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TWISTED LUBIN-TATE FORMAL GROUP LAWS,  
RAMIFIED WITT VECTORS AND (RAMIFIED)  
ARTIN-HASSE EXPONENTIALS

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*Erasmus*

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# TWISTED LUBIN-TATE FORMAL GROUP LAWS, RAMIFIED WITT VECTORS AND (RAMIFIED) ARTIN-HASSE EXPONENTIALS.

by Michiel Hazewinkel

## ABSTRACT.

For any ring  $R$  let  $\Lambda(R)$  denote the multiplicative group of power series of the form  $1 + a_1 t + \dots$  with coefficients in  $R$ . The Artin-Hasse exponential mappings are homomorphisms  $W_{p,\infty}(k) \rightarrow \Lambda(W_{p,\infty}(k))$ , which satisfy certain additional properties. Somewhat reformulated the Artin-Hasse exponentials turn out to be special cases of a functorial ring homomorphism  $E: W_{p,\infty}(-) \rightarrow W_{p,\infty}(W_{p,\infty}(-))$ , where  $W_{p,\infty}$  is the functor of infinite length Witt vectors associated to the prime  $p$ . In this paper we present ramified versions of both  $W_{p,\infty}(-)$  and  $E$ , with  $W_{p,\infty}(-)$  replaced by a functor  $W_{q,\infty}^F(-)$ , which is essentially the functor of  $q$ -typical curves in a (twisted) Lubin-Tate formal group law over  $A$ , where  $A$  is a discrete valuation ring, which admits a Frobenius like endomorphism  $\sigma$  (we require  $\sigma(a) \equiv a^q \pmod{\mathfrak{m}}$  for all  $a \in A$ , where  $\mathfrak{m}$  is the maximal ideal of  $A$ ). These ramified-Witt-vector functors  $W_{q,\infty}^F(-)$  do indeed have the property that, if  $k = A/\mathfrak{m}$  is perfect,  $A$  is complete, and  $\ell/k$  is a finite extension of  $k$ , then  $W_{q,\infty}^F(\ell)$  is the ring of integers of the unique unramified extension  $L/K$  covering  $\ell/k$ .

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## 1. INTRODUCTION.

For each ring  $R$  (commutative with unit element 1) let  $\Lambda(R)$  be the abelian group of power series of the form  $1 + r_1 t + r_2 t^2 + \dots$ . Let  $W_{p,\infty}(R)$  be the ring of Witt vectors of infinite length associated to the prime  $p$  with coefficients in  $R$ . Then the "classical" Artin-Hasse exponential mapping is a map

$$E: W_{p,\infty}(k) \rightarrow \Lambda(W_{p,\infty}(k))$$

defined for all perfect fields  $k$  as follows. (Cf e.g. [1] and [13]). Let  $\Phi(y)$  be the power series

$$\Phi(y) = \prod_{(p,n)=1} (1-y^n)^{\mu(n)/n},$$

where  $\mu(n)$  is the Möbius function. Then  $\Phi(y)$  has its coefficients in  $\mathbb{Z}_p$ , cf e.g. [13]. Because  $k$  is perfect every element of  $W_{p,\infty}(k)$  can be written in the form  $\underline{b} = \sum_{i=1}^{\infty} \tau(c_i) p^i$ , with  $c_i \in k$ , and  $\tau: k \rightarrow W_{p,\infty}(k)$  the unique system of multiplicative representants. One now defines

$$E: W_{p,\infty}(k) \rightarrow \Lambda(W_{p,\infty}(k)), \quad E(\underline{b}) = \prod_{i=0}^{\infty} \Phi(\tau(c_i) t) p^i$$

Now let  $W(-)$  be the ring functor of big Witt vectors. Then  $W(-)$  and  $\Lambda(-)$  are isomorphic, the isomorphism being given by  $(a_1, a_2, \dots) \rightarrow \prod_{i=1}^{\infty} (1 - a_i t^i)$ , cf [2]. Now there is a canonical quotient map  $W(-) \rightarrow W_{p,\infty}(-)$  and composing  $E$  with  $\Lambda(-) \simeq W(-)$  and  $W(-) \rightarrow W_{p,\infty}(-)$  we find a Artin-Hasse exponential

$$E: W_{p,\infty}(k) \rightarrow W_{p,\infty}(W_{p,\infty}(k))$$

Theorem. There exists a unique functorial homomorphism of ring-valued functors  $E: W_{p,\infty}(-) \rightarrow W_{p,\infty}(W_{p,\infty}(-))$  such that for all  $n = 0, 1, 2, \dots$   $w_{p,n} \circ E = \underline{f}^n$ , where  $\underline{f}$  is the Frobenius endomorphism of  $W_{p,\infty}(-)$  and where  $w_{p,n}: W_{p,\infty}(-) \rightarrow W_{p,\infty}(-)$  is the ring homomorphism which assigns to the sequence  $(\underline{b}_0, \underline{b}_1, \dots)$  of Witt-vectors the Witt-vector

$$b_0^p + p b_1^{p^{n-1}} + \dots + p^{n-1} b_{n-1}^p + p^n b_n.$$

It should be noted the classical definition of  $E$  given above works only for perfect fields of characteristic  $p > 0$ . In this form theorem 1.1 is probably due to Cartier, cf. [5].

Now let  $A$  be a complete discrete valuation ring with residue field of characteristic  $p$ , such that there exists a power  $q$  of  $p$  and an automorphism  $\sigma$  of  $K$ , the quotient field of  $A$ , such that  $\sigma(a) \equiv a^q \pmod{\mathfrak{m}}$  for all  $a \in A$ , where  $\mathfrak{m}$  is the maximal ideal of  $A$ . It is the purpose of the present paper to define ramified Witt vector functors

$$W_{q,\infty}^F(-): \underline{\text{Alg}}_A \rightarrow \underline{\text{Alg}}_A,$$

where  $\underline{\text{Alg}}_A$  is the category of  $A$ -algebras, and a ramified Artin-Hasse exponential mapping

$$E: W_{q,\infty}^F(-) \rightarrow W_{q,\infty}^F(W_{q,\infty}^F(-)).$$

There is such a ramified Witt-vector functor  $W_{q,\infty}^F$  associated to every twisted Lubin-Tate formal group law  $F(X,Y)$  over  $A$ . This last notion is defined as follows: let  $f(X) = X + a_2 X^2 + \dots \in K[[X]]$  and suppose that  $a_i \in A$  if  $q$  does not divide  $i$  and  $a_{qi} = \omega^{-1} \tau(a_i) \in A$  for all  $i$  for a certain fixed uniformizing element  $\omega$ . Then  $F(X,Y) = f^{-1}(f(X) + f(Y)) \in A[[X,Y]]$ , and the formal group laws thus obtained are what we call twisted Lubin-Tate formal group laws. The Witt-vector-functors  $W_{q,\infty}^F(-)$  for varying  $F$  are isomorphic if the formal group laws are strictly isomorphic. Now every twisted Lubin-Tate formal group law is strictly isomorphic to one of the form

$$G_\omega(X,Y) = g_\omega^{-1}(g_\omega(X) + g_\omega(Y)) \text{ with } g_\omega(X) = X + \omega^{-1} X^q + \omega^{-1} \sigma(\omega)^{-1} X^{q^2} + \omega^{-1} \sigma(\omega)^{-1} \sigma^2(\omega)^{-1} X^{q^3} + \dots \text{ which permits us to concentrate on the case } F(X,Y) = G_\omega(X,Y) \text{ for some } \omega; \text{ the formulas are more pleasing in this case, especially because the only constants which then appear are the } \sigma^i(\omega), \text{ which is esthetically attractive, because } \omega \text{ is an invariant of the strict isomorphism class of } F(X,Y).$$

The functors  $W_{q,\infty}^F$  and the functor morphisms  $E$  are Witt-vector-like and Artin-Hasse-exponential-like in that

- (i)  $W_{q,\infty}^F(B) = \{(b_0, b_1, \dots) \mid b_i \in B\}$  as a set valued functor and the A-algebra structure can be defined via suitable Witt-like polynomials  $w_{q,n}^F(Z_0, \dots, Z_n)$ ; cf below for more details.
- (ii) There exist a  $\sigma$ -semilinear A-algebra homomorphism  $\underline{f}$  (Frobenius) and a  $\sigma^{-1}$ -semilinear A-module homomorphism  $\underline{v}$  (Verschiebung) with the expected properties, e.g.  $\underline{f}\underline{v} = \omega$  where  $\omega$  is the uniformizing element of A associated to F, and  $\underline{f}(\underline{b}) \equiv \underline{b}^q \pmod{\omega} W_{q,\infty}^F(B)$ .
- (iii) If k, the residue field of A, is perfect and  $\ell/k$  is a finite field extension, then  $W_{q,\infty}^F(\ell) = B$ , the ring of integers of the unique unramified extension L/K which covers  $\ell/k$ .
- (iv) The Artin-Hasse exponential E is characterized by  $w_{q,n}^F \circ E = \underline{f}^n$  for all  $n = 0, 1, 2, \dots$

I hope that these constructions will also be useful in a class-field theory setting. Meanwhile they have been important in formal A-module theory; the results in question have been announced in two notes, [9] and [10], and I now propose to take half a page or so to try to explain these results to some extent.

