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ABSTRACT

This paper shows conditions and procedures for deriving direct utility functions (D.U.F.) from indirect utility functions (I.U.F.) - provided that the latter can be written explicitly. For the fractional expenditure allocation model (F.E.A.M.) the I.U.F. appears to become a fairly simple integral.

Derivation of D.U.F. from the I.U.F., however, appears to impose rather severe restrictions on parameters of the F.E.A.M., if not the number of commodities distinguished.

For the "power" version of the F.E.A.M., this implies either constraints on the number and different values of the "power" parameters or the requirement that one of them become 1; mutatis mutandis, the same applies to the generalized version of that model, viz. cross-bred with the linear expenditure system.

In other cases approximate but still acceptable analytical D.U.F. might be derived provided that this would require but minor adjustments in one or more parameter estimates.

Contents

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Preliminary and confidential

1. Preliminaries

1.1. On the utility of direct analytical utility functions

The main purpose of this paper is to present analytical utility functions underlying a particular kind of "proper" consumer demand functions. In this context, "proper" means: demand functions (to be) derived from maximizing utility (in terms of quantities consumed per period) subject to a budget constraint. In any case, this implies additivity of the demand equations - identically summing up to total consumption -, hence incorporating all prices and total consumption (or "income"), and such that zero-homogeneity of those functions in these arguments is ensured. Indeed, some of the popular demand functions, such as the Klein-Rubin-Samuelson-Stone Linear Expenditure System (L.E.S.), the Intriligator-Gamaletsos Generalized Linear Expenditure System (G.L.E.S.) or the Quadratic Utility (Q.U.) set of demand equations, have been, or can be derived in that way.

Some authors - covertly if not openly - apparently consider the explicit knowledge of direct (analytical) utility functions underlying demand models as one of their attractive properties, making them superior to others (cf. Gamaletsos 1973). Complementarily, inability to produce analytical, direct utility functions as the foundation of demand functions looks like a lingering stigma on the presumably illegitimate birth of the latter. In particular, this applies to an important class of demand functions competing with those formerly mentioned for empirical application, viz. the Fractional Expenditure Allocation Model (F.E.A.M.) (cf., inter alia, Somermeyer a.o. (1956)); a sub-class of these models has also become known as the Indirect Addilog Expenditure System (I.A.E.S.)(cf. Houthakker (1960)).

Conversely, the latter type of model exemplifies the severity of the restrictions which the requirement of the existence of analytical expressions for direct utility functions appears to impose on the admitted class of demand functions resulting from the formers' conditional maximization.

Still, concern about the form of the direct utility functions is not without its justifications. In particular, one should ensure that the marginal utilities remain positive, at least within the relevant sub-space of the quantity vectors, i.e. corresponding to optimum allocation of consumption. A paper by Van Driel (1974) shows that this may require "breaks" in the utility function, at least for the F.E.A.M.

1.2. Purpose of the present paper

The main - and ultimate - purpose of the present paper is to show that and how direct utility functions (D.U.F.) of the fractional expenditure allocation model (F.E.A.M.) can be derived and expressed analytically - provided that a number of conditions are satisfied. These requirements appear to be severely restrictive. Still, their scope is wider than has been indicated in a previous paper by Somermeyer (1973), as will be outlined in section 1.3 and elaborated upon in section 4.3. Meanwhile, for a few particular specifications of the F.E.A.M. van Driel (1974) already presented analytical D.U.F.

1.3. Line of reasoning

As well-known, indirect utility functions (I.U.F.) can be specified (analytically) more easily and under less restrictive requirements than direct utility functions; Gorman (1961) even states that for the former possibility it is necessary and sufficient that the integrability conditions be satisfied.

For this reason, it seems appropriate to choose I.U.F. as a starting point for the derivation of D.U.F. In particular, this applies to a particular ("power") specification of the E.A.M., for which the relativity simple corresponding I.U.F. gave Houthakker (1960) cause to label this model - originally proposed by Leser (1941) - as the indirect addi-log expenditure system (I.A.E.S.).

