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STOCHASTIC COMPACTNESS OF SAMPLE EXTREMES

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Stochastic compactness of sample extremes

by

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Summary

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with common distribution function F and let $X_n = \max\{Y_1, \ldots, Y_n\}$ for $n = 1, 2, \ldots$ Necessary and sufficient conditions (in terms of F) are derived for the existence of a sequence of positive constants $\{a_n\}$ such that the sequence $\{\frac{X_n}{a_n}\}$ is stochastically compact.

Moreover the relation between the stochastic compactness of partial maxima and partial sums of the Y_n's is investigated.

Contents

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Stochastic compactness of sample extremes

1. Introduction

In the following compactness properties of sequences of sample maxima are investigated. This parallels results of Feller with respect to compactness properties of sequences of partial sums [Feller (1965)]. It will become clear, that, in the case of sample maxima, it is not evident how the concept of stochastic compactness should be defined. Therefore this concept will be defined in two different ways (sections 2,4). For both definitions necessary and sufficient conditions are obtained (sections 3,4). These conditions are in terms of the distribution function, in contrast to those for the compactness of sample sums. They resemble a generalisation of the concept of regular variation called R-O-variation [Seneta (1976)]. The relation between the present results and the theory of R-O-variation will be considered in section 4. The relation between the compactness and the weak convergence of sample maxima and the relation between the compactness of partial maxima and the compactness of partial sums will be considered in sections 5 and 6 respectively. In section 7 some examples and counterexamples are presented.

2. Stochastic boundedness and compactness

2.1. Definitions

Let $\{X_n\}$ be a sequence of real-valued random variables with distribution functions (d.f.) $\{F_n\}$. We define:

Definition 2.1.1 (stochastic boundedness). The sequence of random variables $\{X_n\}$ is stochastically bounded if for all $\varepsilon > 0$ there is an $x_0 > 0$ such that $P(|X_n| > x_0) < \varepsilon$ for all n.

Remark 2.1.1. This definition is equivalent with each of the following assertions.

1. Every subsequence $\{X_{n_k}\}$ of $\{X_n\}$ contains a further subsequence $\{X_{n_r}\}$ such, that $\{X_{n_r}\} \overset{w}{\to} X$ with X a non-defective random variable.

2. $\lim_{x\to\infty} P(|X_n| > x) = 0$ uniformly in n.

We now define the concept of stochastic compactness of sample maxima. For that purpose we consider a sequence of identically and independently distributed (i.i.d.) random variables $\{Y_n\}$ with common d.f. F. We suppose, that:

1) F is continuous (see remark 2.2.2)

2) $F(0) = 0$ (see remark 2.2.1)

3) $F(x) < 1$ for all x (this is to avoid trivialities).

We now define for $n = 1, 2, \ldots$

$$X_n = \max \{Y_1, \ldots, Y_n\}.$$

<u>Definition 2.1.2</u> (stochastic compactness). The sequence of sample maxima $\{X_n\}$ is <u>stochastically compact</u> if there is a sequence of positive constants $\{a_n\}$ such, that the sequences $\{\frac{X_n}{a_n}\}$ and $\{\frac{a_n}{X_n}\}$ are stochastically bounded.

<u>Remark 2.1.2.</u>

1. An alternative formulation of this definition is: the sequence of sample maxima $\{X_n\}$ is stochastically compact if there is a sequence of positive constants $\{a_n\}$ such that every convergent subsequence of $\{\frac{X_n}{a_n}\}$ has a limit distribution concentrated on $(0,\infty)$.
2. Instead of $\{\frac{X_n}{a_n}\}$ we can consider the sequence $\{\ln(\frac{X_n}{a_n})\}$. This is possible because $\{X_n\}$ is a sequence of positive random variables. The stochastic compactness of the sequence of sample maxima $\{X_n\}$ then is equivalent with the assertion that there is a sequence of real constants $\{b_n\}$ such, that the sequence $\{\ln X_n - b_n\}$ is stochastically bounded. This is an alternative way to present the results because $\ln X_n = \max \{(\ln Y_1, \ldots, \ln Y_n\}$.
3. Helly's theorem implies, that every subsequence of $\{\frac{X_n}{a_n}\}$ contains a further subsequence, that is weakly convergent. In the definition of stochastic

compactness however certain limit distributions are excluded. In the definition above limit distributions which assign positive probability to the points 0 or ∞, are not admitted (note that one can always choose the constants $\{a_n\}$ such, that e.g. $\frac{X_n}{a_n} \xrightarrow{w} 0$). In his definition of stochastic compactness of partial sums Feller excludes degenerate limit distributions. In the sequel we will consider an alternative definition of stochastic compactness of maxima where also a larger class of limit distributions is excluded.