Let R be a  $\mathbb{Z}_{(p)}$ -algebra and let  $\text{Cart}_p(R)$  be the Cartier-Dieudonné ring. This is a ring "generated" by two symbols  $f, v$  over  $W_{p,\infty}(R)$  subject to "the relations suggested by the notation used". For each formal group  $F(X, Y)$  over R let  $C_p(F; R)$  be its  $\text{Cart}_p(R)$  module of p-typical curves. Finally let  $\hat{W}_{p,\infty}(-)$  be the formal completion of the functor  $W_{p,\infty}(-)$ . Then one has

- (a) The functor  $F \mapsto C_p(F; R)$  is representable by  $\hat{W}_{p,\infty}$  ([3])
- (b) The functor  $F \mapsto C_p(F; R)$  is an equivalence of categories between the category of formal groups over R and a certain (explicitly describable) subcategory of  $\text{Cart}_p(R)$  modules ([3]).
- (c) There exists a theory of "lifting" formal groups, in which the Artin-Hasse exponential  $E: W_{p,\infty}(-) \rightarrow W_{p,\infty}(W_{p,\infty}(-))$  plays an important rôle. These results relate to the so-called "Tapis de Cartier" and relate to certain conjectures of Grothendieck concerning crystalline cohomology, ([4] and [5]).

Now let  $A$  be a complete discrete valuation ring with residue field  $k$  of  $q$ -elements (for simplicity and/or nontriviality of the theory). A formal  $A$ -module over  $B \in \underline{\text{Alg}}_A$  is a formal group law  $F(X, Y)$  over  $B$  together with a ring homomorphism  $\rho_F: A \rightarrow \text{End}_B(F(X, Y))$ , such that  $\rho_F(a) \equiv aX \pmod{(\text{degree } 2)}$ . Then there exist complete analogues of (a), (b), (c) above for the category of formal  $A$ -modules over  $B$ . Here the rôle of  $C_p(F; R)$  is taken over by the  $q$ -typical curves  $C_q(F; B)$ ,  $W_{p, \infty}(-)$  and  $\hat{W}_{p, \infty}$  are replaced by ramified-Witt vector functors  $W_{q, \infty}^\pi(-)$  and  $\hat{W}_{q, \infty}^\pi(-)$  associated to an untwisted, i.e.  $\sigma = \text{id}$ , Lubin-Tate formal group law over  $A$  with associated uniformizing element  $\pi$ . Finally, the rôle of  $E$  in (c) is taken over by the ramified Hasse-Witt exponential  $W_{q, \infty}^\pi(-) \rightarrow W_{q, \infty}^\pi(W_{q, \infty}^\pi(-))$ .

As we remarked in (i) above, it is perfectly possible to define and analyse  $W_{q, \infty}^F(-)$  by starting with the polynomials  $w_{q, n}^F(Z)$  and then proceeding along the lines of Witt's original paper. And, in fact, in the untwisted case, where  $k$  is a field of  $q$ -elements, this has been done, independantly of this paper, and independantly of each other by E. Ditters ([7]), V. Drinfel'd ([8]), J. Casey (unpublished) and, very possibly, several others. In this case the relevant polynomials are of course the polynomials  $X_0^{q^n} + \pi X_1^{q^{n-1}} + \dots + \pi^{n-1} X_{n-1}^{q^1} + \pi^n X_n$ .

Of course the twisted version is necessary if one wants to describe also all ramified discrete valuation rings with not necessarily finite residue fields. A second main reason for considering "twisted formal  $A$ -modules" is that there exist no nontrivial formal  $A$ -modules if the residue field of  $A$  is infinite.

Let me add, that, in my opinion, the formal group law approach to (ramified) Witt-vectors is technically and conceptually easier. Witness, e.g. the proof of theorem 6.6 and the ease with which one defines Artin-Hasse exponentials in this setting (cf. sections 6.1 and 6.5 below). Also this approach removes some of the mystery and exclusive status of the particular Witt polynomials

$$X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^n X_n \quad (\text{unramified case}), \quad X_0^{q^n} + \pi X_1^{q^{n-1}} + \dots + \pi^n X_n \\ (\text{untwisted ramified case}), \quad X_0^{q^n} + \sigma^{n-1}(\omega) X_1^{q^{n-1}} + \sigma^{n-1}(\omega) \sigma^{n-2}(\omega) X_2^{q^{n-2}} + \dots + \sigma^{n-1}(\omega) \dots \sigma(\omega) \omega X_n \quad (\text{twisted ramified case}).$$

From the

theoretical (if not the esthetical and/or computational) point of view all polynomials  $\tilde{w}_{q,n}(X_0, \dots, X_n) = a_n^{-1}(a_n X_0^q + a_{n-1} X_1^q + \dots + a_0 X_n^q) \in A[X]$  are equally good, provided  $a_0 = 1$ ,  $a_2, a_3, \dots$  is a sequence of elements of  $K$  such that

$a_i - \omega^{-1} \sigma(a_{i-1}) \in A$  for all  $i = 1, 2, \dots$ . Cf. in this connection also [6].

## 2. THE FUNCTIONAL-EQUATION-INTEGRALITY LEMMA.

**2.1. The Setting.** Let  $A$  be a discrete valuation ring with maximal ideal  $\mathfrak{m}$ , residue field  $k$  of characteristic  $p > 0$  and field of quotients  $K$ . Both characteristic zero and characteristic  $p > 0$  are allowed for  $K$ . We use  $v$  to denote the normalized exponential valuation on  $K$  and  $\omega$  always denotes a uniformizing element, i.e.  $v(\omega) = 1$  and  $\mathfrak{m} = \omega A$ . We assume that there exists a power  $q$  of  $p$  and an automorphism  $\sigma$  of  $K$  such that

$$(2.2) \quad \sigma(\mathfrak{m}) = \mathfrak{m}, \quad \sigma a \equiv a^q \pmod{\mathfrak{m}} \text{ for all } a \in A.$$

The ring  $A$  does not need not be complete.

Further let  $B \in \underline{\text{Alg}}_A$ , the category of  $A$ -algebras. We suppose that  $B$  is  $A$ -torsion free (i.e. that the natural map  $B \rightarrow B \otimes_A K$  is injective) and we suppose that there exists an endomorphism  $\tau : B \otimes_A K \rightarrow B \otimes_A K$  such that

$$(2.3) \quad \tau(b) \equiv b^q \pmod{\mathfrak{m}B} \text{ for all } b \in B$$

Finally let  $f(X)$  be any power series over  $B \otimes_A K$  of the form

$$(2.4) \quad f(X) = b_1 X + b_2 X^2 + \dots, \quad b_i \in B, \quad b_1 \text{ a unit of } B$$

for which there exists a uniformizing element  $\omega \in A$  such that

$$(2.5) \quad f(X) - \omega^{-1} \tau_* f(X^q) \in B[[X]]$$

where  $\tau_*$  means "apply  $\tau$  to the coefficients". In terms of the coefficients  $b_i$  of  $f(X)$  condition (2.5) means that



(2.6)  $b_i \in B[[X]]$  if  $q$  does not divide  $i$ ,

$$b_{qi} - \omega^{-1} \tau(b_i) \in B[[X]] \text{ for all } i = 1, 2, \dots$$

2.7. Functional-equation lemma. Let  $A, B, \sigma, \tau, K, p, q, f(X), \omega$  be as in 2.1 above such that (2.2) - (2.6) hold. Then we have

- (i)  $F(X, Y) = f^{-1}(f(X) + f(Y))$  has its coefficients in  $B$  and hence is a commutative one dimensional formal group law over  $B$ .  
(Here  $f^{-1}(X)$  is the "inverse function" power series of  $f(X)$ ; i.e.  $f^{-1}(f(X)) = X$ ).
- (ii) If  $g(X) \in B[[X]]$ ,  $g(0) = 0$  and  $h(X) = f(g(X))$  then we have  $h(X) - \omega^{-1} \tau_*(h(X^q)) \in B[[X]]$ .
- (iii) If  $h(X) \in B \otimes_A K[[X]]$ ,  $h(0) = 0$  and  $h(X) - \omega^{-1} \tau_*(h(X^q)) \in B[[X]]$ , then  $f^{-1}(h(X)) \in B[[X]]$ .
- (iv) If  $\alpha(X) \in B[[X]]$ ,  $\beta(X) \in B \otimes_A K[[X]]$ ,  $\alpha(0) = \beta(0) = 0$ , and  $r, m \in \mathbb{N} = \{1, 2, \dots\}$ , then  $\alpha(X) \equiv \beta(X) \pmod{\omega^r B, \text{ degree } m} \iff f(\alpha(X)) \equiv f(\beta(X)) \pmod{\omega^r B, \text{ degree } m}$ .

