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APPLICATION OF NON-LINEAR PROGRAMMING TO PLANE GEOMETRY

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ABSTRACT

In this paper proofs are given of two inequalities, involving elements of a triangle. The methods employed are non-geometric by nature and find their origin in the Kuhn-Tucker theory of non-linear programming.

CONTENTS

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1. INTRODUCTION

The purpose of this paper is to show how one may obtain geometric inequalities by means of purely non-geometric methods. An advantage of this approach is that those inequalities may be viewed in a somewhat wider setting than that given by mere plane geometry. Although we intend to prove only two inequalities, we strongly feel that others may be found in a similar fashion. The method to be used is taken from the field of non-linear programming, to be more specific, we shall employ an adopted version of the Kuhn-Tucker theorem.

To illustrate our point, we have selected the following inequalities:

$$(1.1)\qquad ab + bc + ca < k_1(a+b+c)^2$$

$$\textit{with}\quad k_1 = -\frac{5}{2} + 2\sqrt{2}$$

and

$$(1.2)\qquad (a\beta-b\alpha)^2 + (b\gamma-c\beta)^2 + (c\alpha-a\gamma)^2 < k_2(a+b+c)^2$$

$$\textit{with}\quad k_2 = \tfrac{1}{4}\pi^2$$

In (1.1) the quantities a,b and c stand for the sides of an *obtuse* triangle and in (1.2) a,b and c are the sides and α,

β and γ are the corresponding angles (measured in radials) of an *arbitrary* triangle.

The first inequality is proved in [3] by means of an entirely geometric argument. Note that (1.1) with constant $k_1 = \frac{1}{3}$ holds for any non-equilateral triangle. However in that case the inequality becomes rather trivial (cf.[2], 1.1. p.11).

The second inequality has more stature. A proof may be found in [5]. This proof uses both geometric and non-geometric methods. See also [2], 3.5. p.38.

2. THE KUHN - TUCKER THEOREM

Let $f, g_1, \ldots, g_m$ be real-valued functions defined on a subset X of $\mathbb{R}^n$. Optimization problems, which can be put into the form

$$(2.1) \quad \text{Maximize } f(x), \text{ subject to } \begin{cases} g_i(x) \geqq 0 \text{ for } i = 1,\ldots,m_1 \\ g_i(x) = 0 \text{ for } i = m_1+1,\ldots,m \end{cases} \quad x \in X$$

are the subject matter of what is known as *programming* ; *linear* programming when the functions $f, g_1, \ldots, g_m$ are all linear functions and *non-linear* programming otherwise.

We define the set C as follows:

$$(2.2) \quad C = \{x \in X \mid g_i(x) \geqq 0 \text{ for } i=1,\ldots,m_1 \wedge g_i(x) = 0 \text{ for } i=m_1+1,\ldots,m\}$$

and we shall always assume that this so-called *constraint set* is non-empty.

If C is compact (i.e. closed and bounded) and f continuous, the existence of a solution to problem (2.1) is garanteed by the following well-known theorem:

THEOREM A (Weierstrass).

Let C *be a compact subset of* $\mathbb{R}^n$ *and suppose that the function* $f:\mathbb{R}^n \to \mathbb{R}$ *is continuous. Then the restriction of* f *to* C *attains a (global) maximum and a (global) minimum.*

Often the constraint set is unbounded. In that case it is not always easy, if at all possible, to prove the existence of a solution to (2.1). The only existence theorems known for such a situation relate to concave (or convex) programming and quadratic programming.

We suppose for the moment that a solution does exist. In order to find the maximal value of f attained on C, the following theorem could be of some use, although in practice it is not often applied in a constructive way.

THEOREM B (Kuhn - Tucker).