2.2. Necessary and sufficient conditions for stochastic compactness

From remark 2.1.1 it follows that the stochastic compactness of the sequence of sample maxima $\{X_n\}$ is equivalent with the following two assertions:

1) $\lim_{x\to\infty} \overline{\lim_{n\to\infty}} P(X_n > a_n x) = 0 \Leftrightarrow \lim_{x\to\infty} \underline{\lim_{n\to\infty}} F^n(a_n x) = 1.$ (2.2.1)

2) $\lim_{x\downarrow 0} \overline{\lim_{n\to\infty}} P(X_n \leq a_n x) = 0 \Leftrightarrow \lim_{x\downarrow 0} \overline{\lim_{n\to\infty}} F^n(a_n x) = 0.$ (2.2.2)

Remark 2.1.1. From (2.2.1) and (2.2.2) we see that the requirement $F(0) = 0$ does not result in a loss of generality. For if $0 < F(0) < 1$ then for $x > 0$:

$$P(|X_n| > a_n x) = 1 - F^n(a_n x) + F^n(a_n x-). \qquad (2.2.3)$$

However the last term on the right-hand side of (2.2.3) clearly converges to 0 as $n \to \infty$.

We need the following lemma:

Lemma 2.2.1.

1. $\lim\limits_{x\to\infty} \underline{\lim}_{n\to\infty} F^n(a_n x) = 1 \Leftrightarrow \lim\limits_{x\to\infty} \overline{\lim}_{n\to\infty} n(1 - F(a_n x)) = 0.$

2. $\lim\limits_{x\downarrow 0} \overline{\lim}_{n\to\infty} F^n(a_n x) = 0 \Leftrightarrow \lim\limits_{x\downarrow 0} \underline{\lim}_{n\to\infty} n(1 - F(a_n x)) = \infty.$

Proof. We only prove part 1; part 2 can be proved analogously.

We have

$$\underline{\lim}_{n\to\infty} F^n(a_n x) = \underline{\lim}_{n\to\infty} \left(1 - \frac{n(1 - F(a_n x))}{n}\right)^n = e^{-\overline{\lim}_{n\to\infty} n(1 - F(a_n x))}.$$

Hence:

$$\lim_{x\to\infty} \underline{\lim}_{n\to\infty} F^n(a_n x) = 1 \Leftrightarrow \lim_{x\to\infty} \overline{\lim}_{n\to\infty} n(1 - F(a_n x)) = 0.$$

Necessary and sufficient conditions for the stochastic compactness of a sequence of partial maxima are given by the following theorem.

Theorem 2.2.1. The following assertions are equivalent:

1) Let $\{Y_n\}$ be a sequence of i.i.d. random variables with common d.f. F. The sequence $\{X_n\}$ of sample maxima from $\{Y_n\}$ is stochastically compact.

2) $\lim\limits_{x\to\infty} \overline{\lim}_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = 0.$

3) $\exists\ t_0, x_0 > 0,\ 0 < M < 1$ such that $\frac{1 - F(tx)}{1 - F(t)} \leq M$ for $t \geq t_0$, $x \geq x_0$.

4) $\exists C, \rho > 0$ such that $\frac{1 - F(tx)}{1 - F(t)} \leq Cx^{-\rho}$ for $x \geq 1$, $t \geq t_0$.

5) $\int_1^\infty \frac{1 - F(s)}{s}\, ds < \infty$ and $\underline{\lim}_{x\to\infty} \frac{1 - F(x)}{H(x)} > 0$ with $H(x) = \int_x^\infty \frac{1 - F(s)}{s}\, ds$.

6) $\int_1^\infty \frac{1 - F(s)}{s} ds < \infty$ and $\exists t_0, \eta > 0$ such that $\frac{H(tx)}{H(t)} \le x^{-\eta}$

for $t \ge t_0$, $x \ge 1$.

Remark 2.2.2. It is thus necessary for stochastic compactness that

$E \ln(1 + Y_1) < \infty$.

Proof. Here we only prove $1 \Leftrightarrow 2$. The other implications follow from the results of the next section (take $K(x) = 1 - F(x)$).

$1 \Rightarrow 2$.