Proof. This lemma is a quite special case of the functional equation lemmas of [11], cf sections 2.2 and 10.2. There are also infinite dimensional versions. Here is a quick proof. First notice that (2.6) implies (with induction) that

$$(2.8) \quad b_j \in \omega^{-i} B, \text{ if } j \text{ is not divisible by } q^{i+1}.$$

We now first prove a more general form of (ii). Let  $g(Z) = g(Z_1, \dots, Z_m) \in B[[Z_1, \dots, Z_m]]$ ,  $g(0) = 0$ . Then by the hypotheses of 2.1 we have

$$(2.9) \quad g(Z_1, \dots, Z_m)^{q^{rn}} \equiv \tau(g(Z_1^q, \dots, Z_m^q))^{q^{r-1}n} \pmod{\omega^r B}$$

Combining (2.8) and (2.9) and using (2.6) we see that  $\text{mod}(B[[X]])$  we have

$$\begin{aligned} h(Z) = f(g(Z)) &= \sum_{i=1}^{\infty} b_i g(Z)^i \equiv \sum_{j=1}^{\infty} b_{qj} g(Z)^{qj} \equiv \omega^{-1} \sum_{j=1}^{\infty} \tau(b_j) g(Z)^{qj} \\ &\equiv \omega^{-1} \sum_{j=1}^{\infty} \tau(b_j) \tau_*(g(Z^q))^j = \omega^{-1} \tau_*(f(\tau_*(g(Z^q)))) = \omega^{-1} \tau_*(h(Z^q)). \end{aligned}$$

This proves (ii). To prove (i) we write  $F(X,Y) = F_1(X,Y) + F_2(X,Y) + \dots$ , where  $F_n(X,Y)$  is homogeneous of degree  $n$ . We now prove by induction that  $F_n(X,Y) \in B[X,Y]$  for all  $n = 1, 2, \dots$ . The induction starts because  $F_1(X,Y) = X + Y$ . Now assume that  $F_1(X,Y), \dots, F_m(X,Y) \in B[X,Y]$ . Mod(degree  $m+2$ ) we have that  $f(F(X,Y)) \equiv b_1 F_{m+1}(X,Y) + f(g(X,Y))$ , where  $g(X,Y) = F_1(X,Y) + \dots + F_m(X,Y)$ . Hence, using the more general form of (ii) proved just above, we find mod  $(B[[X,Y]], \text{degree } m+2)$ .

$$\begin{aligned}
 f(F(X,Y)) &\equiv b_1 F_{m+1}(X,Y) + f(g(X,Y)) \equiv b_1 F_{m+1}(X,Y) + \omega^{-1} \tau_* f(\tau_* g(X^q, Y^q)) \equiv \\
 &\equiv b_1 F_{m+1}(X,Y) + \omega^{-1} \tau_* f(\tau_* F(X^q, Y^q)) \\
 &= b_1 F_{m+1}(X,Y) + \omega^{-1} \tau_* f(X^q) + \omega^{-1} \tau_* f(Y^q) \\
 &\equiv b_1 F_{m+1}(X,Y) + f(X) + f(Y) = b_1 F_{m+1}(X,Y) + f(F(X,Y))
 \end{aligned}$$

where we have used the defining relation  $f(F(X,Y)) = f(X) + f(Y)$ , which implies  $\tau_* f(\tau_* F(X^q, Y^q)) = \tau_* f(X^q) + \tau_* f(Y^q)$ , and where we have also used that  $F(X,Y) \equiv g(X,Y) \pmod{\text{degree } m+1} \Rightarrow F(X^q, Y^q) \equiv g(X^q, Y^q) \pmod{\text{degree } m+2}$ . It follows that  $b_1 F_{m+1}(X,Y) \equiv 0 \pmod{(B[[X,Y]], \text{degree } m+2)}$  and hence  $F_{m+1}(X,Y) \in B[X,Y]$  because  $b_1$  is a unit.

The proof of (iii) is completely analogous to the proof of (i).

The implication  $\Rightarrow$  of (iv) is easy to prove. If  $\alpha(X) \equiv \beta(X) \pmod{\omega^r B, \text{degree } m}$  and  $\alpha(X) \in B[[X]]$  then  $\alpha(X)^{q^i j} \equiv \beta(X)^{q^i j} \pmod{\omega^{r+i} B, \text{degree } m}$ , which, combined with (2.8), proves that  $f(\alpha(X)) \equiv f(\beta(X)) \pmod{\omega^r B, \text{degree } m}$ . To prove the inverse implication  $\Leftarrow$  of (iv) we first do the special case  $f(\beta(X)) \equiv 0 \pmod{\omega^r B, \text{degree } m} \Rightarrow \beta(X) \equiv 0 \pmod{\omega^r B, \text{degree } m}$ . Now  $\beta(X) \equiv 0 \pmod{\text{degree } 1}$ , hence  $f(\beta(X)) = b_1 \beta(X) + b_2 \beta(X)^2 + \dots \equiv 0 \pmod{\omega^r B, \text{degree } m}$ , implies  $\beta(X) \equiv 0 \pmod{\omega^r B, \text{degree } 2}$ , if  $m \geq 2$  (if  $m = 1$  there is nothing to prove), because  $b_1$  is a unit. Now assume with induction that  $\beta(X) \equiv 0 \pmod{\omega^r B, \text{degree } n}$  for some  $n < m$ . Then, because  $\beta(X) \equiv 0 \pmod{\text{degree } 1}$  we have  $\beta(X)^i \equiv 0 \pmod{\omega^{r+i} B, \text{degree } (n+i-1)}$  and hence  $b_j \beta(X)^j \equiv 0 \pmod{\omega^r B, \text{degree } n+1}$  if  $j \geq 2$ . Hence  $f(\beta(X)) \equiv 0 \pmod{\omega^r B, \text{degree } m}$  then gives  $b_1 \beta(X) \equiv 0 \pmod{\omega^r B, \text{degree } n+1}$ , so that  $\beta(X) \equiv 0 \pmod{\omega^r B, \text{degree } n+1}$  because  $b_1$  is a unit. This proves this special case of (iv). Now let  $f(\alpha(X)) \equiv f(\beta(X)) \pmod{\omega^r B, \text{degree } m}$ .

Write  $\gamma(X) = f(\beta(X)) - f(\alpha(X))$  and  $\delta(X) = f^{-1}(\gamma(X))$ . Then  $\delta(X) \equiv 0 \pmod{\omega^r B}$ , degree  $m$  by the special case just proved, and  $\beta(X) = f^{-1}(f(\alpha(X)) + f(\delta(X))) = F(\alpha(X), \delta(X)) \equiv \alpha(X) \pmod{\omega^r B}$ , degree  $m$  because  $F(X,Y)$  has integral coefficients,  $F(X,0) = 0$  and because  $\alpha(X)$  is integral. This concludes the proof of the functional equation lemma 2.7.

### 3. TWISTED LUBIN-TATE FORMAL A-MODULES.

3.1. Construction and Definition. Let  $A, K, k, p, \mathfrak{m}, \sigma, q$  be as in 2.1 above. We consider power series  $f(X) = X + c_2 X^2 + \dots \in K[[X]]$  such that there exists a uniformizing element  $\omega \in \mathfrak{m}$  such that

$$(3.2) \quad f(X) - \omega^{-1} \tau_* f(X^q) \in A[[X]]$$

There are many such power series. The simplest are obtained as follows: choose a uniformizing element  $\omega$  of  $A$ . Define

$$(3.3) \quad g_\omega(X) = X + \omega^{-1} X^q + \omega^{-1} \tau(\omega)^{-1} X^{q^2} + \omega^{-1} \sigma(\omega)^{-1} \sigma^2(\omega)^{-1} X^{q^3} + \dots$$

Given such a power series  $f(X)$ , part (i) of the functional equation lemma says that

$$(3.4) \quad F(X,Y) = f^{-1}(f(X) + f(Y))$$

has its coefficients in  $A$ , and hence is a one dimensional formal group law over  $A$ . We shall call the formal group laws thus obtained twisted Lubin-Tate formal A-modules over A. The twisted Lubin-Tate formal A-module is called q-typical if the power series  $f(X)$ , from which it is obtained, is of the form

$$(3.5) \quad f(X) = X + a_1 X^q + a_2 X^{q^2} + \dots$$

From now on all twisted Lubin-Tate formal A-modules will be assumed to be q-typical. This is hardly a restriction because of lemma 3.6 below.

