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A FAMILY OF IMPROVED ORDINARY RIDGE ESTIMATORS

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A FAMILY OF IMPROVED ORDINARY RIDGE ESTIMATORS

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Abstract

This paper studies the Mean Squared Error (MSE) properties of a proposed family of Ordinary Ridge Estimators (OREs) of the coefficients in the linear regression. We make extensive use of G( ) functions to provide both exact and asymptotic approximations to the MSE. Using these results we propose a new set of OREs whose MSE is smaller than that of the Ordinary least squares (OLS) estimator. These improved estimators can be used when faced with the multicollinearity problem. A simulation study is also done to further analyse the MSE of the proposed estimators compared with some of the existing OREs.

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1. INTRODUCTION

Hoerl and Kennard [1970] proposed a biased estimator called the Ridge Regression estimator for the general linear model which dominates the OLS estimator. Vinod [1978] surveyed the growing literature on the subject. When the biasing parameters k_i of Ridge Regression are all equal to say k, we have the so called "Ordinary" Ridge Regression. Recently various authors, e.g., Hoerl et al. [1975], Lawless and Wang [1976] have considered the problem of determining the biasing parameter k in the ORE. Invariably, different formulae presented by various authors for k have depended on unknown parameters, namely, the regression coefficient and residual variance. The usual practice then is to estimate k by substituting sample estimates of the unknown parameters involved. Use of this estimated k certainly makes the ORE operational. However, since k is now stochastic it is no longer definite that ORE will dominate OLS. Vinod's survey [1978] notes that some Monte Carlo studies have been carried out to see the performance of the MSE of ORE for different choices of k.[1] Though these studies do indicate a fairly good performance of ORE, the researchers are still neither certain about the operational range of the parameter k nor its optimal value. Further, in the authors' knowledge, no analytical study is available which analyses exact or approximate MSE of the ORE.

In this paper we present various ways of determining k and then develop a family of "double h-class" estimates of k. This double h-class estimate depends on two arbitrary scalars h_1 and h_2 which could be stochastic or non-stochastic.[2] Using the double h-class estimate of k in ORE we formulate a new family of double h-class OREs. It has been noted that many of the biased estimators in the literature can be considered as special cases of this new family. Section 3 studies the MSE properties of the double h-class estimator using G ( ) functions. Further, using the approximate expressions for MSE, we propose a range of values for h_1 and h_2 for which the double h-class ORE dominates OLS. When multicollinearity is severe in the sense that the smallest eigenvalue (λ_m) of the correlation matrix among regressors is close to zero there is ample theoretical and Monte Carlo evidence which suggests that some form of ridge estimator will reduce the MSE of OLS. The values of h_1 and h_2 obtained show that even when multicollinearity is moderate ($d > 2$ defined in (3.16)) it is possible to reduce the MSE of OLS. Many other interesting results together with comparisons among various well-known members of this family are summarised in Section 3.4. To analyse further the MSE of the proposed estimators we give a brief description of a simulation study. It has been found that (Table 2) the percentage of occurances, where the proposed estimators, have strictly a lower MSE than OLS, is always larger than corresponding percentage for the estimators by Hoerl, Kennard and Baldwin [1975] and Lawless and Wang [1976]. Also, it is indicated that the proposed estimators perform better than Hoerl et al., and Lawless and Wang estimators for large values of noncentrality parameters.

2. THE MODEL AND ESTIMATORS

Let us write the standard linear regression model as

(2.1) $$y = X\beta + u$$

where y is a T x 1 vector of observations on the dependent variable, X is a T x p matrix of p explanatory variables, β is a p x 1 vector of unknown regression coefficients and u is a T x 1 vector of unknown disturbances.

We state the following conventional assumptions:

Assumption 1 The matrix of explanatory variables is nonstochastic and of rank p

Assumption 2 The disturbance vector u is distributed as multivariate normal with mean vector zero and variance covariance matrix $\sigma^2 I$, i.e.,

$$u \sim (0, \sigma^2 I)$$

Assumption 3 The sample size T is greater than the total number of explanatory variables p in (2.1).

The ordinary least squares (OLS) estimator or maximum likelihood estimator of β in (2.1) is written as

(2.2) $$b = (X'X)^{-1}X'y$$

such that

(2.3) $$Eb = \beta, \quad V(b) = E(b-\beta)(b-\beta)' = \sigma^2(X'X)^{-1}$$

and

(2.4) $$MSE(b) = E(b-\beta)'(b-\beta) = \sigma^2 Tr(X'X)^{-1}$$

where Tr represents the trace of a matrix. Further an unbiased estimator of σ^2 is given by

(2.5) $$s^2 = \frac{1}{n}\hat{u}'\hat{u}, \qquad n = T - p$$

where

(2.6) $$\hat{u} = y - Xb .$$

The model in (2.1) can be written in the canonical form as

(2.7) $y = Z\alpha + u$

where

(2.8) $Z = XG, \quad \alpha = G'\beta$

and G is a p x p matrix of normalized eigenvectors corresponding to the eigenvalues of X'X such that

(2.9) $G'G = GG' = I$

and

(2.10) $Z'Z = \Lambda, \; X'X = G \Lambda G'$

where Λ, a p x p diagonal matrix of eigenvalues of X'X, is

(2.11) $\Lambda = \text{Diag.}[\lambda_1,\ldots,\lambda_p]$.

The OLS estimator of α in (2.7) is

(2.12) $a = \Lambda^{-1}Z'y$

and its mean, variance and MSE are, respectively, given as

(2.13) $Ea = \alpha$, $V(a) = \sigma^2 \Lambda^{-1}$

and

(2.14) $MSE(a) = \sigma^2 \sum_1^p 1/\lambda_i$.