Let $f, g_1, \ldots, g_m$ *be real-valued totally differentiable functions defined on a non-empty open subset* X *of* $\mathbb{R}^n$*. Further, let* C *be defined as in* (2.2)*. For every* $x \in C$*, we define* E(x) *to be the set of all indices* $j \in \{1, \ldots, m_1\}$ *for which* $g_j(x) = 0$*. More over, let* f *attain a local maximum on* C *in the point* $\hat{x}$*. Assume that at least one of the following regularity conditions is satisfied:*

R1. *All constraint functions* g_i *are linear*

R2. *The set of gradient vectors* $\{\nabla g_i(\hat{x}) \mid i \in E(\hat{x}) \vee i \in \{m_1+1, \ldots, m\}\}$ *is linear independent.*

*Then the following conditions (*first order *conditions or* Kuhn-Tucker *conditions) are fulfilled:*

There exist real numbers $\hat{\lambda}_1,\dots,\hat{\lambda}_m$ *such that*

$$(2.3)\quad \begin{cases} \nabla f(\hat{x}) + \sum_{i=1}^{m} \hat{\lambda}_i \, g_i(\hat{x}) = 0 \\ \hat{\lambda}_i g_i(\hat{x}) = 0, \; i = 1,\dots,m \\ g_i(\hat{x}) \geqq 0 \text{ and } \hat{\lambda}_i \geqq 0, \; i = 1,\dots,m_1 \\ g_i(\hat{x}) = 0, \; i = m_1+1,\dots,m \end{cases}$$

REMARKS.

The notation $\nabla f(\hat{x})$ stands for "the gradient of f in $\hat{x}$" i.e. $\nabla f(\hat{x}) := (\partial f/\partial x_1,\dots,\partial f/\partial x_n)_{x=\hat{x}}$.

Proofs of theorem B can be found in various places e.g. [1], p.121.

There exist a wide variety of regularity conditions (cf. [4], chapter 1, section D). We have chosen R1. and R2. merely, because they prove sufficient for the applications selected.

On reversing the relevant inequality signs and replacing the phrase "local maximum" by "local minimum" in theorem B, we obtain an analoguous theorem for the problem:

$$(2.1)' \quad \text{Minimize } f(x), \text{ subject to } \begin{cases} g_i(x) \leqq 0 \text{ for } i=1,\dots,m_1 \\ g_i(x) = 0 \text{ for } i=m_1+1,\dots,m \end{cases} \quad x \in X$$

3. APPLICATIONS TO PLANE GEOMETRY

In this section we shall give proofs of the inequalities mentioned in the introduction.

LEMMA 1.

The problem $\quad$ *max* $f(x) = x_1x_2 + x_2x_3 + x_3x_1$

subject to $\quad x_1 \geqq 0,\; x_2 \geqq 0,\; x_3 \geqq 0$

$$x_1^2 \geqq x_2^2 + x_3^2$$

$$x_1 + x_2 + x_3 = 1$$

has a solution. This maximum is attained in one point only, namely $\hat{x} = (-1+\sqrt{2}, 1-\frac{1}{2}\sqrt{2}, 1-\frac{1}{2}\sqrt{2})$ *and* $f(\hat{x}) = -\frac{5}{2} + 2\sqrt{2}$.

Proof.

Clearly f is a continuous function on $\mathbb{R}^3$ and the constraint set C is compact. This shows the existence of a solution M, attained in a point $\hat{x} = (x_1, x_2, x_3)$ say. Since $\hat{x} \in C$, it is clear that $x_1 \neq 0$. More over, if $x_2 = x_3 = 0$, then $M = 0$. However, f is not identically zero on C. So x_2 and x_3 cannot vanish simultaneously. Now suppose that $x_2 x_3 = 0$. Because of symmetry, we may assume that $x_2 = 0$ and $x_3 \neq 0$. Then $M = x_1 x_3 \leqq \frac{1}{4}$, in view of the relation $x_1 + x_3 = 1$. On the other hand, $f(5t,4t,3t) = 47t^2$ and $(5t,4t,3t) \in C$ iff $t = 1/12$. But $f(5/12, 4/12, 3/12) > \frac{1}{4}$. Consequently, $x_2 x_3 \neq 0$. It is now easy to check that condition R2. of theorem B is satisfied in $\hat{x}$. Hence, real numbers $\lambda_1, \lambda_2, \lambda_3, \mu, \nu$ exist such that (see (2.3)):

$$
\begin{array}{|l}
\qquad x_2 + x_3 + \lambda_1 + 2\mu x_1 + \nu = 0 \\
\qquad x_1 + x_3 + \lambda_2 - 2\mu x_2 + \nu = 0 \\
\qquad x_1 + x_2 + \lambda_3 - 2\mu x_3 + \nu = 0 \\
x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0 \qquad \lambda_1 x_1 = \lambda_2 x_2 = \lambda_3 x_3 = 0 \\
x_1^2 \geqq x_2^2 + x_3^2 \qquad \mu(x_1^2 - x_2^2 - x_3^2) = 0 \\
x_1 + x_2 + x_3 = 1 \qquad \nu(x_1 + x_2 + x_3 - 1) = 0 \\
\qquad \lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ \lambda_3 \geqq 0,\ \mu \geqq 0.
\end{array}
$$