3.6. Lemma. Let  $f(X) = X + c_2 X^2 + \dots \in K[[X]]$  be such that (3.2) holds. Let  $\hat{f}(X) = \sum_{i=0}^{\infty} a_i X^{q^i}$  with  $a_0 = 1$ ,  $a_i = c_{\frac{i}{q}}$ . Then  $u(X) = \hat{f}^{-1}(f(X)) \in A[[X]]$

so that  $F(X,Y)$  and  $\hat{F}(X,Y)$  are strictly isomorphic formal group laws over  $A$ .

Proof. It follows from the definition of  $\hat{f}(X)$ , that  $\hat{f}(X)$  also satisfies (3.2). The integrality of  $u(X)$  now follows from part (iii) of the functional equation lemma.

3.7. Remarks. Let  $k$ , the residue field of  $K$ , be finite with  $q$  elements, and let  $\tau = \text{id}$ . Then the twisted Lubin-Tate formal  $A$ -modules over  $A$  as defined above are precisely the Lubin-Tate formal group laws defined in [12], i.e. they are precisely the formal  $A$ -modules of  $A$ -height 1. If  $k$  is infinite there exist no nontrivial formal  $A$ -modules (cf [11], corollary 21.4.23). This is a main reason for considering also twisted Lubin-Tate formal group laws.

3.8. Remark. Let  $f(X) \in K[[X]]$  be such that (3.2) holds for a certain uniformizing element  $\omega$ . Then  $\omega$  is uniquely determined by  $f(X)$ , because  $a_i - \omega^{-1}\tau(a_{i-1}) \in A \Rightarrow \omega \equiv a_i^{-1}\tau(a_{i-1}) \pmod{\omega^{2i}A}$  as  $v(a_i) = -i$ . Using parts (ii) and (iii) of the functional equation lemma we see that  $\omega$  is in fact an invariant of the strict isomorphism class of  $F(X,Y)$ . Inversely given  $\omega$  we can construct  $g_\omega(X)$  as in 3.3 and then  $g_\omega^{-1}(f(X)) = u(X)$  is integral so that  $F(X,Y)$  and  $G_\omega(X,Y) = g_\omega^{-1}(g_\omega(X) + g_\omega(Y))$  are strictly isomorphic formal group laws. In case  $\mathbb{X}k = q$  and  $\tau = \text{id}$ ,  $\omega$  is in fact an invariant of the isomorphism class of  $F(X,Y)$ . For some more results on isomorphisms and endomorphisms of twisted Lubin-Tate formal  $A$ -modules cf [11], especially sections 8.3, 20.1, 21.8, 24.5.

#### 4. CURVES AND $q$ -TYPICAL CURVES.

Let  $F(X,Y)$  be a  $q$ -typical twisted Lubin-Tate formal  $A$ -module obtained via (3.4) from a power series  $f(X) = X + a_1X^q + a_2X^{q^2} + \dots$ .

4.1. Curves. Let  $\underline{\text{Alg}}_A$  be the category of  $A$ -algebras. Let  $B \in \underline{\text{Alg}}_A$ . A curve in  $F$  over  $B$  is simply a power series  $\gamma(t) \in B[[t]]$  such that  $\gamma(0) = 0$ . Two curves can be added by the formula  $\gamma_1(t) +_F \gamma_2(t) = F(\gamma_1(t), \gamma_2(t))$ , giving us an abelian group  $C(F;B)$ . Further, if  $\phi: B_1 \rightarrow B_2$  is in  $\underline{\text{Alg}}_A$ , then  $\gamma(t) \mapsto \phi_*\gamma(t)$  (= "apply  $\phi$  to the coefficients") defines a homomorphism of abelian groups  $C(F;B_1) \rightarrow C(F;B_2)$ . This defines us an abelian group valued functor  $C(F;-): \underline{\text{Alg}}_A \rightarrow \underline{\text{Ab}}$ . There is a natural filtration on  $C(F;-)$  defined by the filtration subgroups  $C^n(F;B) = \{\gamma(t) \in C(F;B) \mid \gamma(t) \equiv 0 \pmod{\text{degree } n}\}$ . The groups  $C(F;B)$  are complete with respect to the topology defined by the filtration

$C^n(F;B)$ ,  $n = 1, 2, \dots$

The functor  $C(F; -)$  is representable by the  $A$ -algebra  $A[S] = A[S_1, S_2, \dots]$ . The isomorphism  $\text{Alg}_A(A[S], B) \xrightarrow{\sim} C(F; -)$  is given by  $\phi \mapsto \sum_{i=1}^{\infty} F \phi(S_i) t^i$ , i.e.

by  $\phi \mapsto \phi_* \gamma_S(t)$ , where  $\gamma_S(t)$  is the "universal curve"

$$\gamma_S(t) = \sum_{i=1}^{\infty} F S_i t^i \in C(F; A[S]).$$

**4.2. g-typification.** Let  $\gamma_S(t) \in C(F; A[S])$  be the universal curve. Consider the power series.

$$h(t) = f(\gamma_S(t)) = \sum_{i=1}^{\infty} x_i(S) t^i$$

Let  $\tau: K[S] \rightarrow K[S]$  be the ring endomorphism defined by  $\tau(a) = \sigma(a)$  for  $a \in K$  and  $\tau(S_i) = S_i^q$  for  $i = 1, 2, \dots$ . Then the hypotheses of 2.1 are fulfilled and it follows from part (ii) of the functional equation lemma that  $h(t) - \omega^{-1} \tau_* h(t^q) \in A[S][[t]]$ . Now let

$$\hat{h}(t) = \sum_{i=0}^{\infty} x_i^q(S) t^{q^i}$$

Then, obviously, also  $\hat{h}(t) - \omega^{-1} \tau_* \hat{h}(t^q) \in A[S][[t]]$  and by part (iii) of the functional equation lemma it follows that

$$(4.3) \quad \varepsilon_q \gamma_S(t) = f^{-1} \left( \sum_{i=0}^{\infty} x_i^q(S) t^{q^i} \right)$$

is an element of  $A[S][[t]]$ . We now define a functorial group homomorphism  $\varepsilon_q: C(F; -) \rightarrow C(F; -)$  by the formula

$$(4.4) \quad \varepsilon_q \gamma(t) = (\phi_\gamma)_* (\varepsilon_q \gamma_S(t))$$

for  $\gamma(t) \in C(F; B)$ , where  $\phi_\gamma: A[S] \rightarrow B$  is the unique  $A$ -algebra homomorphism such that  $\phi_\gamma \gamma_S(t) = \gamma(t)$ .

**4.5. Lemma.** Let  $B$  be  $A$ -torsion free so that  $B \rightarrow B \otimes_A K$  is injective. Then we have for all  $\gamma(t) \in C(F; B)$

$$(4.6) \quad f(\gamma(t)) = \sum_{i=1}^{\infty} b_i t^i \Rightarrow f(\varepsilon_q \gamma(t)) = \sum_{j=0}^{\infty} b_j t^{q^j}$$

and  $\epsilon_q C(F; B) = \{\gamma(t) \in C(F; B) \mid f(\gamma(t)) = \sum c_j t^{q^j} \text{ for certain } c_j \in B \otimes_A K\}$

Proof. Immediate from (4.3) and (4.4).

4.7. Lemma.  $\epsilon_q$  is a functorial, idempotent, group endomorphism of  $C(F; -)$ .

Proof.  $\epsilon_q$  is functorial by definition. The facts that  $\epsilon_q \epsilon_q = \epsilon_q$  and that  $\epsilon_q$  is a group homomorphism are obvious from Lemma 4.5 in case  $B$  is  $A$ -torsion free. Functoriality then implies that these properties hold for all  $A$ -algebras  $B$ .

4.8. The functor  $C_q(F; -)$  of  $q$ -typical curves. We now define the abelian group valued functor  $C_q(F; -)$  as

$$(4.9) \quad C_q(F; -) = \epsilon_q C(F; -)$$

For each  $n \in \mathbb{N} \cup \{0\}$  let  $C_q^{(n)}(F; B)$  be the subgroup  $C_q(F; B) \cap C^{q^n}(F; B)$ . These groups define a filtration on  $C_q(F; B)$ , and  $C_q(F; B)$  is complete with respect to the topology defined by this filtration.