Since the I.U.F., like demand, is zero-homogeneous in prices and income, hence reads in terms of price-income ratios, these functions can serve as stepping stones on the way to deriving the D.U.F., provided that the "inverse" of the allocation problem, viz. expressing those ratios in terms of quantities consumed, can be solved. For the F.E.A.M., this can be done provided that particular conditions are imposed on parameters of the model.

Consequently, section 2 presents the F.E.A.M., first in a general form (section 2.1), and next by a number of particular, operationally useful, specifications (section 2.2). Section 3 deals with the I.U.F. In section 3.1 the reader is reminded of some prime properties of the I.U.F., required for deriving a lemma on I.U.F. relating to the general F.E.A.M. (in section 3.2), with applications to specific F.E.A.M. (in section 3.3).

Section 4.1 outlines the derivation of D.U.F. from I.U.F. in general, with reference to linear and hyperbolic specification of the F.E.A.M. basic functions in section 4.2, and ditto for pertinent power specifications in section 4.3. For the latter two (special) cases are distinguished, viz.:

a) where the "powers" in the basic functions assume values that relate to each other in a limited number of simple proportions, and
b) where one (and only one) of these powers assumes the value of 1.

These cases are dealt with in (sub-)sections 4.3.1 and 4.3.2, respectively.

Finally, in section 5, suggestions are made for deriving approximate but still acceptable analytical D.U.F., provided that the distribution of "power" parameters is appropriate for this purpose.

2. Fractional expenditure allocation model

2.1. General form

The allocation model for consumer expenditures may be expressed in a general manner by:

$$(2.1.1)\quad q_k = \frac{CF_k(p_k,C)}{p_k \sum_{h=1}^{K} F_h(p_h,C)} \qquad \text{for } k = 1, \ldots, K,$$

with:

q_k = quantities consumed of commodities k,

p_k = their corresponding prices,

while:

C represents the total amount spent on consumption - by an individual, within a specific period, and

F_k denote functions of p_k and C.

Evidently, (2.1.1) satisfies the additivity condition:

$$(2.1.2)\quad C = \sum_{k=1}^{K} p_k q_k .$$

In order to ensure the logical requirement:

$$(3.1.3)\quad q_k \geq 0 \text{ for all positive vectors } (p_1 \ldots p_K,C) \text{ and for all } k,$$

$$(2.1.4)\quad F_k(p_k,C) \geq 0$$[1]

should hold good, with the equality sign applying to at most K - 1 items k.

In order that the allocation of C according to (2.1.1) is optimal in the sense that this system of demand equations results from maximizing a utility function:

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[1] Or, equivalently, of course $F_k(p_k,C) \leq 0\ \forall k$ under the same conditions.

(2.1.5) $u = u(q_1, \ldots, q_K)$

subject to the budget constraint (2.1.2), functions F_k should satisfy:

(2.1.6) $F_k(p_k, C) = f_k(r_k)$,

with $r_k = p_k/C$ (cf. Somermeyer a.o. (1962), (1972)).

Substitution of (2.1.6) into (2.1.1) yields:

(2.1.7) $$q_k = \frac{f_k(r_k)}{r_k \sum_{h=1}^{K} f_h(r_h)}.$$

Of course, the f_k should be specified in such a manner that also the second-order conditions for a constrained utility maximum are complied with.