Further, we note from (2.8) that

(2.15) $a = G'b$

such that

(2.16) $MSE(a) = MSE(b)$.

The generalized ridge estimator (GRE) of α in (2.7), given by Hoerl and Kennard [1970], is formed by adding $k_1,\ldots,k_p$, the "additive eigenvalue inflation factors" to $\lambda_1,\ldots,\lambda_p$. Specifically, it is written as

(2.17) $\hat{\alpha}_K = [\Lambda + K]^{-1} Z'y$

where K is a p x p diagonal matrix of constants as

(2.18) $K = \text{Diag.}\ [k_1, \ldots, k_p]$.

For $k_1 = k_2 = \ldots = k_p = k$ the ordinary ridge estimator (ORE) can be written as

(2.19) $\hat{\alpha}_k = [\Lambda + kI]^{-1} Z'y$

$= \Delta a$

where Δ is a p x p diagonal matrix as

(2.20) $\Delta = \text{Diag.}\ [\lambda_1/\lambda_1 + k, \ldots, \lambda_k/\lambda_p + k]$

and a is the OLS estimator given in (2.12).

Using (2.8) we can write the GRE and ORE, respectively, of β in (2.1) as

(2.21) $\hat{\beta}_K = G\hat{\alpha}_K = (X'X + GKG')^{-1} X'y$

and

(2.22) $\hat{\beta}_k = (X'X + kI)^{-1} X'y$.

2.1 Determination of k

Let us write the MSE of a_i, the i^{th} component of the OLS estimator a, from (2.12) as

(2.23) $MSE(a_i) = \sigma^2/\lambda_i$, $i=1,\ldots,p$.

Similarly, it can be easily verified that

(2.24) $MSE(\hat{\alpha}_{k_i}) = \dfrac{k_i^2\, \alpha_i^2}{(\lambda_i + k_i)^2} + \dfrac{\sigma^2\, \lambda_i}{(\lambda_i + k_i)^2}$

where $\hat{\alpha}_{k_i}$ is the i^{th} component of the GRE $\hat{\alpha}_K$. Thus

$$(2.25)\qquad MSE(\hat{\alpha}_{k_i}) - MSE(a_i) = \frac{k_i\lambda_i(k_i\alpha_i^2 - 2\sigma^2) - \sigma^2 k_i^2}{\lambda_i(\lambda_i + k_i)^2} .$$

It follows from (2.25) that for

$$(2.26)\qquad 0 < k_i < \frac{2\sigma^2}{\alpha_i^2 - \sigma^2\lambda_i^{-1}}$$

$MSE(\hat{\alpha}_{k_i})$ is smaller than $MSE(a_i)$. In fact a sufficient condition under which $MSE(\hat{\alpha}_{k_i}) < MSE(a_i)$ is that

$$(2.27)\qquad 0 < k_i < \frac{2\sigma^2}{\alpha_i^2} .$$

The upper bound of k_i in (2.27), however, is more conservative than in (2.26). An alternative conservative range of k_i, given in Hoerl and Kennard [1970], is

$$(2.28)\quad 0 < k_i < \frac{\sigma^2}{\alpha_i^2} .$$

It is to be noted that $k_i = \sigma^2/\alpha_i^2$ is the optimal value of k_i for which $MSE(\hat{\alpha}_{k_i})$ is minimum.

The k_i values in (2.26), (2.27) and (2.28) for which the GRE dominates OLS estimator can be written in a compact and general form as below:

$$(2.29)\qquad k_i = \frac{h_1\sigma^2}{\alpha_i^2 - h_2\sigma^2\lambda_i^{-1}} \qquad , \quad i=1,\ldots,p$$

where h_1 and h_2 are arbitrary scalars which could be stochastic or nonstochastic. For $0 < h_1 < 2$ and $h_2 = 1$ we get (2.26) and similarly (2.27) and (2.28) are special cases of (2.29).

In the case of ORE where $k_1 = \ldots = k_p = k$ we can determine k by considering the harmonic mean of k_i in (2.29). This can be verified as

$$(2.30) \qquad k = \frac{p\, h_1\, \sigma^2}{\alpha'\alpha - h_2\sigma^2\, \Sigma\, \lambda_i^{-1}}$$

where $\sigma^2\, \Sigma\, \lambda_i^{-1} = \mathrm{Tr}\, V(a)$, $V(a)$ is as given in (2.13).

Different choices of k suggested for ORE in the literature can be seen as special cases of (2.30). For example if $h_1 = 1$ and $h_2 = 0$, $k = p\sigma^2/\alpha'\alpha$, which is the harmonic mean of $k_i = \sigma^2/\alpha_i^2$. This is given by Hoerl, et al. [1975]. If $h_1 = 1/p$ and $h_2 = 0$ we get $k = \sigma^2/\alpha'\alpha$, which is suggested by Farebrother [1975]. Further, if $0 < h_1 < \frac{2}{p}$ and $h_2 = 0$ we have $0 < k < \frac{2\sigma^2}{\alpha'\alpha}$ as given by Theobald [1974]. It has been shown by Theobald that for this range $MSE(\hat{\alpha}_k)$ is smaller than MSE(a). Also, the range of k suggested by Hoerl and Kennard, i.e., $0 < k < \frac{\sigma^2}{\alpha_{max}^2}$ can be seen as a special case of (2.30).