The functor  $C_q(F; -)$  is representable by the  $A$ -algebra  $A[T] = A[T_0, T_1, \dots]$ .

Indeed, writing  $f(X) = \sum_{i=0}^{\infty} a_i X^{q^i}$  we have

$$f(\gamma_S(t)) = f\left(\sum_{i=1}^{\infty} F_{S_i} t^{q^i}\right) = \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} a_j S_i^{q^j} t^{q^j q^i}$$

and it follows that

$$\epsilon_q \gamma_S(t) = \sum_{j=0}^{\infty} F_{S_j} t^{q^j}$$

From this one easily obtains that the functor  $C_q(F; -)$  is representable by  $A[T]$ . The isomorphism  $\text{Alg}_A(A[T], B) \xrightarrow{\sim} C_q(F; B)$  is given by

$\phi \mapsto \sum_{i=0}^{\infty} F_{\phi(T_i)} t^{q^i} = \phi_*(\gamma_T(t))$ , where  $\gamma_T(t)$  is the universal  $q$ -typical curve

$$(4.10) \quad \gamma_T(t) = \sum_{i=0}^{\infty} F_{T_i} t^{q^i} \in C_q(F; A[T])$$

4.11. Remarks. The explicit formulas of 4.8 above depend on the fact



that  $F$  was supposed to be  $q$ -typical. In general slightly more complicated formulae hold. For arbitrary formal groups  $q$ -typification (i.e.  $\varepsilon_q$ ) is not defined (unless  $q=p$ ). But a similar notion of  $q$ -typification exists for formal  $A$ -modules of any height and any dimension if  $k = q$ .

## 5. THE $A$ -ALGEBRA STRUCTURE ON $C_q(F; -)$ , FROBENIUS AND VERSCHIEBUNG.

5.1. From now on we assume that  $f(X) = g_\omega(X) = X + \omega^{-1}X^q + \omega^{-1}\sigma(\omega)^{-1}X^{q^2} + \dots$  for a certain uniformizing element  $\omega$ . Otherwise we keep the notations and assumptions of section 4. Thus we now have  $a_i^{-1} = \omega\sigma(\omega) \dots \sigma^{i-1}(\omega)$ ,  $a_0 = 1$ . This restriction to "logarithms"  $f(X)$  of the form  $g_\omega(X)$  is not very serious, because every twisted Lubin-Tate formal  $A$ -module over  $A$  is strictly isomorphic to a  $G_\omega(X, Y)$ , (cf. remark 3.8), and one can use the strict isomorphism  $g_\omega^{-1}(f(X))$  to transport all the extra structure on  $C_q(F; -)$  which we shall define in this section. The restriction  $f(X) = g_\omega(X)$  does have the advantage of simplifying the defining formulas (5.4), (5.5), (5.8),... somewhat, and it makes them look rather more natural especially in view of the fact that  $\omega$ , the only "constant" which appears, is an invariant of strict isomorphism classes of twisted Lubin-Tate formal  $A$ -modules; cf. remark 3.8 above.

In this section we shall define an  $A$ -algebra structure on the functor  $C_q(F; -)$  and two endomorphisms  $\underline{f}_\omega$  and  $\underline{v}_q$ . These constructions all follow the same pattern, the same pattern as was used to define and analyse  $\varepsilon_q$  in section 4 above. First one defines the desired operations for universal curves like  $\gamma_T(t)$  which are defined over rings like  $A[T]$ , which, and this is the crucial point, admit an endomorphism  $\tau: K[T] \rightarrow K[T]$ , viz.  $\tau(a) = \sigma(a)$ ,  $\tau(T_i) = T_i^q$ , which extends  $\sigma$  and which is such that  $\tau(x) \equiv x^q \pmod{\omega A[T]}$ . In such a setting the functional equation lemma assures us that our constructions do not take us out of  $C(F; -)$  or  $C_q(F; -)$ . Second, the definitions are extended via representability and functoriality, and thirdly, one derives a characterization which holds over  $A$ -torsion free rings, and using this, one proves the various desired properties like associativity of products,  $\sigma$ -semilinearity of  $\underline{f}_\omega$ , etc...

5.2. Constructions. Let  $\gamma_T(t)$  be the universal  $q$ -typical curve (4.9). We write

$$(5.3.) \quad f(\gamma_T(t)) = \sum_{i=0}^{\infty} x_i(T) t^{q^i}$$

Let  $f(X) = g_{\omega}(X) = \sum_{i=0}^{\infty} a_i X^{q^i}$ ; i.e.  $a_i = \omega^{-1} \sigma(\omega)^{-1} \dots \sigma^{i-1}(\omega)^{-1}$  and let  $a \in A$ .

We define

$$(5.4) \quad \{a\}_{F\gamma_T}(t) = f^{-1} \left( \sum_{i=0}^{\infty} \sigma^i(a) x_i(T) t^{q^i} \right)$$

$$(5.5) \quad \underline{f}_{\omega}\gamma_T(t) = f^{-1} \left( \sum_{i=0}^{\infty} \sigma^i(\omega) x_{i+1}(T) t^{q^i} \right)$$

The functional equation lemma now assures us that (5.4) and (5.5) define elements of  $C(F;A[T])$ , which then are in  $C_q(F;A[T])$  by lemma 4.5. To illustrate this we check the hypotheses necessary to apply (iii) of 2.7 in the case of  $\underline{f}_{\omega}$ . Let  $\tau : K[T] \rightarrow K[T]$  be as in 5.1 above. Then by part (ii) of the functional equation lemma we know that

$$x_0 \in A[T], \quad x_{i+1} - \omega^{-1} \tau(x_i) = c_i \in A[T]$$

It follows by induction that

$$(5.6) \quad x_i \in \omega^{-i} A[T]$$

and we also know that

$$(5.7) \quad v(a_i^{-1}) = v(\omega \sigma(\omega) \dots \sigma^{i-1}(\omega)) = i$$

where  $v$  is the normalized exponential valuation on  $K$ . We thus have

$\sigma^0(\omega)x_1 = \omega x_1 \in A[T]$  and  $\sigma^i(\omega)x_{i+1} - \omega^{-1} \tau(\sigma^{i-1}(\omega)x_i) = \sigma^i(\omega)c_i + \sigma^i(\omega)\omega^{-1} \tau(x_i) - \omega^{-1} \tau(\sigma^{i-1}(\omega)x_i) = \sigma^i(\omega)c_i \in A[T]$ . Hence part (iii) of the functional equation lemma says that  $\underline{f}_{\omega}\gamma_T(t) \in C(F;A[T])$ .

To define the multiplication on  $C_q(F;-)$  we need two independant

universal  $q$ -typical curves. Let

$$\gamma_T(t) = \sum^{\mathbb{F}} r_i t^{q^i}, \quad \delta_{\hat{T}}(t) = \sum^{\mathbb{F}} \hat{r}_i t^{q^i} \in C_q(F; A[T; \hat{T}]).$$

We define

$$(5.8) \quad \gamma_T(t) * \delta_{\hat{T}}(t) = f^{-1} \left( \sum_{i=0}^{\infty} a_i^{-1} x_i y_i t^{q^i} \right)$$

where  $f(\gamma_T(t)) = \sum x_i t^{q^i}$ ,  $f(\delta_{\hat{T}}(t)) = \sum y_i t^{q^i}$ . To prove that (5.8)

defines something integral we proceed as usual. We have  $x_o, y_o \in A[T; \hat{T}]$ ,

$x_{i+1} - \omega^{-1} \tau(x_i) = c_i \in A[T; \hat{T}]$ ,  $y_{i+1} - \omega^{-1} \tau(y_i) = d_i \in A[T; \hat{T}]$ , where

$\tau: K[T; \hat{T}] \rightarrow K[T; \hat{T}]$  is defined by  $\tau(a) = a$  for  $a \in K$ , and  $\tau(T_i) = T_i^q$ ,  $\tau(\hat{T}_i) = \hat{T}_i^q$ ,  $i = 0, 1, 2, \dots$ .