If the sequence $\{X_n\}$ is stochastically compact, it follows from lemma 2.2.1 that

$$\lim_{x \downarrow 0} \underline{\lim}_{n\to\infty} n(1 - F(a_n x)) = \infty.$$

Thus in particular there is an $x_1 > 0$ such that

$$\underline{\lim}_{n\to\infty} n(1 - F(a_n x_1)) > 0.$$

Also according to lemma 2.2.1 we have:

$$\lim_{x\to\infty} \overline{\lim}_{n\to\infty} n(1 - F(a_n x)) = 0.$$

Define for $t > 0$

$$n(t) = \min \{m | a_{m+1} > t\}$$

so that

$$a_{n(t)} \le t < a_{n(t)+1}$$

and

$$\frac{1 - F(tx_1 \cdot \frac{x}{x_1})}{1 - F(tx_1)} \leq \frac{[1 - F(a_{n(t)}x)]n(t)}{[1 - F(a_{n(t)+1}x_1)][n(t)+1]} \cdot \frac{n(t) + 1}{n(t)}$$

Because $n(t) \to \infty$ as $t \to \infty$ it follows

$$\lim_{x\to\infty} \overline{\lim_{t\to\infty}} \frac{1 - F(tx)}{1 - F(t)} = 0.$$

$2 \Rightarrow 1.$

Define for $n > 1$ the sequence $\{\alpha_n\}$ by:

$$n(1 - F(\alpha_n)) = 1.$$

This is possible, because F is continuous.

Then:

$$0 \leq \lim_{x\to\infty} \overline{\lim_{n\to\infty}} n(1 - F(\alpha_n x)) \leq \lim_{x\to\infty} \overline{\lim_{t\to\infty}} \frac{1 - F(tx)}{1 - F(t)} = 0.$$

Analogously $\lim_{x\downarrow 0} \underline{\lim}_{n\to\infty} n(1 - F(\alpha_n x)) = \infty$ follows from $\lim_{x\downarrow 0} \underline{\lim}_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = \infty.$

Remark 2.2.3. The assumption that the d.f. F is continuous plays an important role in this proof. It is possible however to replace this assumption by a slightly weaker one namely the condition that there are an $x_1 > 0$ and an n_0 such that for $n \geq n_0$

$$n(1 - F(\beta_n x_1)) > 0$$

with

$$\beta_n = \inf \{X \mid 1- F(x) \leq \frac{1}{n}\} ; \quad n = 1, 2, \ldots$$

Since in case the sequence $\{X_n\}$ is stochastically compact there are $x_1, x_2 > 0$ and an n_0 such that for $n \geq n_0$

$$n(1 - F(a_n x_1)) > 1$$
$$n(1 - F(a_n x_2)) \leq 1,$$

it follows that

$$x_1 \leq \frac{\beta_n}{a_n} \leq x_2 .$$

Thus we can replace the sequence of constants $\{a_n\}$ from the definition of stochastic compactness by the sequence $\{\beta_n\}$. It is now not difficult to show that the above-mentioned conditions are necessary and sufficient.

3. The class of asymptotically decreasing functions

In this section we investigate the class of non-increasing functions which satisfy assertion 3 of theorem 2.2.1.

Definition 3.1 (asymptotically decreasing). Let K be a positive and non-increasing function defined on $\mathbb{R}^+$. The function K is asymptotically decreasing if there are $x_0, t_0 > 0$ and $0 < M < 1$ such that

$$\frac{K(tx)}{K(t)} \leq M \quad \text{for} \quad x \geq x_0, \quad t \geq t_0 .$$

Remark 3.1.

1. From this definition it follows that $\lim_{x\to\infty} K(x) = 0$.

2. If we define $K^*(x) = \sup\{K(y) | y > x\}$ it can easily be shown that K^* is right-continuous and that K is asymptotically decreasing if K^* is asymptotically decreasing. Thus we can suppose without loss of generality that K is right-continuous.
3. The sequence of sample maxima $\{X_n\}$ is stochastically compact iff $1 - F$ is asymptotically decreasing.
4. The above-defined concept is a one-sided generalisation of the concept of regular variation.

   In the definition of R-O-variation (a two-sided generalisation of regular variation) two inequalities appear: A positive and measurable function K is said to be R.O.-varying if there are $a > 1$, $t_0 > 0$, $m > 0$ and $M < \infty$ such that

$$m \leq \frac{K(tx)}{K(t)} \leq M \quad \text{for} \quad 1 \leq x \leq a \quad \text{and} \quad t \geq t_0$$

   (See e.g. [Seneta (1976)]).

   Another (different) generalisation of regular variation is the concept of dominated variation which is used by Feller [Feller (1965)]. A non-increasing, positive function K is dominatedly varying if there are $C, \rho, t_0 > 0$ such that

$$\frac{K(tx)}{K(t)} \geq Cx^{-\rho} \quad \text{for} \quad t \geq t_0,\ x \geq 1$$

   (cf. assertion 3 of theorem 3.1)
5. It should be noted that in definition 3.1 we have assumed that the function K is non-increasing. This assumption plays an important role in the

derivation of the necessary and sufficient conditions for a function to be asymptotically decreasing. It is however possible to define a similar concept for measurable, but not necessarily non-increasing functions. In that case the necessary and sufficient conditions are different from those given in theorem 3.1 (see section 4).