We note that, since k in (2.30) depends on unknown parameters α and σ^2 it would not determine ORE $\hat{\alpha}_k$ or $\hat{\beta}_k$ in (2.19) and (2.22). However, we can use unbiased estimators a of α and s^2 of σ^2 as given in (2.12) and (2.5), respectively. The k in (2.30) can then be determined as

$$(2.31) \qquad \hat{k}_{h_1,h_2} = \hat{k} = \frac{p\, h_1\, s^2}{a'a - h_2 s^2\, \Sigma\, \lambda_i^{-1}}\ , \quad s^2 = \frac{1}{n}\,\hat{u}'\hat{u}, \quad \hat{u} = y - Za.$$

Again if $h_1 = 1$ and $h_2 = 0$ we get $\hat{k}$ as suggested by Hoerl, et al. [1975]. If $h_1 = \frac{1}{p}$ and $h_2 = 0$ we get $\hat{k}$ as given by Farebrother [1975]. Further, if $h_1 = \frac{1}{p}$ and $h_2 = a'(I - \Lambda)a/s^2\, \Sigma\, \lambda_i^{-1}$, which is stochastic, then we obtain $\hat{k}$ which is suggested by Lawless and Wang [1976] in the Bayesian framework.

2.2 A Family of Double h-Class ORE

If we substitute (2.31) in (2.19) we can write a family of operational ordinary ridge estimator (OORE) as

(2.32) $$\tilde{\alpha}_{h_1,h_2} = \hat{\Delta}\, a$$

where

(2.33) $$\hat{\Delta} = \text{Diag.}\ [\hat{\delta}_1,\ldots,\hat{\delta}_p]$$

such that for i=1,...,p

(2.34) $$\hat{\delta}_i = [1 - \frac{h_{1i}\ \hat{u}'\hat{u}}{a'a + h_{2i}\hat{u}'\hat{u}}]$$

and

(2.35) $$h_{1i} = \frac{h_1 p}{n\lambda_i}\ ;\quad h_{2i} = \frac{1}{n}\ [\frac{h_1 p}{\lambda_i} - h_2 \Sigma\ \lambda_i^{-1}]$$

We note that h_1 and h_2 are arbitrary scalars which could be stochastic or nonstochastic and the same is true for h_{1i} and h_{2i}. The estimator in (2.32) can be termed as a family of double h-class ordinary ridge estimators.

The i^{th} component of this double h-class estimators can be written as

(2.36) $$(\tilde{\alpha}_{h_1,h_2})_i = [1 - \frac{h_{1i}\ \hat{u}'\hat{u}}{a'a + h_{2i}\hat{u}'\hat{u}}]\ a_i\ .$$

If h_2 is stochastic and given by

(2.37) $$h_2 = \hat{h} = \frac{a'(I - \Lambda)a}{s^2\ \Sigma\ \lambda_i^{-1}}$$

then the i^{th} component of $\tilde{\alpha}_{h_1,h_2}$ can be written as

(2.38) $$(\tilde{\alpha}_{h_1,\hat{h}})_i = [1 - \frac{h_{1i}\hat{u}'\hat{u}}{a'\ \Lambda\ a + h_{1i}\hat{u}'\hat{u}}]\ a_i$$

This can be considered as a family of empirical Bayes estimator which depends on h_1 only.

If $h_2 = 0$ we get another set of estimators which depend on h_1 only; it is written as

$$(2.39) \qquad (\tilde{\alpha}_{h_1,0})_i = [1 - \frac{h_{1i}\hat{u}'\hat{u}}{a'a + h_{1i}\hat{u}'\hat{u}}] a_i$$

It is interesting to note that for $h_1 = 0$, (2.32) is an OLS estimator, i.e., $\tilde{\alpha}_{0,h_2} = a$. For $h_1 = 1$ and $h_2 = 0$, $\tilde{\alpha}_{1,0} = \tilde{\alpha}_{HKB}$ (HKB represents Hoerl, Kennard and Baldwin [1975] estimator) whose i^{th} component can be written as

$$(2.40) \qquad (\tilde{\alpha}_{HKB})_i = [1 - \frac{p\,\hat{u}'\hat{u}}{n\,\lambda_i a'a + p\,\hat{u}'\hat{u}}] a_i = \frac{\lambda_i a'a}{\lambda_i a'a + ps^2} a_i .$$

Next, for $h_1 = 1/p$ and $h_2 = 0$, $\tilde{\alpha}_{1/p,0}$ is the estimator suggested by Farebrother, FB, [1975]. Its i^{th} component can be written as

$$(2.41) \qquad (\tilde{\alpha}_{FB})_i = \frac{\lambda_i a'a}{\lambda_i a'a + s^2} a_i$$

Further, for $h_1 = 1$ and h_2 as given in (2.37), the i^{th} component of $\tilde{\alpha}_{1,\hat{h}}$ can be written as

$$(2.42) \qquad (\tilde{\alpha}_{LW})_i = (\tilde{\alpha}_{1,\hat{h}})_i = [1 - \frac{p\hat{u}'\hat{u}}{n\lambda_i a'\,\Lambda\,a + p\,\hat{u}'\hat{u}}] a_i$$

This is the estimator suggested by Lawless and Wang [1976].