2.2. Specific forms

Inter alia, the following specifications of f_k meet the requirements stated in section 2.1:

(2.2.1) $f_k(r_k) = a_{1k} + b_{1k} r_k \quad (a_{1k}, b_{1k} \geq 0)$: linear

(2.2.2) $f_k(r_k) = a_{2k} + b_{2k} r_k^{-1} \quad (a_{2k}, b_{2k} \geq 0)$: hyperbolic

(2.2.3) $f_k(r_k) = c_k r_k^{\alpha_k} \quad (c_k \geq 0,\ \alpha_k \leq 1)$: power

(2.2.4) $f_k(r_k) = d_k \exp(\beta_k r_k) \quad (d_k \geq 0,\ \ _k \leq 0)$: exponential

Also "mixed" specifications are feasible; examples are:

(2.2.5) $f_k(r_k) = a_k + b_{1k} r_k + b_{2k} r_k^{-1} \quad (a_k, b_{1k}, b_{2k} \geq 0)$:

(2.2.6) $f_k(r_k) = e_k r_k^{\alpha_k} \exp(\beta_k r_k) \quad (e_k \geq 0,\ \ _k \leq 1,\ \ _k \leq 0)$;[2]

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[2] The latter form has been used, inter alia, by Athanasopoulos (1962).

the inequality parts of the $\geq$ signs stated above should hold good for at least one of the items k[3]. Because of the zero-homogeneity of the q_k in terms of the f_k, f_h-functions, the coefficients in their specifications above are determinate but for an arbitrary multiplicative factor.

Properties and the ensueing advantages and disavantages of adopting these alternative forms are dealt with by Somermeyer (1973). The properties of the E.A.M. with the "power" specification (2.2.3) are more elaborately discussed and illustrated by Somermeyer and Langhout (1972).

3. Indirect utility functions (I.U.F.)

3.1. Definition and properties

Indirect utility functions are defined by:

(3.1.1) $u^* = u^*(p_1, \ldots, p_K, C)$.

Conceptually, they may be derived from the D.U.F. by substitution of the general demand functions:

(3.1.2) $q_k = q_k(p_1, \ldots, p_K, C)$

into (2.1.5).

Since demand functions (3.1.2) are homogeneous of degree zero in their arguments - as exemplified by (2.1.7) -, I.U.F. (3.1.1) may also be written as:

(3.1.3) $u^* = u^*(r_1, \ldots, r_K)$.

These functions imply:

(3.1.4) $q_k = - \frac{\partial u^*}{\partial p_k} / \frac{\partial u^*}{\partial C}$ (cf. Roy (1943)).

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[3] For a more precise specification of the conditions imposed on the parameters, see Van Driel (1974, section 4.7). That one and only one of the α_k is allowed to equal 1 exactly, derives from the fact that the semi-negative-definite Slutsky matrix corresponding to the "power" specification (2.2.3) has rank K-1, i.e. one less than its number of rows or columns. Section 4.3.2 makes use of this property.

3.2. I.U.F. corresponding to the general F.E.A.M.

The latter property comes in handy for deriving indirect utility functions corresponding to the expenditure allocation functions (2.1.7).

First, we reduce the partial derivatives of u* with respect to the p_k and C separately to derivatives with respect to their ratios r_k only:

$$(3.2.1) \quad \frac{\partial u^*}{\partial p_k} = \frac{\partial u^*}{\partial r_k} \cdot \frac{\partial r_k}{\partial p_k} = C^{-1} \frac{\partial u^*}{\partial r_k}, \text{ and}$$

$$(3.2.2) \quad \frac{\partial u^*}{\partial C} = \sum_{h=1}^{K} \frac{\partial u^*}{\partial r_h} \cdot \frac{\partial r_h}{\partial C} = -C^{-1} \sum_{h=1}^{K} r_h \frac{\partial u^*}{\partial r_h}.$$

In order that the last member of (3.2.1) equals $-q_k$ times the last member of (3.2.2) - as required by (2.2.4):

$$(3.2.3) \quad \frac{\partial u^*}{\partial r_h} = \nu r_h^{-1} f_h(r_h) < 0,$$

with ν a negative but otherwise arbitrary factor, should hold good for all h.