The necessary and sufficient conditions are given by the following theorem.

Theorem 3.1. The following assertions are equivalent:

1) The function K is asymptotically decreasing.

2) $\lim\limits_{x\to\infty} \overline{\lim\limits_{t\to\infty}} \dfrac{K(tx)}{K(t)} = 0,$

3) $\exists C, \rho, t_0 > 0$ such that $\dfrac{K(tx)}{K(t)} \leq Cx^{-\rho}$ for $x \geq 1, t \geq t_0$.

4) $\int_1^\infty \dfrac{K(s)}{s}\,ds < \infty$ and $\overline{\lim\limits_{x\to\infty}} \dfrac{H(x)}{K(x)} < \infty$ with $H(x) = \int_x^\infty \dfrac{K(s)}{s}\,ds.$

5) $\int_1^\infty \dfrac{K(s)}{s}\,ds < \infty$ and $\exists \eta, t_0 > 0$ such that

$$\frac{H(tx)}{H(t)} \leq x^{-\eta} \quad \text{for} \quad t \geq t_0,\ x \geq 1.$$

Proof. 1 ⇒ 3 [cf. Feller (1965) p. 385].

If K is asymptotically decreasing, there are $0 < M < 1$, $x_0 > 1$ and $t_0 > 0$ such that

$$\frac{K(tx_0)}{K(t)} \leq M \qquad \text{for} \quad t \geq t_0.$$

Thus with induction

$$\frac{K(tx_0^n)}{K(t)} \leq M^n = x_0^{-n\rho} \quad \text{with } \rho = \frac{-\ln M}{\ln x_0} > 0.$$

Then for $x_0^n \leq x < x_0^n$ and $t \geq t_0$

$$\frac{K(tx)}{K(t)} \leq \frac{K(tx_0^{n-1})}{K(t)} \leq x_0^{-(n-1)\rho} \leq Cx^{-\rho}$$

with $C = x_0^{\rho}$.

$3 \Rightarrow 2$ and $2 \Rightarrow 1$: trivial.

Next we prove that if K is asymptotically decreasing then $H(1) < \infty$. Assertion 3 of theorem 3.1 implies that there is a $t_0 > 0$ and $x_1 > 1$ such that

$$\frac{K(tx)}{K(t)} \leq \left(\frac{x}{x_1} \right)^{-\rho} \quad \text{for } x \geq 1, \ t \geq t_0.$$

Let $A > \max \{x_1, t_0\}$ and define for $n = 2, 3, \ldots$

$$I_n = \int_{A^n}^{A^{n+1}} \frac{K(s)}{s} \, ds.$$

Then

$$I_n = \int_{A^{n-1}}^{A^n} \frac{K(sA)}{s} \, ds \leq \int_{A^{n-1}}^{A^n} \frac{K(s)}{s} \left(\frac{A}{x_1} \right)^{-\rho} ds = \left(\frac{A}{x_1} \right)^{-\rho} I_{n-1}.$$

Thus

$$I_n = \left(\frac{A}{x_1} \right)^{-(n-2)\rho} I_2$$

and hence $H(1) < \infty$.

We now prove the other implications.

$3 \Rightarrow 4$.

A slight extension of Fatou's lemma gives

$$\overline{\lim_{x\to\infty}} \frac{H(x)}{K(x)} = \overline{\lim_{x\to\infty}} \int_1^\infty \frac{K(sx)}{K(x)} \, ds \le \int_1^\infty \overline{\lim_{x\to\infty}} \frac{K(sx)}{K(x)} \, ds \le \frac{C}{\rho} < \infty.$$

$4 \Rightarrow 2$.

Let $N(x) = K(1) - K(x)$. Then for $x \ge 1$ and $b > 1$ (K is assumed to be right continuous)

$$\infty > a \ge \frac{\int_x^\infty \frac{K(t)}{t} \, dt}{K(x)} = \frac{\int_1^\infty \int_{tx}^\infty dN(s) \frac{dt}{t}}{K(x)} \ge \frac{\int_1^b \int_{bx}^\infty dN(s) \frac{dt}{t}}{K(x)} \ge \frac{K(bx)}{K(x)} \ln b .$$

Hence

$$\frac{K(bx)}{K(b)} \le \frac{a}{\ln b} .$$

$4 \Rightarrow 5$.