Finally, if h_{1i} and h_{2i} are any arbitrary stochastic or nonstochastic values, not constrained by (2.35), then for $h_{1i} = c_1$ for all i=1,...p and $h_{2i} = a'(\Lambda - I)a/\hat{u}'\hat{u}$ for all i, the estimator $(\tilde{\alpha}_{h_1,h_2})_i$ is the James and Stein [1961] estimator in the regression context. This can be written as

$$(2.43) \qquad (\tilde{\alpha}_{JS})_i = [1 - \frac{c_1 \hat{u}'\hat{u}}{a'\Lambda \quad a}]a_i$$

The difference between the Stein-rule estimators and the double h-class ORE is then with respect to the coefficients of a_i for i=1,...,p. While in Stein-rule these are the same, in double h-class they differ. The MSE properties of the JS estimator, in the regression context, have been analysed extensively in Ullah and Ullah [1978], Baranchik [1964] and Zellner and Vandaele [1975] among others, therefore it will not be studied in this paper.

3. THE BIAS AND MSE OF ORE

In this section we shall analyze the exact and appromimate bias and MSE of the double h-class ORE $\tilde{\alpha}_{h_1,h_2}$ in two different cases. We first present results for the case when $h_2 = \hat{h}$ is stochastic and then more generally, when h_2 is any nonstochastic scalar. We note again that $\tilde{\alpha}_{LW}$ in (2.42) is a member of $\tilde{\alpha}_{h_1,\hat{h}}$, while $\tilde{\alpha}_{HKB}$ and $\tilde{\alpha}_{FB}$ in (2.40) and (2.41), respectively, are members of $\tilde{\alpha}_{h_1,0}$. Finally we point out that though both the exact and approximate results are presented for $\tilde{\alpha}_{h_1,\hat{h}}$, only approximate results using small σ-approach of Kadane [1970, 1971], are obtained for $\tilde{\alpha}_{h_1,h_2}$. This is because the exact results are complicated to be analyzed, although they can be obtained for $\tilde{\alpha}_{h_1,h_2}$ in the same way as for $\tilde{\alpha}_{h_1,\hat{h}}$.

3.1 The Exact Results for $\tilde{\alpha}_{h_1,\hat{h}}$

Firstly, we write the sampling error of the estimator in (2.38) as

$$(3.1) \qquad (\tilde{\alpha}_{h_1,\hat{h}} - \alpha)_i = (a - \alpha)_i - h_{1i} c_i a_i$$

where, using the fact that $a' \wedge a = y'y - \hat{u}'\hat{u}$,

$$(3.2) \qquad c_i = \frac{y' M y}{y' N_i y} ,$$

and M and N_i are both TxT matrices given as

$$(3.3) \qquad M = I - Z \wedge^{-1} Z' , \quad N_i = I - (1-h_{1i})M .$$

Further, M is an idempotent matrix with rank n and N_i is a non-negative definite matrix for $h_{1i} \geq 0$.

Secondly, according to Assumption 2 we observe that

(3.4) $y \sim N(\bar{y}, \sigma^2 I), \quad \bar{y} = Z\alpha$

From (3.1) we note that to derive the bias and MSE of $\tilde{\alpha}_{h_1,\hat{h}}$ we require the expectations of $c_i a_i$, $c_i^2 a_i^2$, etc. The conditions for the existence of these expectations and their values can be obtained from Ullah and Ullah [1978]. They are not reproduced here for the sake of space.

We now introduce the following notations and functions for the sake of simplicity of exposition:[3]

(3.5) $$g_{i,\mu,\nu} = G(1 - h_{1i}, \theta; \frac{T}{2} + \mu, \frac{n}{2} + \nu); \; \mu,\nu = 0,1,\ldots$$

$$= 2\int_{-\infty}^{0} \frac{\exp[2\ \theta\ t/1-2t]}{[1-2t]^{\frac{p}{2}+\mu-\nu}[1-2h_{1i}t]^{\frac{n}{2}+\nu}}$$

where

(3.6) $$\theta = \frac{\alpha' \Lambda \alpha}{2\sigma^2}$$

is a noncentrality parameter. The following results can then be stated:

<u>Theorem 1. The exact bias of the double h-class ORE of α for $h_1 > 0$ exists; it is given by</u>

(3.7) $$E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha) = -\frac{h_1 p}{2} g'_{2,1} \Lambda^{-1}\alpha$$

<u>where $g_{2,1}$ is the column vector of $g_{1,2,1},\ldots,g_{p,2,1}$; $g_{i,2,1}$ is as given in (3.5) for $\mu = 2$ and $\nu = 1$.</u>

<u>Note</u>: The i^{th} component of (3.7) can be written as

(3.8) $$E(\tilde{\alpha}_{h_1\hat{h}} - \alpha)_i = -n\frac{h_{1i}}{2} g_{i,2,1}\alpha_i$$

It can be shown that the relative bias in (3.8), for a given sample size, lies in the following range

$$- \frac{n\, h_{1i}}{T+2} \le E \frac{(\tilde{\alpha}_{h_1,\hat{h}} - \alpha)_i}{\alpha_i} \le 0 \quad , \quad i=1,\ldots,p \; .$$

Further, the relative bias is an increasing function of the noncentrality parameter θ.

<u>Theorem 2</u>. <u>The exact MSE of a component of the double h-class ORE of</u> α <u>for</u> $h_1 > 0$ <u>exists; it is given by</u>

$$(3.9) \quad E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha)_i^2 = \frac{\sigma^2}{\lambda_i}[1 - n\, h_{1i} g_{i,2,1} - h_{1i}^2 \frac{n(n+2)}{4}(g_{i,3,2} - g_{i,2,2})]$$

$$- \alpha_i^2 [n\, h_{1i}(g_{i,3,1} - g_{i,2,1}) + \frac{h_{1i}^2 n(n+2)}{4}(g_{i,4,2} - g_{i,3,2})] \; .$$

<u>where</u> $h_{1i} = h_1 p/n\lambda_i$ <u>as given in</u> (2.35).