Since $x_1 x_2 x_3 \neq 0$, it follows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. From the first three equations we obtain by addition

$$0 = 2(x_1+x_2+x_3) + \lambda_1 + \lambda_2 + \lambda_3 + 2\mu(x_1-x_2-x_3) + 3\nu =$$

$$= 2 + 2\mu(x_1-x_2-x_3) + 3\nu.$$

If $\mu = 0$, then $\nu = -2/3$ and thus $1 - x_1 = 1 - x_2 = 1 - x_3 = 2/3$.

Hence $x_1 = x_2 = x_3 = 1/3$, but this contradicts $x_1^2 \geqq x_2^2 + x_3^2$. Thus $\mu > 0$ and consequently $x_1^2 = x_2^2 + x_3^2$. From

$$\nu + 1 = x_1(1-2\mu) = x_2(1+2\mu) = x_3(1+2\mu)$$

it follows that $x_2 = x_3$ and hence $x_1^2 = 2x_2^2$ which gives $x_1 = x_2\sqrt{2}$. Then $x_1 + x_2 + x_3 = 1$ shows that $x_1 = -1 + \sqrt{2}$ and $x_2 = x_3 = 1 - \frac{1}{2}\sqrt{2}$. We also find $\mu = \frac{3}{2} - \sqrt{2}$ and $\nu = 5 - 4\sqrt{2}$.

□

From this lemma, the following theorem can be easily deduced.

THEOREM 1.

Let a, b *and* c *be the sides of an obtuse triangle. Then inequality* (1.1) *holds and the constant* k_1 *is best possible.*

Proof.

Put $x_1 = \frac{a}{a+b+c}$, $x_2 = \frac{b}{a+b+c}$ and $x_3 = \frac{c}{a+b+c}$. The quantities x_1, x_2, x_3 satisfy

$x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $x_1 + x_2 + x_3 = 1$ and $x_1^2 > x_2^2 + x_3^2$

if we assume, without loss of generality, that $a = \max(a,b,c)$.

Lemma 1 shows that equality can only be reached in a right isosceles triangle with $b = c = \frac{1}{2}a\sqrt{2}$. That k_1 is best possible also follows from the observation that for each sufficiently small positive number δ, the triangle with sides $a = -1 + \sqrt{2} + \delta$, $b = c = 1 - \frac{1}{2}\sqrt{2} - \frac{1}{2}\delta$ is obtuse.

□

Inequality (1.2) is somewhat harder to prove. We need the following lemma.

LEMMA 2. *The problem*

$$\max f(x;y) = (x_1y_2-x_2y_1)^2 + (x_2y_3-x_3y_2)^2 + (x_3y_1-x_1y_3)^2$$

subject to

$$\begin{array}{ll} x_1 + x_2 + x_3 = 1 \quad , & y_1 + y_2 + y_3 = 1 \\ -x_1 + \tfrac{1}{2} \geqq 0 & \\ x_1 - x_2 \geqq 0 & y_1 - y_2 \geqq 0 \\ x_2 - x_3 \geqq 0 & y_2 - y_3 \geqq 0 \\ x_3 \geqq 0 & y_3 \geqq 0 \end{array}$$

is solvable. The maximum $M = \frac{1}{4}$ *is attained at* $\hat{x} = (\frac{1}{2},\frac{1}{2},0;1,0,0)$ *and at no other point of* C.

Proof.