Hence:

$$(3.2.4) \quad u^* = \sum_{h=1}^{K} \int \frac{\partial u^*}{\partial r_h} \cdot dr_h = -\sum_{h=1}^{K} \int r_h^{-1} f_h(r_h) \cdot dr_h,$$

with ν equated to -1[4] provided that the integral exists.

This means that analytical expressions for the indirect utility functions can be obtained if the f_h are specified in such a manner that the indefinite integral in the last member of (3.2.4) can be written explicitly.

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[4] This may be done without any loss of generality, since the demand functions resulting from maximizing utility subject to a budget constraint are invariant against monotonically increasing transformation of the (direct or indirect) utility function. For the same reason, the integrals in (3.2.4) need not - and should not - be definite.

3.3. I.U.F. corresponding to some specific F.E.A.M. models

Application of (3.2.4) to specifications (2.2.1) through (2.2.3) yields:

$$(3.3.1)\quad u^* = -\sum_{k=1}^{K} (a_{1k} \ln r_k + b_{1k} r_k)$$

corresponding to linear functions (2.2.1);

$$(3.3.2)\quad u^* = -\sum_{k=1}^{K} (a_{2k} \ln r_k - b_{2k} r_k^{-1}),$$

corresponding to hyperbolic functions (2.2.2); and

$$(3.3.3)\quad u^* = -\sum_{k=1}^{K} c_k \left\{ \alpha_k^{-1} r_k^{\alpha_k} (1 - \delta_{\alpha_k,o}) + \ln r_k \cdot \delta_{\alpha_k,o} \right\},^{5}$$

with $\delta_{\alpha_k,o} = 1$ for $\alpha_k = 0$

$= 0$ for $\alpha_k \neq 0$,

and the indeterminateness of $\alpha_k^{-1}(1-\delta_{\alpha_k,o})$ for $\alpha_k = 0$ to be raised according to the value(s) of $\alpha_{k'}$, $(k' \neq k)$, corresponding to power functions (2.2.3).

For the exponential f_k-specification (2.2.4), however, the integration implied by (2.3.4) cannot, in general, be carried out analytically.

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[5] Cf. also Houthakker (1960), who does not consider, however, the possibility $\alpha_k = 0$ for one or more k.

4. Direct utility functions

4.1. Uniqueness of the derivation of the D.U.F. from the I.U.F.

In principle, the I.U.F. (2.3.4) can serve as stepping stones for the derivation of the direct utility functions if the r_k can be expressed in terms of the q_k, i.e. "solved" from (2.1.7), and substituted into (3.2.4), or any of the subsequent more specific expressions. Actually, however, such a procedure will succeed, i.e. yield analytical D.U.F. if and only if particular conditions are satisfied.

A general condition is the existence of a one-to-one relationship between the q's and the r's, i.e. such that one and only one relevant (to wit: positive) r-vector corresponds to any given positive q-vector, like one and only one positive q-vector corresponds to any positive r-vector according to (2.1.7). Uniqueness of the relevant solution of the r-vector in terms of the $q_1 \ldots q_K$ is ensured if the functions:

$$(4.1.1) \quad g_k(r_k) = r_k^{-1} f_k(r_k)$$

are all monotonic and all in the same direction, i.e. that the g_k are either all monotonically increasing or all monotonically decreasing, for all k.

Indeed all four specifications (2.2.1) through (2.2.4), subject to the restrictions imposed on their parameters, imply the like-directed monotonicity of the functions (4.1.1) for $r_k > 0$; hence, at least in those cases, the "one-to-one-ness" of the relationships between the r-vector and the q-vector is ensured.

4.2. Linear and hyperbolic specifications of f_k

First, (2.1.7) is specified further by means of (2.2.1), yielding:

$$(4.2.1)\quad q_k = \frac{a_{1k} + b_{1k} r_k}{r_k \sum_{h=1}^{K} (a_{1h} + b_{1h} r_h)} \qquad \text{for } k = 1, \ldots, K.$$

This system may be rewritten as:

$$(4.2.2)\quad q_k \sum_{h=1}^{K} b_{1h} r_h r_k + (a_1 q_k - b_{1k}) r_k = a_{1k} \quad \forall k,$$

with $a_1 = \sum_{h=1}^{K} a_{1h}$,

i.e. a set of K equations quadratic in $r_1 \ldots r_k$.