<u>Note</u>:

$$(3.10) \qquad E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha)'(\tilde{\alpha}_{h_1,\hat{h}} - \alpha) = \sum_{i=1}^{p} E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha_i)^2 \; .$$

3.2 <u>Large</u> θ<u>-Asymptotic Expansion for</u> $\tilde{\alpha}_{h_1,\hat{h}}$

In the earlier section we were able to analyze the exact bias of $\tilde{\alpha}_{h_1,\hat{h}}$. However, the expression for MSE was too complicated to draw any meaningful result. To analyze MSE, we therefore obtain the asymptotic expansion in terms of the inverse of θ. We note, however, that the results below require sufficiently large θ, which according to (3.6) means sufficiently small σ^2. Thus, we consider the terms of order $1/\theta$, $1/\theta^2$ as the terms of order σ^2 and σ^4, respectively.

The following results can now be stated.[4]

Theorem 3. The asymptotic expansion of the bias of the double h-class ORE estimator of α in (3.7) up to order $1/\theta$ is given by

$$(3.11) \qquad E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha) = -\frac{h_1 p}{2\theta}\Lambda^{-1}\alpha$$

when $h_1 > 0$.

Theorem 4. The asymtotic expansion of the MSE of a component of the double h-class ORE of α in (3.9) for $h_1 > 0$, up to order $1/\theta^2$, is given by

$$(3.12) \qquad E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha)_i^2 = \frac{\sigma^2}{\lambda_i} + \frac{nh_{1i}}{4\theta^2}[\alpha_i^2\{4 + h_{1i}(n+2)\} - 2\frac{\alpha'\Lambda\alpha}{\lambda_i}]$$

where $h_{1i} = h_1 p/n\lambda_i$.

Note: Using (3.12), the MSE of $\tilde{\alpha}_{h_1,\hat{h}}$ can be written as

$$(3.13) \qquad E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha)'(\tilde{\alpha}_{h_1,\hat{h}} - \alpha) = \sigma^2 T_r\Lambda^{-1} + \frac{h_1 p}{4\theta^2}[4\alpha'\Lambda^{-1}\alpha + h_1\frac{p(n+2)}{n}\alpha'\Lambda^{-2}\alpha - 2\alpha'\Lambda\alpha T_r\Lambda^{-2}]$$

or alternatively as

$$(3.14) \qquad E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha)'(\tilde{\alpha}_{h_1,\hat{h}} - \alpha) = \sigma^2 T_r\Lambda^{-1} + \frac{h_1 p}{4\theta^2}\alpha'\Lambda^{-2}\alpha[h_1\frac{p(n+2)}{n} - 2\frac{\alpha'\Lambda^{-1}\alpha}{\alpha'\Lambda^{-2}\alpha}\{\frac{\alpha'\Lambda\alpha}{\alpha'\Lambda^{-1}\alpha}T_r\Lambda^{-2} - 2\}].$$

Corollary. The double h-class estimator of α in (3.7) dominates ordinary least squares estimator a in (2.12) in large θ asymptotics up to the order $1/\theta^2$, in the sense that

$$(3.15) \qquad \lim_{\theta\to\infty} \theta^2[E(\tilde{\alpha}_{h_1,\hat{h}} - \alpha)'(\tilde{\alpha}_{h_1,\hat{h}} - \alpha) - E(a - \alpha)'(a - \alpha)] < 0$$

for

$$(3.16) \qquad 0 < h_1 \le \frac{2n\lambda_m}{p(n+2)}(d - 2); \quad d = (\sum_{i=1}^{p} 1/\lambda_i^2)\lambda_m^2 > 2$$

where λ_m is the minimum value of $\lambda_1, \ldots, \lambda_p$.[5]

3.3 Large θ - Asymptotic Expantion for $\tilde{\alpha}_{h_1,h_2}$

Let us write the sampling error of the estimator in (2.41) as

$$(3.17) \qquad (\tilde{\alpha}_{h_1,h_2} - \alpha)_i = (a - \alpha)_i - h_{1i} d_i a_i$$

where

$$(3.18) \qquad d_i = \frac{y'My}{y'R_i y},$$

$M = I - Z\Lambda^{-1}Z'$ and

$$(3.19) \qquad R_i = Z\Lambda^{-2}Z' + h_{2i}M.$$

If we compare (3.17) from (3.1) we can observe that $\tilde{\alpha}_{h_1,\hat{h}}$ and $\tilde{\alpha}_{h_1,h_2}$ differ only with respect to the matrices N_i and R_i. Thus, one could obtain the exact moments of $\tilde{\alpha}_{h_1,h_2}$ by using the results in the Appendix of Ullah and Ullah [1978], though it may not be as straight forward as in the case of $\tilde{\alpha}_{h_1,\hat{h}}$. However, since the exact moments would again be in terms of the complicated mathematical functions from which any useful result is difficult to be obtained, we present here the asymptotic expansion of the moments by using Kadane's small σ expansion approach.[6] The following theorems can therefore be stated for $h_{2i} > 0$. We require $h_{2i} > 0$ for the existence of the moments.

Theorem 5. The asymptotic expansion of the bias of the double h-class ORE $\tilde{\alpha}_{h_1,h_2}$ of α, up to order $1/\theta_1$ is given by

$$(3.20) \qquad E(\tilde{\alpha}_{h_1,h_2} - \alpha) = -\frac{h_1 p}{2\theta_1}\Lambda^{-1}\alpha$$

where $\theta_1 = \alpha'\alpha/2\sigma^2$.