The function f is continuous on $\mathbb{R}^6$ and the constraint set C is compact. Let f attain its maximum M on C in the point $\hat{x} = (x_1,x_2,x_3;y_1,y_2,y_3)$. Since all constraint functions are linear, the regularity condition R1. of theorem B is fulfilled. Hence, there exist real numbers α_0, α_1, α_2, α_3, β_1, β_2, β_3, λ ,μ such that (see (2.3) of theorem B):

$$\left|\begin{array}{llll} \frac{\partial f}{\partial x_1} - \alpha_0 + \alpha_1 + \lambda = 0 & , \quad \frac{\partial f}{\partial y_1} + \beta_1 + \mu = 0 & & \\ \frac{\partial f}{\partial x_2} - \alpha_1 + \alpha_2 + \lambda = 0 & \quad \frac{\partial f}{\partial y_2} - \beta_1 + \beta_2 + \mu = 0 & & \\ \frac{\partial f}{\partial x_3} - \alpha_2 + \alpha_3 + \lambda = 0 & \quad \frac{\partial f}{\partial y_3} - \beta_2 + \beta_3 + \mu = 0 & & \\ & & & \\ -x_1 + \frac{1}{2} \geqq 0 \ , & \alpha_0(-x_1+\frac{1}{2}) = 0 & & \\ x_1 - x_2 \geqq 0 & \alpha_1(x_1-x_2) = 0 \ , & y_1 - y_2 \geqq 0 \ , & \beta_1(y_1-y_2) = 0 \\ x_2 - x_3 \geqq 0 & \alpha_2(x_2-x_3) = 0 & y_2 - y_3 \geqq 0 & \beta_2(y_2-y_3) = 0 \\ x_3 \geqq 0 & \alpha_3 x_3 = 0 & y_3 \geqq 0 & \beta_3 y_3 = 0 \\ x_1 + x_2 + x_3 = 1 & & y_1 + y_2 + y_3 = 1 & \\ \multicolumn{4}{l}{\alpha_0 \geqq 0,\ \alpha_1 \geqq 0,\ \alpha_2 \geqq 0,\ \alpha_3 \geqq 0,\ \beta_1 \geqq 0,\ \beta_2 \geqq 0,\ \beta_3 \geqq 0} \end{array}\right.$$

First of all we note that $f(\frac{1}{2},\frac{1}{2},0;1,0,0) = \frac{1}{4}$. Hence $M = \max f \geqq \frac{1}{4}$.

Since $3y_3 \leqq y_1 + y_2 + y_3 = 1$ and $0 \leqq x_2 \leqq x_1 \leqq \frac{1}{2}$, we have

$$\beta_2 - \beta_3 - \mu = \frac{\partial f}{\partial y_3} = 2y_3(x_1^2+x_2^2) - 2x_3(x_1y_1+x_2y_2) \leqq \frac{1}{3}.$$

More over, as a function of y_1, y_2, y_3 alone, the function f is homogeneous of degree 2. Hence, by Euler's theorem

$$2f = y_1\frac{\partial f}{\partial y_1} + y_2\frac{\partial f}{\partial y_2} + y_3\frac{\partial f}{\partial y_3} = -\mu.$$

Combining these two results, we obtain

$$2f + \beta_2 - \beta_3 \leqq \frac{1}{3}.$$

Now, if $\beta_3 = 0$ then $\beta_2 \geqq 0$ implies that $f \leqq \frac{1}{6}$. This means that we may assume that $\beta_3 > 0$. But then $y_3 = 0$.

As a function of x_1, x_2, x_3 alone, the function f is also homogeneous of degree 2. Hence, as before,

$$2f = x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} = -\lambda + \tfrac{1}{2}\alpha_0.$$

Further, $\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} = \alpha_0 - \alpha_2 - 2\lambda = -\alpha_2 + 4f$ and also

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} = 2x_1(y_1-y_2)^2 - 2y_1(x_1-x_2)(y_1-y_2) \leqq 1,$$

since $y_3 = 0$. Thus

$$-\alpha_2 + 4f \leqq 1.$$

Suppose now that $\alpha_2 = 0$. Because we are only interested in values of $f \geqq \frac{1}{4}$, it follows from the above that

$$1 \leqq 4f = 2x_1(y_1-y_2)^2 - 2y_1(x_1-x_2)(y_1-y_2) \leqq 1$$

and this means that

$$2x_1(y_1-y_2)^2 = 1 \text{ and } 2y_1(x_1-x_2)(y_1-y_2) = 0.$$

This is only possible when $x_1 = x_2 = \frac{1}{2}$ and $y_1 = 1$, $y_2 = 0$. Consequently, $x_3 = 0$.