In general, an analytical solution yielding the single set of positive values $r_1 \ldots r_K$ is feasible only if K does not exceed 2, i.e. equals 2. Thus, one of the two r's, say r_1, can be expressed in terms of the other (r_2) by solving one of the two quadratic equations; then, substitution of this intermediate result into the other equation yields a fourth-degree polynomial in r_2. One and only one of its roots will be relevant (real and positive). Evidently, the fact that polynomial equations have explicit solutions up to and including but not exceeding the fourth degree requires the restriction of K to 2 (cf. Birkhoff and Mac Lane (1953)). Analogous considerations and conclusions apply to the hyperbolic specification (2.2.2).

4.3. "Power" specification of f_k

Second, we may try to derive the direct utility function underlying:

$$(4.3.1)\quad q_k = \frac{c_k r_k^{\alpha_k - 1}}{\sum_{h=1}^{K} c_h r_h^{\alpha_h}}\text{, with } \alpha_k \leq 1,\ \forall K \text{ and } \alpha_k = 1 \text{ for at most one } k.$$

Assuming that the $\alpha_k(\alpha_h)$ are all rational numbers, (4.3.1) may be rewritten as a system of polynomial equations for $r_1, \ldots, r_K$, expressed in terms of the $q_k (k = 1, \ldots, K)$. In general, these polynomials are of a high degree, obviating the attainment of analytical D.U.F. Still, under special conditions, to be dealt with below, the latter may be derived.

4.3.1. Particular proportions between the "powers"

In this case, we assume $\alpha_k < 1$ for all k. Taking some item k, say 1, as the "base" commodity, we derive from (4.3.1):

$$(4.3.1.1)\quad r_k^{\alpha_k - 1} = \frac{q_k/c_k}{q_1/c_1} \cdot r_1^{\alpha_1 - 1} \qquad \forall k.$$

Next, substitution of (4.3.1.1) into (3.3.3) yields:

$$(4.3.1.2)\quad u^* = -\sum_{k=1}^{K} (c_k/\alpha_k) r_k^{\alpha_k}$$

$$= -\sum_{k=1}^{K} (c_k/\alpha_k) \left(\frac{q_k/c_k}{q_1/c_1}\right)^{\frac{\alpha_k}{\alpha_k - 1}} r_1^{\frac{\alpha_k(\alpha_1 - 1)}{\alpha_k - 1}} .$$

Thus, u* is expressed in ratios of all q_k to q_1, and a single r_k, viz. r_1, only. Hence, the last member of (4.3.1.2) represents "almost" a D.U.F., i.e. but for the r_1. Hence, in order to convert (4.3.1.2) into a complete, "pure" D.U.F., i.e.

in terms of q_k $\forall k$, r_1 has to be expressed in these quantities alone. In principle, this may be done through substitution of (4.3.1.1) into (4.3.1) for k = 1, resulting in:

$$q_1 \sum_{k=1}^{K} c_k \left(\frac{q_k/c_k}{q_1/c_1}\right)^{\frac{\alpha_k}{\alpha_k-1}} \cdot r_1^{\frac{\alpha_k(\alpha_1-1)}{\alpha_k-1}} = c_1 r_1^{\alpha_1-1},$$

or, more simply:

$$\text{(4.3.1.3)} \quad q_1 \sum_{k=1}^{K} c_k \left(\frac{q_k/c_k}{q_1/c_1}\right)^{\frac{\alpha_k}{\alpha_k-1}} \cdot r_1^{\frac{\alpha_1-1}{\alpha_k-1}} = c_1$$

An analytical solution of (4.3.1.3) is possible if and only if this equation can be written in the form of a polynomial of at most the fourth degree (cf. Birkhoff and Mac Lane (1953)).