Theorem 6. The asymptotic expansion for a component of the MSE of the estimator $\tilde{\alpha}_{h_1,h_2}$, up to order σ^4, is given by

$$(3.21)\qquad E(\tilde{\alpha}_{h_1,h_2} - \alpha)_i^2 = \frac{\sigma^2}{\lambda_i} + \frac{\sigma^4 h_{1i}}{\alpha'\alpha}\left[h_{1i}\frac{n(n+2)}{\alpha'\alpha}\alpha_i^2 + \frac{2n}{\lambda_i}\left(2\frac{\alpha_i^2}{\alpha'\alpha} - 1\right)\right]$$

Note: Using (3.21), the MSE of $\tilde{\alpha}_{h_1,h_2}$ up to the order $1/\theta_1^2$ (or σ^4) can be written as

$$(3.22)\qquad E(\tilde{\alpha}_{h_1,h_2} - \alpha)'(\tilde{\alpha}_{h_1,h_2} - \alpha) = \sigma^2 T_r \Lambda^{-1} + \frac{h_1 p}{4\theta_1^2}\left[4\alpha'\Lambda^{-2}\alpha + h_1\frac{p(n+2)}{n}\alpha'\Lambda^{-2}\alpha - 2\alpha'\alpha \,\mathrm{Tr}\,\Lambda^{-2}\right]$$

where θ_1 is as defined in Theorem 5.

Corollary. The double h-class ORE $\tilde{\alpha}_{h_1,h_2}$ dominates ordinary least squares estimator a in large θ asymptotics up to the order $1/\theta_1^2$, in the sense that

$$(3.23)\qquad \lim_{\theta_1\to\infty} \theta_1^2[E(\tilde{\alpha}_{h_1,h_2} - \alpha)'(\tilde{\alpha}_{h_1,h_2} - \alpha) - E(a - \alpha)'(a - \alpha)] < 0$$

for

$$(3.24)\qquad 0 < h_1 \le \frac{2n}{p(n+2)}(d - 2), \quad d > 2$$

where d is as given in (3.16).[7]

3.4 Conclusion: In Section 3 we have first analysed the bias and MSE of $\tilde{\alpha}_{h_1,h_2}$ when $h_2 = \hat{h}$ is a stochastic variable and then when h_2 is any nonstochastic scalar. The h_1 in both cases was a nonstochastic scalar. For $h_2 = \hat{h}$, $\tilde{\alpha}_{h_1,\hat{h}}$ is a set of empirical Bayes estimators which belong to $\tilde{\alpha}_{h_1,h_2}$. Since the form of $\tilde{\alpha}_{h_1,\hat{h}}$ gets slightly different from that of $\tilde{\alpha}_{h_1,h_2}$ the results are presented separately.

With respect to $\tilde{\alpha}_{h_1,\hat{h}}$ we have noted in the corollary that it dominates the OLS estimator for the values of h_1 in

$$0 < h_1 \leqslant \frac{2n\lambda_m}{p(n+2)}(d - 2), \quad d > 2,$$

where d, as given in (3.16), can be considered as a measure of overall multicollinearity in the data matrix.

In the case of zero multicollinearity d=p and when there is a high degree of collinearity d is close to zero. The range of h_1 given above suggests that even for moderate multicollinearity (d > 2) there exist estimators $\tilde{\alpha}_{h_1,\hat{h}}$, whose MSE are smaller than the OLS. We note here that the estimator $\tilde{\alpha}_{1,\hat{h}} = \tilde{\alpha}_{LW}$, which is a member of $\tilde{\alpha}_{h_1,\hat{h}}$, would dominate OLS if, for given n, the data matrix is such that the upper bound of h_1 is at least one. Thus the estimator $\tilde{\alpha}_{1,\hat{h}}$ may not always do better than OLS. We therefore suggest that h_1 could be chosen from the above mentioned range since this ensures a lower MSE as compared to OLS. In fact, the simulation study in the following section indicates that $\tilde{\alpha}_{h_1,\hat{h}}$ for a choice of $h_1 = n\lambda_m(d - 2)/p(n + 2)$ performs better than OLS as well as $\tilde{\alpha}_{1,\hat{h}}$. The optimal h_1 in the given range, however, remains to be a subject of future studies.

It is interesting to note that in the case of zero multicollinearity, i.e., when d = p the range of h_1 in (3.16) reduces to

$$(3.25) \qquad 0 < h_1 \le \frac{2n(p-2)}{(n+2)p}, \quad p \ge 3$$

Now with regard to $\tilde{\alpha}_{h_1,h_2}$ we have shown in the corollary that it dominates the OLS estimator for the values of h_1 in

$$0 < h_1 \le \frac{2n}{p(n+2)}(d - 2)$$

and for any nonstochastic h_2 for which h_{2i} in (2.35) remains positive. Thus $\tilde{\alpha}_{h_1,0}$ also dominates OLS. Considering the two cases $h_1 = 1$ and $h_1 = 1/p$ we observe that, for some data matrices they may not be in the range of h_1 given above. Thus $\tilde{\alpha}_{1,0} = \tilde{\alpha}_{HKB}$ and $\tilde{\alpha}_{1/p,0} = \tilde{\alpha}_{FB}$ might not always dominate the OLS estimator. However, since $h_1 = 1/p < 1$ gives shorter range for k as compared to $h_1 = 1$, $\tilde{\alpha}_{FB}$ would be more conservative than $\tilde{\alpha}_{HKB}$. Like in the case of $\tilde{\alpha}_{h_1,\hat{h}}$ the simulation study indicates that $\tilde{\alpha}_{h_1,0}$ for a choice of $h_1 = n(d - 2)/p(n + 2)$ performs better than both OLS and $\tilde{\alpha}_{1,0}$.