Then  $a_o x_o y_o = x_o y_o \in A[T; T]$  and  $a_{i+1}^{-1} x_{i+1} y_{i+1} - \omega^{-1} \tau(a_i^{-1} x_i y_i) =$

$$\omega \sigma(a_i)^{-1} (c_i + \omega^{-1} \tau(x_i)) (d_i + \omega^{-1} \tau(y_i)) - \omega^{-1} \sigma(a_i^{-1}) \tau(x_i) \tau(y_i) =$$

$$\omega \sigma(a_i^{-1}) c_i d_i + \sigma(a_i)^{-1} (c_i \tau(y_i) + d_i \tau(x_i)) \in A[T; \hat{T}] \text{ by (5.6) and (5.7).}$$

**5.9. Definitions.** Let  $\gamma(t)$ ,  $\delta(t)$  be two  $q$ -typical curves in  $F$  over  $B \in \underline{\text{Alg}}_A$ . Let  $\phi: A[T] \rightarrow B$  be the unique  $A$ -algebra homomorphism such that  $\phi_* \gamma_T(t) = \gamma(t)$ , and let  $\psi: A[T; \hat{T}] \rightarrow B$  be the unique  $A$ -algebra homomorphism such that  $\psi \gamma_T(t) = \gamma(t)$ ,  $\psi_* \delta_{\hat{T}}(t) = \delta(t)$ . Let  $a \in A$ . We define

$$(5.10) \quad \{a\}_F \gamma(t) = \phi_* (\{a\}_F \gamma_T(t))$$

$$(5.11) \quad \underline{f}_{\omega} \gamma(t) = \phi_* (\underline{f}_{\omega} \gamma_T(t))$$

$$(5.12) \quad \gamma(t) * \delta(t) = \psi_* (\gamma_T(t) * \delta_T(t))$$

**5.13. Characterizations.** Let  $B$  be an  $A$ -torsion free  $A$ -algebra; i.e.  $B \rightarrow B \otimes_A K$  is injective, then the definitions (5.10) - (5.12) are characterized by the implications

$$(5.14) \quad f(\gamma(t)) = \sum_{i=0}^{\infty} x_i t^{q^i} \Rightarrow f(\{a\}_F \gamma(t)) = \sum_{i=0}^{\infty} \sigma^i(a) x_i t^{q^i}$$

$$(5.15) \quad f(\gamma(t)) = \sum_{i=0}^{\infty} x_i t^{q^i} \Rightarrow f(\underline{f}_{\omega} \gamma(t)) = \sum_{i=0}^{\infty} \sigma^i(\omega) x_{i+1} t^{q^i}$$

$$(5.16) \quad f(\gamma(t)) = \sum_{i=0}^{\infty} x_i t^{q^i}, \quad f(\delta(t)) = \sum_{i=0}^{\infty} y_i t^{q^i} \Rightarrow$$

$$f(\gamma(t)*\delta(t)) = \sum_{i=0}^{\infty} a_i^{-1} x_i y_i t^{q^i}$$

This follows immediately from (5.4), (5.5), (5.8) compared with (5.10) - (5.12), because  $\phi_*$  and  $\psi_*$  are defined by applying  $\phi$  and  $\psi$  to coefficients, and because  $\gamma(t) \mapsto f(\gamma(t))$  is injective, if  $B$  is  $A$ -torsion free.

5.17. Theorem. The operators  $\{a\}_F$  defined by (5.10) define a functorial  $A$ -module structure on  $C_q(F; -)$ . The multiplication  $*$  defined by (5.12) then makes  $C_q(F; -)$  an  $A$ -algebra valued functor, with as unit element the  $q$ -typical curve  $\gamma_0(t) = t$ . The operator  $f_{\omega}$  is a  $\sigma$ -semilinear  $A$ -algebra homomorphism; i.e.  $f_{\omega}$  is a unit and multiplication preserving group endomorphism such that  $f_{\omega}\{a\}_F = \{\sigma(a)\}_{F \neq \omega}$ .

Proof. In case  $B$  is  $A$ -torsion free the various identities in  $C_q(F; B)$  like  $\{a\}_F \gamma(t) * \delta(t) = \{a\}_F (\gamma(t) * \delta(t))$ ,

$$\gamma(t) * (\delta(t) +_F \varepsilon(t)) = (\gamma(t) * \delta(t)) +_F (\gamma(t) * \varepsilon(t)), \dots$$

are obvious from the characterizations (5.14) - (5.16). The theorem then follows by functoriality.

5.18. Verschiebung. We now define the Verschiebung operator  $V_q$  on  $C_q(F; -)$  by the formula  $V_q \gamma(t) = \gamma(t^q)$ . (It is obvious from lemma 4.5 that this takes  $q$ -typical curves into  $q$ -typical curves). In terms of the logarithm  $f(X)$  one has for curves  $\gamma(t)$  over  $A$ -torsion free  $A$ -algebras  $B$

$$(5.19) \quad f(\gamma(t)) = \sum_{i=0}^{\infty} x_i t^{q^i} \Rightarrow f(V_q \gamma(t)) = \sum_{i=0}^{\infty} x_i t^{q^{i+1}}$$

5.20. Theorem. For  $q$ -typical curves  $\gamma(t)$  in  $F$  over an  $A$ -algebra  $B$

$$(5.21) \quad f_{\omega} V_q \gamma(t) = \{\omega\}_F \gamma(t)$$

$$(5.22) \quad f_{\omega} \gamma(t) = \gamma(t)^{*q} \bmod \{\omega\}_F C_q(F; B)$$

Proof. (5.21) is immediate from (5.14), (5.15) and (5.19) in the case of  $A$ -torsion free  $B$  and then follows in general by functoriality.

The proof of (5.22) is a bit longer. It suffices to prove (5.22) for curves  $\gamma(t) \in C_q(F; A[[T]])$ . In fact it suffices to prove (5.22)

for  $\gamma(t) = \gamma_T(t)$ , the universal curve of (4.9). Let

$$(5.23) \quad \delta(t) = f^{-1} \left( \sum_{i=0}^{\infty} y_i t^{q^i} \right), \quad y_i = x_{i+1} - \sigma^i(\omega)^{-1} a_i a_i^{-q} x_i^q$$

where the  $x_i$ ,  $i = 0, 1, 2, \dots$  are determined by  $f(\gamma(t)) = \sum x_i t^{q^i}$ .

It then follows from (5.14) - (5.16) that indeed

$\underline{f}_\omega \gamma(t) - \gamma(t)^{*q} = \{\omega\}_F \delta(t)$ , provided that we can show that  $\delta(t)$  is integral, i.e. that  $\delta(t) \in C_q(F; A[T])$ . To see this it suffices to show that  $y_0 \in A[T]$  and  $y_{i+1} - \omega^{-1} \tau(y_i) \in A[T]$  because of part (iii) of the functional equation lemma. Let  $c_i = x_{i+1} - \omega^{-1} \tau(x_i) \in A[T]$ . Then

$$y_0 = x_1 - \sigma^0(\omega)^{-1} x_0^q = c_0 + \omega^{-1} \tau(x_0) - \omega^{-1} x_0^q \in A[T]$$

because  $\tau(x_0) \equiv x_0^q \pmod{\omega A[T]}$ . Further from  $x_{i+1} = c_i + \omega^{-1} \tau(x_i)$  we find

$$\begin{aligned} a_{i+1}^{-1} x_{i+1} &= \omega \sigma(\omega) \dots \sigma^i(\omega) c_i + \sigma(\omega) \dots \sigma^i(\omega) \tau(x_i) = \\ &= \omega^{i+1} d_i + \tau(a_i^{-1} x_i) \end{aligned}$$

for a certain  $d_i \in A[T]$ , and hence

$$a_{i+1}^{-q} x_{i+1}^q = \tau(a_i^{-q} x_i^q) + \omega^{i+2} e_i$$

for a certain  $e_i \in A[T]$ . It follows that

$$\begin{aligned} y_{i+1} - \omega^{-1} \tau(y_i) &= x_{i+2} - \sigma^{i+1}(\omega)^{-1} a_{i+1} a_{i+1}^{-q} x_{i+1}^q - \omega^{-1} \tau(x_{i+1}) + \\ &+ \omega^{-1} \tau(\sigma^i(\omega)^{-1} a_i a_i^{-q} x_i^q) \\ &= c_{i+1} - \sigma^{i+1}(\omega)^{-1} (a_{i+1} a_{i+1}^{-q} x_{i+1}^q - \omega^{-1} \sigma(a_i) \tau(a_i^{-q} x_i^q)) \\ &= c_{i+1} - \sigma^{i+1}(\omega)^{-1} a_{i+1} (a_{i+1}^{-q} x_{i+1}^q - \tau(a_i^{-q} x_i^q)) \in A[T] \end{aligned}$$

because  $a_{i+1} = \omega^{-1} \sigma(a_i)$  and because of (5.23). (Recall that  $v(a_{i+1}) = -i - 1$  by (5.7)). This concludes the proof of theorem 5.20.

## 6. RAMIFIED WITT VECTORS AND RAMIFIED ARTIN-HASSE EXPONENTIALS.

We keep the assumptions and notations of section 5 above.