This requires that the values of α_k - in relation to the value of $\alpha_1 < 1$ - are restricted by:

$$\text{(4.3.1.4)} \quad \alpha_k = 1 - (1-\alpha_1)/i,$$

where i may be any of the integers 1, 2, 3 or 4, but is not allowed to assume any other value.

Thus, (4.3.1.3) may be rewritten as:

$$\text{(4.3.1.5)} \quad q_1 \sum_{i=1}^{4} \left\{ \sum_{k\in\{i\}} c_k \left(\frac{q_k/c_k}{q_1/c_1}\right)^{\frac{\alpha_k}{\alpha_k-1}} \right\} r_1^{\,i} = c_1,$$

with $k\in\{i\}$ denoting an element of the integer set {i} to which (4.3.1.4) applies.

For one or more feasible values of i, {i} may be empty, meaning that it would not encompass a single k. In particular, the degree of the polynomial might be reduced to a cubic or even a quadratic one if {i} were empty for i = 4, and for i = 3 and 4, respectively, i.e. if the ratios $(1-\alpha_1)/(1-\alpha_k)$ were confined to (at most) 1 : 2 : 3, and 1 : 2, respectively.

By way of example, the following table shows a triplet of such mutually compatible values of α_1 and α_k:

$(k \in \{i\};\ i = 1, 2, 3, 4)$

$\frac{1-\alpha_1}{1-\alpha_k}$	α_1 = -1	α_1 = -2	α_1 = -3
	α_k		
4	1/2	1/4	0
3	1/3	0	-1/3
2	0	-1/2	-1
1	-1	-2	-3

Of course, this table allows for simple interpolation and extrapolation. Anyhow, it shows for various (negative) values of α_1 (= $\min_k \alpha_k$) to what extent the range of corresponding values of α_2, α_3, and α_4 widens and shifts downwards according as α_1 becomes less. It should be noted that because of $\alpha_k < 1$, the powers are ever-positive and finite. Before reverting to this phenomenon in section 5, we first present a simple example.

Take, for instance, $K = 2$, $\alpha_1 = -1$ and $\alpha_2 = 0$; then (4.3.1) implies:

$$(4.3.1.6a)\quad q_1 = \frac{c_1 r_1^{-2}}{c_1 r_1^{-1} + c_2},$$

and

$$(4.3.1.6b)\quad q_2 = \frac{c_2 r_2^{-1}}{c_1 r_1^{-1} + c_2}.$$

For $r_1, r_2 > 0$, these equations imply:

(4.3.1.7a) $r_1 = \frac{1}{2}\{\sqrt{1 + 4c_2/(c_1q_1)} - 1\}c_1/c_2$

and

(4.3.1.7b) $r_2 = \frac{1}{2}[c_1q_1\{1 - \sqrt{1 + 4c_2/(c_1q_1)}\} + 2c_2]/(c_2q_2)$,

obviously satisfying the budget constraint $r_1q_1 + r_2q_2 = 1$.

Substitution of (4.3.1.6) into (3.3.3) yields:

(4.3.1.8) $$u = c_1q_1 + \sqrt{(c_1q_1)^2 + 4c_1c_2q_1} - c_2 \ln \frac{2c_2 + c_1q_1 - \sqrt{(c_1q_1)^2 + 4c_1c_2q_1}}{2c_2q_2}$$

- a fairly complicated expression, compared to the simplicity of the resulting demand equations (3.2.2). For further examples, see Van Driel (1974), who also proves the positiveness of the marginal utilities for analytical D.U.F., such as (4.3.1.8), in general.