Comparing $\tilde{\alpha}_{h_1,0}$ with $\tilde{\alpha}_{h_1,\hat{h}}$ we find that $\tilde{\alpha}_{h_1,\hat{h}}$ would be more or less conservative than $\tilde{\alpha}_{h_1,0}$, depending on the value of λ_m. Since more often λ_m is small, $\tilde{\alpha}_{h_1,\hat{h}}$ would be more conservative. In the special case when $h_1 = 1$, $\tilde{\alpha}_{1,0} = \tilde{\alpha}_{HKB}$ would be better than or worse than $\tilde{\alpha}_{1,\hat{h}} = \tilde{\alpha}_{LW}$ depending upon the values of $\lambda_1,\ldots,\lambda_p$. This follows by comparing the expressions of MSE for $\tilde{\alpha}_{1,0}$ and $\tilde{\alpha}_{1,\hat{h}}$ from (3.22) and (3.14), respectively.

4. A SIMULATION EXPERIMENT

A simulation study was conducted in order to draw some further conclusions about the comparative performance (MSE) of the estimators listed in Table 1 with appropriate references over a wide class of regression problems. Since there is no need to merely verify mathematical proofs the setting of the simulation study is chosen to be intuitively appropriate for regression problems.

Two data sets are used. One is a four regressor ($p=4$, $n=13$) structure due to Hald, and second is a ten-factor ($p=10$, $n=36$) structure due to German and Toman used by Hoerl and Kennard [1970], and many researchers for illustrating ridge regression. This simulation is similar to Hoerl, Kennard and Baldwin [1975], and Vinod [1976 and 1977].

For the purpose of our simulation we consider the following transformation of (2.7)

$$(4.1) \qquad y = Z\alpha + u = H\Lambda^{\frac{1}{2}}\alpha + u = H\eta + u$$

where the regressors are orthogonal: $H'H = I$. Let $L^2 = \eta'\eta = \alpha'\Lambda\alpha$ denote the squared length of the parameter vector $\eta = \Lambda^{\frac{1}{2}}\alpha$. Let θ denote a noncentrality parameter defined by $2\theta = L^2/\sigma^2$. Eight values of θ in the interval (1,450) are chosen. It may be argued that a large part of the action insofar as which of the estimators dominates OLS is taking place on a "sphere" selected to have the same L^2 and σ^2. Hence we select $\sigma^2 = 1$ and $L^2 = 2\theta$ in all our experiments.

Given L^2 we choose p random numbers from a uniform distribution in the range -1 to $+1$ and call them w_i. Then $\eta_i = Lw_i/\left(\Sigma w_i^2\right)^{\frac{1}{2}}$ satisfy $L^2 = \eta'\eta$. Thus, the true η_i are selected from a scaled unit cube. This conforms with the practitioner's notion that the true regression coefficients can be anywhere in the selected range. The alternative procedure of selecting η_i from a scaled unit ball of radius L was not adopted.

Next, a vector u of T random normal deviates with mean zero and variance σ^2 is created using the so called "super duper" random number generator developed at the McGill University, Canada, by G. Marsaglia et al. [1973]. The T values of the dependent variable are found by $y = H\eta + u$ of (4.1). The OLS estimate is $\eta^0 = (H'H)^{-1}H'y = H'y$. Next, the squared length of y, $y'y$, is computed for each case. The squared multiple correlation coefficient is $R^2 = \eta^{0'}H'H\eta^0/y'y = \eta^{0'}\eta^0/y'y$.

For a fixed set of true η_i values 500 sets of random y vectors were created. For each y the weighted sum of squares of error is computed as:

$$\mathrm{WSSE}[b^{(\cdot)}] = \sum_{i=1}^{p}\left(b_i^{(\cdot)} - \beta_i\right)^2 \lambda_i^{-1}$$

where $b^{(\cdot)}$ represents one of five estimators. The first is the usual OLS estimator. Second, HKB is Hoerl, Kennard and Baldwin's [1975] $k = ps^2/a'a$, i.e., $h_1 = 1$ and $h_2 = 0$. Third is LW for Lawless and Wang's [1976] choice $h_1 = 1$ and $h_2 = \hat{h}$. The fourth is Empirical Bayes (EB) estimator of equation (2.38) having $h_1 = n\lambda_m(d - 2)/(pn + 2p)$, where d is from (3.16) and $h_2 = \hat{h}$ from LW. Fifth is a double-h class (DH) estimator with $h_1 = n(d - 2)/(pn + 2p)$, and $h_2 = 0$.

The $\mathrm{MSE}(b^{(\cdot)})$ is then the simple average of $\mathrm{WSSE}(b^{(\cdot)})$ for our 500 replications. To assess the variability of $\mathrm{WSSE}(b^{(\cdot)})$ over 500 replications their standard deviation, $\mathrm{SDE}(b^{(\cdot)})$, was computed.

In Tables 1 and 2 we report the results for the structure having $p = 4$. The $\mathrm{MSE}(b^{(\cdot)})$ is reported for each $b^{(\cdot)}$ in Table 1. There are eight rows associated with each of the eight θ values. A mark (✓) is included in the Table for the lowest MSE in each row.

