty \frac{N(v)}{v} dv \sim L(x)$$

$(x \to \infty)$ by dominated convergence, since

$$\int_1^x \frac{1}{v} \sum_{k \leq v} \frac{\mu(k)}{k} dv = \sum_{k \leq x} \frac{\mu(k)}{k} \int_k^x \frac{dv}{v} = \log x . \sum_{k \leq x} \frac{\mu(k)}{k} - \sum_{k \leq x} \frac{\mu(k)}{k} \log k \to 1 \quad (x \to \infty)$$

(see [15]).

Remark: The proof of lemma 2 implies the following:

If $h(x) = \int_1^x \frac{L(u)}{u} du + L(x)$ where L satisfies the conditions of lemma 1, then $\sum_{m \leq x} \frac{\mu(m)}{m} h\left(\frac{x}{m}\right) \sim \frac{1}{x} \int_1^x s dh(s)$, which is the auxiliary function of h(x). For functions h(x) of the form $\int_1^x \frac{L(u)}{u} du$, L slowly varying the conditions of lemma 1 are not necessary for this result. Compare with theorem 1 of [4] .

The following lemma gives the same statement under different conditions on L(x).

Lemma 3. Suppose $\sum_{m \leq x} \frac{1}{m} n\left(\frac{x}{m}\right) = \int_1^x \frac{L(u)}{u} du + L(x)$ where $L(x) \to \infty$ and $L(x) \in \Pi$

Then $n(x) \sim L(x) \quad (x \to \infty)$

Proof: We write $L(x) = \int_1^x \frac{\gamma(u)}{u} du + o(\gamma(x))$ where γ is slowly varying.

Then Möbius inversion gives:

$$n(x) = \sum_{m \leq x} \frac{\mu(m)}{m} \int_1^{x/m} \frac{L(u) + \gamma(u)}{u} du + \sum_{m \leq x} \frac{\mu(m)}{m} o\left(\gamma\left(\frac{x}{m}\right)\right)$$

The first part is asymptotic to $L(x)$ as in lemma 2 since $\gamma(x) = o(L(x))$ $(x\to\infty)$.

For the second term we proceed as follows:

Suppose $|o(\gamma(x))| < \varepsilon\ \gamma(x)$ for $x \geqq x_0$. Then we have:

$$\left|\sum_{m\leqq x} \frac{\mu(m)}{m}\, o(\gamma(\tfrac{x}{m}))\right| \leqq \varepsilon \sum_{m\leqq \frac{x}{x_0}} \frac{1}{m}\,\gamma(\tfrac{x}{m}) + \sum_{\frac{x}{x_0}<m\leqq x} \frac{1}{m}\left|o(\gamma(\tfrac{x}{m}))\right|$$

$$\leqq \varepsilon \sum_{m\leqq \frac{x}{x_0}} \frac{1}{m}\,\gamma\,(\tfrac{x}{m}) + c \sum_{\frac{x}{x_0}<m\leqq x} \frac{1}{m} < \varepsilon \sum_{m\leqq x} \frac{1}{m}\,\gamma(\tfrac{x}{m}) + O(1)$$

where $c = \sup_{x\varepsilon(1,x_0)} |o(\gamma(x))|$

Now we have $\sum_{m\leqq x} \frac{1}{m}\,\gamma(\frac{x}{m}) \sim L(x)$ $(x\to\infty)$ by theorem 2. This proves the lemma.

<u>Theorem 3</u>. Suppose

(i) $$\int_1^x \frac{L(u)}{u}\,du + o(L(x)) = \sum_{m\leqq x} \frac{1}{m}\, n\,(\tfrac{x}{m})$$

where $L(x) \to \infty$ $(x\to\infty)$ is slowly varying

(ii) $R(x) \equiv \sum_{m\leqq x} \frac{1}{m}\, n\,(\frac{x}{m})$ is non-decreasing

(iii) $L_*(x) \equiv \frac{1}{x}\int_1^x t\, dR(t)$ is non-decreasing

(iv) $\frac{L_*(x)}{L_*(x-1)} \leqq 1 + x^{-\alpha}$ with $\alpha > 0$ for $x \geqslant x_0(\alpha)$

then $n(x) \sim L(x)$ $(x\to\infty)$.

<u>Proof</u>: We have $R(x) = \int_1^x \frac{L_*(t)}{t}\,dt + L_*(x)$ where $L_*(x)$ is defined as above (see [11] theorem 1.3.4.) ; $L_*(x) \sim L(x)$ satisfies the conditions of lemma 2. This proves the theorem.

Theorem 4. Suppose (i) and (ii) of theorem 3 and $L_*(x) \in \Pi$

Then $L(x) \sim n(x) \quad (x \to \infty)$

Proof: We use now lemma 3. The rest of the proof is as in theorem 3.

Combining corollary 1 with the theorems 3 or 4 gives the following Tauberian theorem.

Theorem 5.

If $\tilde{n}(1/s) = \int_1^s \frac{L(u)}{u}\, du + o(L(s)) \quad (s \to \infty)$ with $L(s) \to \infty$

slowly varying and $n : R^+ \to R^+$ satisfies the conditions

(i) $\frac{n(u)}{u}$ is integrable on $(0,R)$ for every $R > 0$

(ii) $R(x) \equiv \sum\limits_{m \leqq x} \frac{1}{m}\, n\left(\frac{x}{m}\right)$ is non-decreasing

(iii) $L_*(x) \equiv \frac{1}{x} \int_1^x t\,dR(t)$ is non-decreasing

and (iv) $L_*(x) \in \Pi$

or (v) $\frac{L_*(x)}{L_*(x-1)} \leqq 1 + x^{-\alpha}$ with $\alpha > 0$ for $x \geqq x_0(\alpha)$

then
$$\frac{\int_1^s \frac{n(t)}{t}\, dt - \int_1^s \frac{L(t)}{t}\, dt}{L(s)} \to 0 \quad (s \to \infty)$$

Proof: Applying corollary 1 and theorem 3 or 4 we have $n(x) \sim L(x)$

Application of theorem 1 yields the result.

For regularly varying functions the following analogue of theorem 5 is well-known.

Theorem 6. If $\tilde{n}(1/s) \sim \zeta\,(\alpha+1)\ \Gamma\,(\alpha+1)\ s^\alpha\, L(s)$ where

$L(s) \in RV_\alpha^{(\infty)}$ with $\alpha > 0$ and $n(u)$ satisfies the following conditions

(i) $\frac{n(u)}{u}$ and $\frac{n(u)}{u} \log u$ are integrable on $(0,R)$ for every $R > 0$

(ii) $n(u)$ is non-decreasing

then $n(u) \sim u^\alpha L(u) \quad (u \to \infty)$

For a proof see [13] theorem 2.

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