After some calculation we find that $0 \leqq \alpha_0 \leqq 1$, $\alpha_1 = \frac{1}{2}(1+\alpha_0)$, $(\alpha_2 = 0)$, $\alpha_3 = \frac{1}{2}(1-\alpha_0)$, $\beta_1 = 0$, $\beta_2 = 1$ and $\beta_3 = \frac{3}{2}$. Hence the first order conditions are satisfied in the point $(\frac{1}{2},\frac{1}{2},0;1,0,0)$.

We continue by assuming that $\hat{x} \neq (\frac{1}{2},\frac{1}{2},0;1,0,0)$. Then clearly $\alpha_2 > 0$ and $x_2 = x_3$. Now $2f + \frac{1}{2}\alpha_0 - \alpha_1 = \frac{\partial f}{\partial x_1} =$ $= 2y_2(x_1y_2-x_2y_1) = 2x_1y_2(y_2-y_1) + 2y_1y_2(x_1-x_2) \leqq y_1y_2 \leqq \frac{1}{4}$, because $y_1 + y_2 = 1$ (recall that $y_3 = 0$). Hence,

$$2f - \alpha_1 \leqq \tfrac{1}{4},$$

in view of $\alpha_0 \geqq 0$. From $\alpha_1 = 0$, it follows that $f \leqq \frac{1}{8}$. Hence suppose that $\alpha_1 > 0$. Then $x_1 = x_2$. Also $x_2 = x_3$ and thus $x_1 = x_2 = x_3 = \frac{1}{3}$. We have

$$4f - \beta_2 = \frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_2} = \frac{2}{9}(y_1+y_2) = \frac{2}{9}.$$

If $\beta_2 = 0$, then $f = 1/18$. And if $\beta_2 > 0$, then $y_2 = y_3 = 0$ and thus $y_1 = 1$. This implies that $2f = \frac{\partial f}{\partial y_1} = \frac{4}{9}$, since $\beta_1 = 0$ $(y_1 \neq y_2)$. But then $f = \frac{2}{9} < \frac{1}{4}$.

This proves the lemma. □

THEOREM 2.

Let a, b, c *be the sides and* α, β, γ *the corresponding angles of a triangle. Then inequality* (1.2) *holds. More over, the constant* k_2 *is best possible.*

Proof.

Put $x_1 = \frac{a}{a+b+c}$, $x_2 = \frac{b}{a+b+c}$, $x_3 = \frac{c}{a+b+c}$ and assume that $a \geqq b \geqq c$. Further, put $y_1 = \frac{\alpha}{\pi}$, $y_2 = \frac{\beta}{\pi}$ and $y_3 = \frac{\gamma}{\pi}$. Then $x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = 1$. Since $b + c > a$, we have also $x_2 + x_3 > x_1$. This shows that $x_1 < \frac{1}{2}$. In view of $a \geqq b \geqq c$, we have $\alpha \geqq \beta \geqq \gamma$ and consequently $x_1 \geqq x_2 \geqq x_3 > 0$ and $y_1 \geqq y_2 \geqq y_3 > 0$.

That k_2 is best possible may be seen as follows (in fact the proof of lemma 2 already gives evidence to that effect):

Let $\delta > 0$. Put $y_1 = \frac{\alpha}{\pi} = 1 - \frac{\delta+\delta^2}{\pi}$, $y_2 = \frac{\beta}{\pi} = \frac{\delta}{\pi}$, $y_3 = \frac{\gamma}{\pi} = \frac{\delta^2}{\pi}$ and $x_1 = a = \frac{\sin(\delta+\delta^2)}{2\delta}$, $x_2 = b = \frac{\sin\delta}{2\delta}$, $x_3 = c = \frac{\sin\delta^2}{2\delta}$.

Now let δ tend to zero. □

4. POSTSCRIPT

The most difficult part of the foregoing method in order to obtain geometric inequalities, lies in the choice of the constraint set. The relations between the elements of a triangle are often given in terms of circle functions. These functions, when appearing in the constraint functions, greatly complicate the determination of points satisfying the first order conditions.

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