It is clear that OLS is a poor estimator for low θ values, and the "room" for improvement over OLS becomes less and less as θ increases. In Table 1 the double h-class estimator DH wins the most number of (✓) marks. For low θ values the HKB or LW estimators have lower MSE than the proposed EB or DH estimators. However, for large θ values the MSE for EB and DH estimators remains always lower than that of OLS, which is not true for HKB or LW. Also for large θ values EB and DH estimators perform better compared to HKB or LW. These findings are consistent with our theoretical conclusions in Section 3.4. Table 2 gives the percentage of Times MSE of various estimators is strictly less than that of OLS. The mark (✓) is placed on the estimator with the highest percentage. Again it is clear that the chances of having higher MSE than OLS with the HKB or LW estimators in a given problem are large. By contrast, our EB or DH estimators improve upon OLS more often. The results for the ten-factor example are essentially similar and omitted to save space.

To assess whether the reduction in MSE by using HKB, LW, EB or DH rather than OLS is merely due to sampling variability we compute SDE

$(\tilde{b}^{(\cdot)})/(500)^{\frac{1}{2}}$ for several MSE($\tilde{b}^{(\cdot)}$) values. Detailed reporting of these results seems unnecessary. We find that SDE values for various $\tilde{b}(\cdot)$ are adequately summarized by those for OLS reported in the last column of Table 1 entitled variability. Had we simulated 10000 times we would have SDE/100 as a measure of variability which is an order of magnitude smaller than ours. However, the cost of such a large sample would be too high. A more ambitious simulation would have varied λ_i over a wider range of possibilities. It is hoped that our simulation does give at least a limited guidance about what might be expected in a practical (regression) settling.

Table 1

Mean Squared Error for p=4 Structure *

θ	OLS $a=\tilde{\alpha}_{0,h_2}$	HKB $\tilde{\alpha}_{1,0}$	LW $\tilde{\alpha}_{1,\hat{h}}$	EB $\tilde{\alpha}_{h_1,\hat{h}}$	DH $\tilde{\alpha}_{h_1,0}$	variability
0.5	4.51	1.32	1.31✓	3.95	3.79	.143
4.5	4.09	3.11	3.10✓	3.92	3.86	.130
12.5	4.21	4.24	4.20	4.21	4.19	.129
24.5	4.01	3.97	3.91✓	4.04	4.02	.141
40.5	4.09	4.42	4.45	4.07✓	4.07✓	.133
100	4.37	4.55	4.61	4.35✓	4.35✓	.129
200	4.35	4.53	4.61	4.34✓	4.34✓	.139
450	4.42✓	4.47	4.46	4.42✓	4.42✓	.142

* NOTE: The eigenvalues (λ_i, i=1,...,4) are 1.4, 1, 0.9 and 0.7. These imply less severe multicollinearity than in the original problem and satisfy d>2, where d is defined in equation (3.16). 2θ has been named as signal to noise ratio in the earlier simulation studies on Ridge estimators.

Table 2

Percent of Time MSE of Various Estimators Strictly Less than OLS in 500 Samples

θ	HKB $\tilde{\alpha}_{1,0}$	LW $\tilde{\alpha}_{1,\hat{h}}$	EB $\tilde{\alpha}_{h_1,\hat{h}}$	DH $\tilde{\alpha}_{h_1,0}$
.5	97	97	99 ✓	99 ✓
4.5	66	66	83 ✓	82 ✓
12.5	55	56	71 ✓	70
24.5	56	57	68 ✓	68 ✓
40.5	48	47	59 ✓	59 ✓
100	49	48	57 ✓	57 ✓
200	47	45	54 ✓	54 ✓
450	45	45	49 ✓	49 ✓

Footnotes

[1] For further detail see Vinod [1978].

[2] The name h-class is reminiscent of Theil's [1971] choice of the notation h instead of Hoerl and Kennard's k.

[3] Alternatively the G( ) function is written as (See Sawa [1972, p. 678], and Ullah and Ullah [1978]),

$$G(k, n; a,c) = c^{-n} \frac{\Gamma(a-1)}{\Gamma(c)} \sum_{h=0}^{\infty} (k)^h \frac{\Gamma(c+h)}{\Gamma(a+h)} {}_1F_1(a-1;a+h,n)$$

where ${}_1F_1(\ )$ is the well known confluent hypergeometric function (see Slater [1960]).

[4] All the results follow by noting that $\sigma^2 = \alpha'\Lambda\alpha/2\theta$, which is of order $1/\theta$ in magnitude and using the asymptotic expansion formula for G( ) as (See Sawa [1972, p. 667]),

$$G(k, n; a, c) = \frac{1}{n} + (ck-a+2)\frac{1}{n^2} + [c(c+1)k^2 -$$

$$- 2c(a-2)k + (a-2)(a-3)]\frac{1}{n^3} \quad .$$

[5] In arriving to this result we have used the result that

$$\underset{\alpha}{\text{Min. of}} \ \frac{\alpha'A\alpha}{\alpha'B\alpha}$$

is the min. value of $|A - \lambda B| = 0$, see Rao [1973, p.74].

6 The proofs and detail are not presented for the sake of space but can be obtained from the authors on request.

7 We note that the values of h_1 for which $\tilde{\alpha}_{h_1,h_2}$ and $\tilde{\alpha}_{h_1,\hat{h}}$ dominate the OLS estimator differ with the value of the corresponding scalar c_1 in the James and Stein estimator given in (2.43). In fact, for the James and Stein estimator to dominate the OLS estimator we require $0 < c_1 < \frac{2(d^*-2)}{(n+2)}$, where $d^* = (\sum_1^p 1/\lambda_i)\lambda_m > 2$ (see e.g., Ullah and Ullah [1978], and Judge and Bock [1976]).

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