Let $a(t) = \dfrac{K(t)}{H(t)}$. Then for $x \ge x_1 > 0$

$$\int_{x_1}^x \frac{a(t)}{t} \, dt = \int_{x_1}^x \frac{K(t)}{t \int_t^\infty \frac{K(s)}{s} \, ds} \, dt = -\ln H(x) + \ln H(x_1).$$

Thus

$$H(x) = \beta \exp\left(- \int_{x_1}^{x} \frac{a(t)}{t} dt \right) \quad \text{for} \quad x \geq x_1 \quad \text{with} \quad \beta = H(x_1).$$

Then for $x \geq 1$

$$\overline{\lim_{t\to\infty}} \frac{H(tx)}{H(t)} = \overline{\lim_{t\to\infty}} \exp\left(- \int_{t}^{tx} \frac{a(s)}{s} ds \right) \leq \exp\left(- \int_{1}^{x} \frac{1}{s} \underline{\lim}_{t\to\infty} a(st)\, ds \right) \leq$$

$$\leq x^{-\eta}$$

because $\underline{\lim}_{s\to\infty} a(s) = \eta > 0.$

$5 \Rightarrow 4.$

There is a $t_0 > 0$ such that for $x \geq 1$ and $t \geq t_0$

$$1 - x^{-\eta} \leq \frac{H(t) - H(tx)}{H(t)} = \int_{t}^{tx} \frac{K(s)}{s\, H(t)} ds \leq \frac{K(t)}{H(t)} \ln x$$

whence

$$\underline{\lim}_{t\to\infty} \frac{K(t)}{H(t)} \geq \frac{1 - x^{-\eta}}{\ln x} \Rightarrow \underline{\lim}_{t\to\infty} \frac{K(t)}{H(t)} \geq \lim_{x\downarrow 1} \frac{1 - x^{-\eta}}{\ln x} = \eta > 0.$$

The condition on H in assertion 5 implies that H is asymptotically decreasing, but it is required that the constant C, appearing in assertion 3, of K equals 1. Without proof we state the equivalence of assertion 4 and assertion 5 for H-functions which are possibly not integrals involving a monotone K-function.

Theorem 3.2. Let L be a non-increasing function and let the derivative of L exist. The function L is asymptotically decreasing with $C = 1$ i.e. $\exists t_0, \eta > 0$ such that $\frac{L(tx)}{L(t)} \leq x^{-\eta}$ for $t \geq t_0$, $x \geq 1$.

iff $\overline{\lim}_{x\to\infty} L_*'(x) < 0$ with $L_*(x) = \ln L(e^x)$.

4. An alternative definition of stochastic compactness of sample maxima

Until now we have assumed that the common d.f. F of the Y_n's which appear in the definition of the sequence of sample maxima $\{X_n\}$ is continuous. This requirement played an important role in the proof of theorem 2.2.1. It is however possible to drop the continuity requirement. As a consequence we will have to exclude in the definition a class of limit distributions different from that excluded in definition 2.1.2. We define:

Definition 4.1 (stochastic compactness[1]). The sequence of sample maxima $\{X_n\}$ is stochastically compact[1] if there is a sequence of positive constants $\{a_n\}$ such that every convergent subsequence of $\{\frac{X_n}{a_n}\}$ converges weakly to a distribution with d.f. G (depending on the subsequence) which satisfies the following requirements

1. $G(\infty) = 1$
2. $G(x) < 1$ for all $x \in \mathbb{R}$ (cf. remark 2.1.1.1).

The requirement that $G(\infty) = 1$, is equivalent with the assertion that $\lim_{x\to\infty} \underline{\lim}_{n\to\infty} F^n(a_n x) = 1$; the requirement that $G(x) < 1$ for all x is equivalent to $\overline{\lim}_{n\to\infty} F^n(a_n x) < 1$ for all $x > 0$. Necessary and sufficient conditions for the stochastic compactness of the sequence $\{X_n\}$ are given by the following theorem.

Theorem 4.1. The following assertions are equivalent:

1) The sequence of partial maxima $\{X_n\}$ is stochastically compact[1].

2) $\lim_{x\to\infty} \overline{\lim}_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = 0$ and $\underline{\lim}_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} > 0$ for all $x > 0$,

3) $\exists x_0,\ t_0,\ 0 < m \leq M < 1$

$$m \leq \frac{1 - F(tx)}{1 - F(t)} \leq M \quad \text{for} \quad x \geq x_0, \quad t \geq t_0.$$

4) $\exists t_0,\ C_1,\ C_2,\ \rho,\ \tau > 0$ such that

$$C_1 x^{-\tau} \leq \frac{1 - F(tx)}{1 - F(t)} \leq C_2 x^{-\rho} \quad \text{for} \quad x \geq 1, \quad t \geq t_0.$$

5) $\int_1^\infty \frac{1 - F(s)}{s}\, ds < \infty$ and $\varliminf_{x\to\infty} \frac{1 - F(x)}{H(x)} > 0$ and $\varlimsup_{x\to\infty} \frac{1 - F(x)}{H(x)} < \infty$

with as before $H(x) = \int_x^\infty \frac{1 - F(s)}{s}\, ds.$

6) $\exists t_0,\ \eta,\ \theta > 0$ such that

$$x^{-\theta} \leq \frac{H(tx)}{H(t)} \leq x^{-\eta} \quad \text{for} \quad t \geq t_0, \quad x \geq 1.$$

Proof. First we prove $4 \Rightarrow 1$.