6.1. A preliminary Artin-Hasse exponential. Let  $B$  be an  $A$ -algebra which is  $A$ -torsion free and which admits an endomorphism  $\tau : B \otimes_A K \rightarrow B \otimes_A K$  which restricts to  $\sigma$  on  $A \otimes_A K = K \subset B \otimes_A K$  and which is such that  $\tau(b) \equiv b^q \pmod{\omega B}$ . We define a map  $\Delta_B : B \rightarrow C_q(F; B)$  as follows

$$(6.2) \quad \Delta_B(b) = f^{-1} \left( \sum_{i=0}^{\infty} \tau^i(b) a_i t^{q^i} \right)$$

This is well defined by part (iii) of the functional equation lemma. A quick check by means of (5.14) - (5.16) shows that  $\Delta_B$  is a homomorphism of  $A$ -algebras such that moreover

$$(6.3) \quad \Delta_B \circ \tau = \underline{f}_{\omega} \circ \Delta_B$$

(because  $\sigma^i(\omega) a_{i+1} = a_i$ ), and that  $\Delta_B$  is functorial in the sense that if  $(B', \tau')$  is a second such  $A$ -algebra with endomorphism  $\tau'$  of  $B' \otimes_A K$  and  $\phi : B \rightarrow B'$  is an  $A$  algebra homomorphism such that  $\tau' \phi = \phi \tau$ , then  $C_q(F; \phi) \circ \Delta_B = \Delta_{B'} \circ \phi$ .

6.4. Remark. Using  $(B, \tau)$  instead of  $(A, \sigma)$  we can view  $F(X, Y)$  as a twisted Lubin-Tate formal  $B$ -module over  $B$ , if we are willing to extend the definition a bit, because, of course,  $B$  need not be a discrete valuation ring, nor is  $B \otimes_A K$  necessarily the quotient field of  $B$ . In fact  $B$  need not even be an integral domain. If we view  $F(X, Y)$  in this way then  $\Delta_B : B \rightarrow C_q(F; B)$  is just the  $B$ -algebra structure map of  $C_q(F; B)$ .

6.5. Now let  $B$  be any  $A$ -algebra. Then  $C_q(F; B)$  is an  $A$ -algebra which admits an endomorphism  $\tau$ , viz.  $\tau = \underline{f}_{\omega}$ , which, as  $\tau x \equiv x^q \pmod{\omega}$  by (5.22), satisfies the hypotheses of 6.1 above (because  $\underline{f}_{\omega}$  is  $\sigma$ -semilinear). It is also immediate from (5.10) and (5.4), cf. also (5.14) that  $C_q(F; B)$  is always  $A$ -torsion free. Substituting  $C_q(F; B)$  for  $B$  in 6.1 we therefore find  $A$ -algebra homomorphisms

$$E_B : C_q(F; B) \rightarrow C_q(F; C_q(F; B))$$



which are functorial in  $B$  because  $f_{\omega}$  is functorial, and because of the functoriality property of the  $\Delta_B$  mentioned in 6.1 above. This functorial  $A$ -algebra homomorphism is in fact the ramified Artin-Hasse exponential we are seeking and, as is shown by the next theorem,  $C_q(F;B)$  is the desired ramified Witt vector functor.

**6.6. Theorem.** Let  $A$  be complete with perfect residue field  $k$ . Let  $B$  be the ring of integers of a finite separable extension  $L$  of  $K$ . Let  $\ell$  be the residue field of  $B$ . Consider the composed map

$$\mu_B: B \xrightarrow{\Delta_B} C_q(F;B) \rightarrow C_q(F;\ell)$$

Then  $\mu_B$  is an isomorphism of  $A$ -algebras. Moreover if  $\tau: B \rightarrow B$  is the unique extension of  $\sigma: A \rightarrow A$  such that  $\tau(b) \equiv b^q \pmod{B}$ , then  $f_{\omega}\mu_B = \mu_B\tau$ , i.e.  $\tau$  and  $f_{\omega}$  correspond under  $\mu_B$ .

*Proof.* Let  $b \in B$ . Consider  $\Delta_B(\omega^r b)$ . Then from (6.2) we see that

$$f(\Delta_B(\omega^r b)) \equiv a_{\tau} \tau^r(\omega^r) \tau^r(b) t^{q^r} \pmod{\omega B, \text{degree } q^{r+1}}$$

By part (iv) of the functional equation lemma 2.7 it follows that

$$\Delta_B(\omega^r b) \equiv y_r \tau^r(b) t^{q^r} \pmod{\omega B, \text{degree } q^{r+1}}$$

where  $y_r = a_{\tau} \tau^r(\omega^r)$  is a unit of  $B$ . It follows that  $\mu_B$  maps the filtration subgroups  $\omega^r B$  of  $B$  into the filtration subgroups  $C_q^{(r)}(F;\ell)$  and that the induced maps

$$\ell \xrightarrow{\sim} \omega^r B / \omega^{r+1} B \xrightarrow{\mu_B} C_q^{(r)}(F;\ell) / C_q^{(r+1)}(F;\ell) \xrightarrow{\sim} \ell$$

are given by  $x \mapsto y_r x^{q^r}$ , for  $x \in \ell$ . (Here  $\ell \xrightarrow{\sim} \omega^r B / \omega^{r+1} B$  is induced by  $\omega^r b \mapsto \bar{b}$  with  $\bar{b}$  the image of  $b$  in  $\ell$  under the canonical projection  $B \rightarrow \ell$ , and  $C_q^{(r)}(F;\ell) / C_q^{(r+1)}(F;\ell) \xrightarrow{\sim} \ell$  is induced by  $C_q^{(r)}(F;\ell) \rightarrow \ell$ ,  $\gamma(t) \mapsto (\text{coefficient of } t^{q^r} \text{ in } \gamma(t))$ ). Because  $\ell$  is perfect and  $\bar{y}_r \neq 0$ , it follows that the induced maps  $\bar{\mu}_B$  are all isomorphisms. Hence  $\mu_B$  is an isomorphism because  $B$  and  $C_q(F;\ell)$  are both complete in their filtration topologies.

The map  $\mu_B$  is an  $A$ -algebra homomorphism because  $\Delta_B$  is an  $A$ -algebra homomorphism and  $C_q(F; -)$  is an  $A$ -algebra valued functor. Finally the last statement of theorem 6.6 follows because both  $\tau$  and  $\mu_B^{-1} \underline{f}_\omega \mu_B$  extend  $\sigma$  and  $\tau(b) \equiv b^q \equiv \mu_B^{-1} \underline{f}_\omega \mu_B(b) \pmod{\omega B}$ .

6.7. The maps  $s_{q,n}$  and  $w_{q,n}^F$ . The last thing to do is to reformulate the definitions of  $C_q(F; B)$  and  $E_B$  in such a way that they more closely resemble the corresponding objects in the unramified case, i.e. in the case of the ordinary Witt-vectors. This is easily done, essentially because  $C_q(F; -)$  is representable.

Indeed, let, as a set valued functor,  $W_{q,\infty}^F: \underline{\underline{Alg}}_A \rightarrow \underline{\underline{Set}}$  be defined by

$$(6.8) \quad W_{q,\infty}^F(B) = \{(b_0, b_1, b_2, \dots) \mid b_i \in B\}, \quad W_{q,\infty}^F(\phi)(b_0, b_1, \dots) = \\ = (\phi(b_0), \phi(b_1), \dots)$$

We now identify the set-valued functors  $W_{q,\infty}^F(-)$  and  $C_q(F; -)$  by the functorial isomorphism

$$(6.9) \quad e_B(b_0, b_1, \dots) = \sum_{i=0}^{\infty} F b_i t^{q^i},$$

and define  $W_{q,\infty}^F(-)$  as an  $A$ -algebra valued functor by transporting the  $A$ -algebra structure on  $C_q(F; B)$  via  $e_B$  for all  $B \in \underline{\underline{Alg}}_B$ . We use  $\underline{f}$  and  $\underline{v}$  to denote the endomorphism of  $W_{q,\infty}^F(-)$  obtained by transporting  $\underline{f}_\omega$  and  $\underline{v}_q$  via  $e_B$ . Then one has immediately

$$(6.10) \quad \underline{v}(b_0, b_1, \dots) = (0, b_0, b_1, \dots)$$

and in fact

$$(6.11) \quad \underline{f}(b_0, b_1, \dots) = (\hat{b}_0, \hat{b}_1, \dots) \Rightarrow \hat{b}_i \equiv b_i^q \pmod{\omega B}$$

(We have not proved the analogon of this for  $\underline{f}_\omega$ ; this is not difficult to do by using part (iv) of the functional equation lemma and the additivity of  $\underline{f}_\omega$ ).