4.3.2. <u>One of the α_k equaling one</u>

Equations (4.3.1) imply:

(4.3.2.1) $$r_k^{\alpha_k - 1} = (q_k/c_k) \sum_{h=1}^{K} c_h r_h^{\alpha_h} \qquad \forall k.$$

Assume that for a single k, say k = 1, the corresponding α_k equals 1:

(4.3.2.2) $\alpha_1 = 1$.

By virtue of (4.3.2.1) and (4.3.2.2):

$$(4.3.2.3)\quad r_k^{\alpha_k} = \left(\frac{r_k^{\alpha_k-1}}{r_1^{\alpha_1-1}}\right)^{\frac{\alpha_k}{\alpha_k-1}} = \left(\frac{q_k/c_k}{q_1/c_1}\right)^{\frac{\alpha_k}{\alpha_k-1}} \qquad \text{for } k = 2, \ldots, K,$$

and

$$(4.3.2.4)\quad c_1 r_1 = (c_1/q_1) - \sum_{h=1}^{K} c_h r_h^{\alpha_h} \qquad \text{for } k = 1.$$

Substitution of (4.3.2.3) and (4.3.2.4) into (4.3.1.2) yields:

$$(4.3.2.5)\quad u^* = -(c_1/q_1) - \sum_{k=2}^{K} c_k\{(1-\alpha_k)/\alpha_k\} \left(\frac{q_k/c_k}{q_1/c_1}\right)^{\alpha_k-1}$$

$$= u,$$

i.e. the indirect utility function re-converted to the direct utility function. The marginal utilities corresponding to (4.3.2.5), evaluated at the consumer optima, appear to be positive for all k.

4.3.3. Possible extensions

Mutatis mutandis, the preceding reasoning may analogously be administered to possible extensions of the F.E.A.M. in its "power" version. In particular, this applies to the following "cross-breeding" between (G.)L.E.S. and F.E.A.M., viz.:

$$(4.3.3.1)\quad q_k = \gamma_k + \frac{c_k(p_k/Z)^{\alpha_k}}{\sum_{k'=1}^{K} c_{k'}(p_{k'}/Z)^{\alpha_{k'}}} \cdot (Z/p_k),$$

with $Z = C - \sum_{k'=1}^{K} \gamma_{k'} p_{k'}$,

γ_k (≥ 0) denoting consumption of k minimally required,

and Z representing "super-numerary income",

respectively (cf. Van Daal and Somermeyer (1977)).

The I.U.F. corresponding to (4.3.3.1) is completely analogous to (3.3.3) or (4.3.1.2), with the $r_k^{(')}$ redefined as p_k/Z; cf. eq. (2.1.7) in Van Daal and Somermeyer (1977). The reason, of course, is the resemblance of (4.3.3.1) to the F.E.A.M. and the linearity of the relationships between the "new" variables $q_k-\gamma_k$ and Z on the one hand, and the original variables q_k, C (and the $p_{k'}$) on the other hand. Also the subsequent considerations and conditions presented in sections 4.3.1 and 4.3.2 remain the same, provided that the q_k are replaced by "surplus" consumption $q_k-\gamma_k$.

5. Approximations

Section 4 showed that the restrictions to be imposed on the power parameters of the "power" version of the F.E.A.M., in order to "merit" an analytical D.U.F., are rather severe. Still, these sets of demand functions might more generally be considered as associated with "approximate" analytical D.U.F. - provided that the α_k could reasonably be arranged into not more than 4 classes according to value, while each class could satisfactorily be represented by a kind of modal or mean value, as exemplified in the table in section 4.3.1.

Alternatively, if one of the α_k were very close to 1, the utility function (4.3.2.5) might well be an acceptable approximative D.U.F. for the set of demand functions in question.

Indeed, both contingencies appear to have an empirical basis (cf. e.g. Somermeyer a.o. (1961)): values of α_k close to (if not exceeding) 1 have been found, as well as minimum values of α_k not less than around -2.5.

Consequently, the restrictions on parameter values for drawing analytical D.U.F. may be less severe in practice than they appear to be in pure theory.

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