From assertion 4 it follows that

$$\frac{1 - F(tx)}{1 - F(t)} \leq \frac{1}{C_1} x^{-\tau} \quad \text{for} \quad t \geq t_0, \quad 0 < x \leq 1.$$

Define for $n = 2, 3, \ldots$

$$a_n = \inf \ \{x \mid 1 - F(x) \leq \frac{1}{n}\}.$$

From the right-continuity of F it follows that $1 - F(a_n) \leq \frac{1}{n}$. We also have that $1 - F(a_n -) \geq \frac{1}{n}$.

Then for $\varepsilon > 0$

$$\frac{1 - F(t(1-\varepsilon))}{1 - F(t)} \leq \frac{1}{C_1} (1-\varepsilon)^{-\tau} \quad \text{for} \quad t \geq t_0.$$

In particular:

$$1 - F(a_n(1-\varepsilon)) \leq \frac{1}{C_1} (1-\varepsilon)^{-\tau} (1 - F(a_n)) \quad \text{for} \quad n \geq n_0.$$

Letting $\varepsilon \downarrow 0$ we conclude that $\frac{1}{n} \geq 1 - F(a_n) \geq \frac{C_1}{n}$ for $n \geq n_0$. Now it can easily be shown (extend lemma 2.2.1 for this case) that $\{X_n\}$ is stochastically compact[1].

For the implications $1 \Rightarrow 2$, $2 \Rightarrow 3$, $3 \Rightarrow 4$, $4 \Rightarrow 5$, $5 \Rightarrow 6$ the reader is referred to the proof of theorem 3.1. No particular difficulties arise from the appearance of left-hand inequalities in the assertions.

Finally we prove $6 \Rightarrow 2$. Assertion 6 gives $\lim_{x\to\infty} \overline{\lim}_{t\to\infty} \frac{H(tx)}{H(t)} = 0$. Using assertion 5 we find:

$$0 \leq \lim_{x\to\infty} \overline{\lim}_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} \leq \overline{\lim}_{t\to\infty} \frac{1 - F(t)}{H(t)} \lim_{x\to\infty} \overline{\lim}_{t\to\infty} \frac{H(tx)}{H(t)} \overline{\lim}_{t\to\infty} \frac{H(t)}{1 - F(t)} = 0.$$

Analogously one proves

$$\underline{\lim}_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} > 0 \quad \text{for all} \quad x > 0.$$

Remark 4.1.

1. With a similar argument as in remark 2.2.2 one can show that, without loss of

generality, the constants a_n in definition 4.1 can be chosen as $a_n = \inf \{x | 1 - F(x) \leq \frac{1}{n}\}$.

2. The sequence $\{X_n\}$ is stochastically compact[1] iff $1 - F$ is R-O-varying with constant $M < 1$ (cf. remark 3.1).

5. Stochastic compactness and convergence

We have investigated the stochastic compactness of the sequence of partial maxima $\{X_n\}$. This sequence can however show a more precise limiting behaviour. It is for instance possible that there exists a sequence of positive comstants $\{a_n\}$ such that $\{\frac{X_n}{a_n}\} \overset{w}{\to} C$. The weak law of large numbers gives necessary and sufficient conditions for this case [Gnedenko (1943)]. Another possibility is that there is a positive sequence $\{a_n\}$ such that $\{\frac{X_n}{a_n}\} \overset{w}{\to} X$ (X non-degenerate). The conditions for the stochastic compactness of the sequence $\{X_n\}$ as well for definition 2.2.1 as for definition 4.1 and those for the weak convergence of that sequence are compared in the following scheme $(H(x) = \int_x^\infty \frac{1 - F(s)}{s} ds)$:

I. Equivalent are

a. $\{X_n\}$ stochastically compact (definition 2.2.1).

b. $\exists C, \rho, t_0 > 0$ such that

$$\frac{1 - F(tx)}{1 - F(t)} \leq Cx^{-\rho} \quad \text{for } x \geq 1, \ t \geq t_0 .$$

c. $\exists$ $c > 0$ such that

$$\frac{1 - F(x)}{H(x)} \geq c \quad \text{for all} \quad x > 0.$$

II. Equivalent are

a. $\{X_n\}$ is stochastically compact[1] (definition 4.1).

b. $\exists C_1, C_2, \tau, \rho, t_0 > 0$ such that

$$C_1 x^{-\tau} \leq \frac{1 - F(tx)}{1 - F(t)} \leq C_2 x^{-\rho} \quad \text{for} \quad t \geq t_0, \quad x \geq 1.$$

c. $\exists C_1 > 0, \; C_2 < \infty$ such that

$$C_2 \geq \frac{1 - F(x)}{H(x)} \geq C_1 \quad \text{for all} \quad x > 0.$$

III. Equivalent are

a. There are constants $\{a_n\}$ such that $\{\frac{X_n}{a_n}\} \xrightarrow{w} X$ with $P\{0 < X < \infty\} = 1$.

b. $\exists\, 0 < \alpha \leq \infty$ such that

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha} \quad \text{for } x > 0.$$

c. $\lim_{x\to\infty} \frac{1 - F(x)}{H(x)} = \alpha \quad (0 < \alpha \leq \infty).$

Note that the case $\alpha = \infty$ corresponds with the weak law of large numbers.

A derivation of the necessary and sufficient conditions for weak convergence can be found in Gnedenko (1943), de Haan (1970).

6. Stochastic compactness of sample maxima and sums

In this section we investigate the relation between the stochastic compactness of partial maxima and partial sums. Let $\{Y_n\}$ be a sequence of i.i.d. random variables with common d.f. F; suppose that the distribution is symmetric about zero. Define for $n = 1, 2, \ldots$

$$S_n = Y_1 + \ldots + Y_n.$$

In Feller (1965) the following definition of the stochastic compactness of the sequence $\{S_n\}$ is given.

Definition 6.1 (stochastic compactness). The sequence of sample sums $\{S_n\}$ is stochastically compact if there is a sequence of positive constants $\{a_n\}$ such that every subsequence of $\{\frac{X_n}{a_n}\}$ contains a further subsequence which weakly converges to a non-degenerate and non-defective distribution.

Necessary and sufficient conditions for the stochastic compactness of the sequence $\{S_n\}$ are given in the following theorem.

Theorem 6.1 (Feller (1965)). The following assertions are equivalent:

1. The sequence of sample sums $\{S_n\}$ is stochastically compact.
2. $\exists\ 0 < \varepsilon \leq 2$ such that

$$\overline{\lim_{t\to\infty}} \frac{G(tx)}{G(t)} \leq x^{2-\varepsilon} \quad \text{for } x \geq 1 \quad \text{with } G(x) = \int_0^x s[1 - F(s)]ds.$$

3. $$\underline{\lim_{x\to\infty}} \frac{G(x)}{x^2[1 - F(x)]} > \tfrac{1}{2} \quad \text{with } G(x) = \int_0^x s\,[1 - F(s)]\,ds.$$

The relation between the stochastic compactness of partial maxima and partial sums is given by the following theorem.

Theorem 6.2. For a d.f. F the following implication holds:

$$\lim_{x\to\infty} \overline{\lim_{t\to\infty}} \frac{1 - F(tx)}{1 - F(t)} = 0 \Rightarrow \underline{\lim_{x\to\infty}} \frac{G(x)}{x^2[1 - F(x)]} > \tfrac{1}{2}$$

Proof. It is given that

$$\overline{\lim_{t\to\infty}} \frac{1 - F(tx)}{1 - F(t)} = \varphi(x) \quad \text{with} \quad \lim_{x\to\infty} \varphi(x) = 0.$$

Hence for $0 < x < 1$

$$\underline{\lim_{t\to\infty}} \frac{1 - F(tx)}{1 - F(t)} = \frac{1}{\varphi(\frac{1}{x})} = k(x).$$

We then have:

$$\underline{\lim_{x\to\infty}} \frac{G(x)}{x^2 [1 - F(x)]} \geq \int_0^1 s\, k(s)\, ds.$$

Because $\lim_{s\downarrow 0} k(s) = \infty$ there is an $0 < s_0 < 1$ such that if $s \leq s_0$ then $k(s) \geq M > 1$.

Hence:

$$\int_0^1 s\, k(s) ds = \int_0^{s_0} s\, k(s)\, ds + \int_{s_0}^1 s\, k(s)\, ds \geq \int_0^{s_0} s\, M\, ds + \int_{s_0}^1 s\, ds =$$

$$= \tfrac{1}{2} s_0^2 (M-1) + \tfrac{1}{2} > \tfrac{1}{2}.$$

Theorem 6.2 shows that the stochastic compactness of a sequence of sample maxima as well according to definition 2.2.1 as according to definition 4.1 implies the stochastic compactness of the corresponding sequence of sample sums. The converse however is not true (see example 7.3). Note that the convergence of the entire sequence $\{\frac{X_n}{a_n}\}$ is equivalent to the convergence of $\{\frac{S_n}{a_n}\}$ in the case of a non-degenerate limit distribution. The well-known technique of considering the partial maxima as functionals on the partial sums process, succesful in the case of convergence of the sequence cannot be used here.

Remark 6.1. A similar relation between partial maxima and partial sums exists in the case of the weak law of large numbers (W.L.L.N.). If F is a d.f. and $\{X_n\}$ the corresponding sequence of partial maxima and $\{S_n\}$ the corresponding sequence of partial sums, then

1) W.L.L.N. for sample sums.

There are constants $\{a_n\}$ such that $\lim_{n\to\infty} P\left(\left|\frac{S_n}{a_n} - 1\right| > \varepsilon\right) = 0$ for all $\varepsilon > 0$ iff

$$\lim_{x\to\infty} \frac{\int_0^x [1 - F(s)]ds}{x[1 - F(x)]} = 0.$$

[Feller (1971)]

2) W.L.L.N. for sample maxima.

There are constants $\{a_n\}$ such that $\lim_{n\to\infty} P\left(\left|\frac{X_n}{a_n} - 1\right| > \varepsilon\right) = 0$

for all $\varepsilon > 0$ iff

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} \begin{cases} = \infty & 0 < x < 1 \\ = 1 & x = 1 \\ = 0 & x > 1. \end{cases}$$

[Gnedenko (1943)]

If the sequence $\{X_n\}$ satisfies the condition of the W.L.L.N. for partial maxima then:

$$\lim_{x\to\infty} \frac{\int_0^x [1 - F(s)]ds}{x[1 - F(x)]} \geq \int_0^1 \lim_{x\to\infty} \frac{1 - F(sx)\ ds}{1 - F(x)} = \infty .$$

And we conclude that $\{S_n\}$ satisfies the condition of the W.L.L.N. for partial sums.

7. Examples and counter examples

In this section we give some examples of d.f.'s for which the corresponding sequence of partial maxima is not stochastically compact. Moreover we show that the converse of theorem 6.2 is not true.

Example 7.1.

$$\text{Let } F(x) = 0 \qquad x < e$$

$$= 1 - \frac{1}{\ln x} \qquad x \geq e .$$

Both the sequence of sample maxima and that of sample sums are not stochastically compact.

Example 7.2.

$$\text{Let } F(x) = 0 \qquad x < e$$

$$= 1 - x^{-\alpha}\ (\sqrt{2} + \sin\ln\ln\ln x) \qquad x \geq e .$$

Then for every sequence $\{t_k\}$ with $t_k \to \infty$ we have:

$$\lim_{k\to\infty} \frac{1 - F(t_k x)}{1 - F(t_k)} = \varphi(x) \Leftrightarrow \lim_{k\to\infty} L_*(s_k+y) - L_*(s_k) = \ln \varphi(e^y)$$

with $L_*(x) = \ln(1 - F(e^x))$, $y = \ln x$ and $s_k = \ln t_k$.

Because $s_k \{ \sin(\ln s_k + \ln(1 + \frac{y}{s_k})) - \sin \ln s_k \} - y \cos\ln s_k \to 0$

as $k \to \infty$ we have:

$$\lim_{k\to\infty} L_*(s_k+y) - L_*(s_k) = -\alpha\sqrt{2}\, x - \alpha x \lim_{k\to\infty} (\sin\ln(s_k+y) + \cos\ln s_k).$$

The limit points of $\frac{1 - F(tx)}{1 - F(t)}$ as $t \to \infty$ are thus given by

$$\varphi(x) = x^{-C} \quad \text{with} \quad C \in [0, 2\alpha\sqrt{2}].$$

Hence:

$$\lim_{x\to\infty} \overline{\lim_{t\to\infty}} \frac{1 - F(tx)}{1 - F(t)} = 1.$$

<u>Example 7.3</u>. The converse of theorem 6.2 is not true. Consider namely example 7.2. with $\alpha = 6$. Then the distribution has a finite variance and according to the central limit theorem the sequence $\{\frac{S_n}{a_n}\}$ converges to a normal distribution. The sequence of partial maxima is however not stochastically compact.

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Stochastic compactness of sample extremes

by

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Summary

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with common distribution function F and let $X_n = \max\{Y_1, \ldots, Y_n\}$ for $n = 1, 2, \ldots$ Necessary and sufficient conditions (in terms of F) are derived for the existence of a sequence of positive constants $\{a_n\}$ such that the sequence $\{\frac{X_n}{a_n}\}$ is stochastically compact.

Moreover the relation between the stochastic compactness of partial maxima and partial sums of the Y_n's is investigated.

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