Next we discuss the analogue of the Witt-polynomials

$X_0^n + pX_1^{n-1} + \dots + p^n X_n$ . We define for the universal curve

$$\gamma_T(t) \in C_q(F; A[T])$$

$$(6.12) \quad s_{q,n}(\gamma_T(t)) = a_n^{-1}(\text{coefficient of } t^{q^n} \text{ in } f(\gamma_T(t)))$$

and, as usual, this is extended functorially for arbitrary curves  $\gamma(t)$  over arbitrary  $A$ -algebras by

$$(6.13) \quad s_{q,n} \gamma(t) = \phi(s_{q,n}(\gamma_T(t)))$$

where  $\phi: A[T] \rightarrow B$  is the unique  $A$ -algebra homomorphism such that  $\phi_* \gamma_T(t) = \gamma(t)$ . If  $B$  is  $A$ -torsion free one has of course that

$$s_{q,n} \gamma(t) = a_n^{-1}(\text{coeff. of } t^{q^n} \text{ in } f(\gamma(t))).$$

Using this one checks that

$$(6.14) \quad \begin{aligned} s_{q,n}(\gamma(t) +_F \delta(t)) &= s_{q,n}(\gamma(t)) + s_{q,n}(\delta(t)), \quad s_{q,n}(\gamma(t) * \delta(t)) = \\ &= s_{q,n}(\gamma(t)) s_{q,n}(\delta(t)), \quad s_{q,n}(\{a\}_F \gamma(t)) = \sigma^n(a) s_{q,n}(\gamma(t)), \\ s_{q,n}(f_\omega \gamma(t)) &= s_{q,n+1}(\gamma(t)), \quad s_{q,n}(\bigvee_{=q} \gamma(t)) = \\ &= \sigma^{n-1}(\omega) s_{q,n-1}(\gamma(t)) \text{ if } \underline{\geq} 1, \quad s_{q,0}(\bigvee_{=q} \gamma(t)) = 0 \\ s_{q,n}(t) &= 1 \text{ for all } n. \end{aligned}$$

Now suppose that we are in the situation of 6.1 above. Then, by the definition of  $\Delta_B$ , we have

$$(6.15) \quad s_{q,n}(\Delta_B(b)) = \tau^n(b)$$

Now define  $w_{q,n}^F(B): W_{q,\infty}^F(B) \rightarrow B$  by  $w_{q,n}^F = s_{q,n} \circ e_B$ . It is not difficult to calculate  $w_{q,n}^F$ . Indeed

$$f(\gamma_T(t)) = f\left(\sum_{i=0}^{\infty} F_{T,i} t^{q^i}\right) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{j,i} T_i^j t^{q^{i+j}} = \sum_{r=0}^{\infty} \left(\sum_{i=0}^r a_{i,r-i} T_i^{q^i}\right) t^{q^r}$$

and it follows that  $w_{q,n}^F$  is the functorial map  $W_{q,\infty}^F(B) \rightarrow B$  defined by the polynomials

$$\begin{aligned}
w_{q,n}^F(Z_0, \dots, Z_n) &= a_n^{-1} \left( \sum_{i=0}^n a_i Z_i^{q^{n-i}} \right) \\
(6.16) \quad &= Z_0^{q^n} + \sigma^{n-1}(\omega) Z_1^{q^{n-1}} + \sigma^{n-1}(\omega) \sigma^{n-2}(\omega) Z_2^{q^{n-2}} + \dots + \\
&\quad + \sigma^{n-1}(\omega) \dots \sigma(\omega) \omega Z_n
\end{aligned}$$

6.17. Theorem. Let  $(A, \sigma)$  be a pair consisting of a discrete valuation ring  $A$  of residue characteristic  $p > 0$  and a Frobenius-like automorphism  $\sigma : K \rightarrow K$  such that (2.2) holds for some power  $q$  of  $p$ . Let  $\omega$  be any uniformizing element of  $A$ , and let  $w_{q,n}^F(Z)$ ,  $n = 0, 1, \dots$  be the polynomials defined by (6.15). Then there exists a unique  $A$ -algebra valued functor  $W_{q,\infty}^F: \underline{\text{Alg}}_A \rightarrow \underline{\text{Alg}}_A$  such that

- (i) as a set-valued functor  $W_{q,\infty}^F(B) = \{(b_0, b_1, b_2, \dots) \mid b_i \in B\}$  and  $W_{q,\infty}^F(\phi)(b_0, b_1, \dots) = (\phi(b_0), \phi(b_1), \dots)$  for all  $\phi: B \rightarrow B'$  in  $\underline{\text{Alg}}_A$
- (ii) the polynomials  $w_{q,n}^F(Z)$  induce functorial  $\sigma^n$ -semilinear  $A$ -algebra homomorphisms  $w_{q,\infty}^F: W_{q,\infty}^F(B) \rightarrow B$ ,  $(b_0, b_1, \dots) \mapsto w_{q,\infty}^F(b_0, \dots, b_n)$ .

Moreover, the functor  $W_{q,\infty}^F(-)$  has  $\sigma^{-1}$ -semilinear  $A$ -module functor endomorphism  $\underline{V}$  and a functorial  $\sigma$ -semilinear  $A$ -algebra endomorphism  $\underline{f}$  which satisfy and are characterized by

- (iii)  $w_{q,n}^F \circ \underline{V} = \sigma^{n-1}(\omega) w_{q,n-1}^F$  if  $n = 1, 2, \dots$ ;  $w_{q,0}^F \circ \underline{V} = 0$
- (iv)  $w_{q,n}^F \circ \underline{f} = w_{q,n+1}^F$

These endomorphisms  $\underline{f}$  and  $\underline{V}$  have (among others) the properties

- (v)  $\underline{f}\underline{V} = \omega$
- (vi)  $\underline{f}b \equiv b^q \pmod{\omega W_{q,\infty}^F(B)}$  for all  $b \in W_{q,\infty}^F(B)$ ,  $B \in \underline{\text{Alg}}_A$
- (vii)  $\underline{V}(b(\underline{f}c)) = (\underline{V}b)\underline{c}$  for all  $b, c \in W_{q,\infty}^F(B)$ ,  $B \in \underline{\text{Alg}}_A$

Further there exists a unique functorial  $A$ -algebra homomorphism

$$E: W_{q,\infty}^F(-) \rightarrow W_{q,\infty}^F(W_{q,\infty}^F(-))$$

which satisfies and is characterized by

- (viii)  $w_{q,n}^F \circ E = \underline{f}^n$  for all  $n = 0, 1, 2, \dots$

(Here  $w_{q,n}^F: W_{q,\infty}^F(W_{q,\infty}^F(B)) \rightarrow W_{q,\infty}^F(B)$  is short for  $w_{q,n,w_{q,\infty}^F(B)}^F$ , i.e. it is the map which assigns to a sequence  $(\underline{b}_0, \underline{b}_1, \dots)$  of elements of  $W_{q,\infty}^F(B)$  the element  $w_{q,n}^F(\underline{b}_0, \underline{b}_1, \dots) \in W_{q,\infty}^F(B)$ ). The functor homomorphism  $E$  further satisfies

$$(ix) \quad w_{q,\infty}^F(w_{q,n}^F) \circ E = \underline{f}^n,$$

where  $w_{q,\infty}^F(w_{q,n}^F): W_{q,\infty}^F(W_{q,\infty}^F(B)) \rightarrow W_{q,\infty}^F(B)$  assigns to a sequence

$$(\underline{b}_0, \underline{b}_1, \dots) \text{ of elements of } W_{q,\infty}^F(B) \text{ the sequence } (w_{q,n}^F(\underline{b}_0), w_{q,n}^F(\underline{b}_1), \dots) \\ \in W_{q,\infty}^F(B)$$

Finally if  $A$  is complete with perfect residue field  $k$  and  $\ell/k$  is a finite separable extension, then  $W_{q,\infty}^F(\ell)$  is the ring of integers  $B$  of the unique unramified extension  $L/K$  covering the residue field extension  $\ell/k$  and under this  $A$ -algebra isomorphism  $\underline{f}$  corresponds to the unique extension of  $\sigma$  to  $\tau: B \rightarrow B$  which satisfies  $\tau(b) \equiv b^q \pmod{\omega B}$ . In particular  $W_{q,\infty}^F(k) \simeq A$  with  $\underline{f}$  corresponding to  $\sigma$ .

Proof. Existence of  $W_{q,\infty}^F(-)$ ,  $\underline{V}$ ,  $\underline{f}$ ,  $E$  such that (i), (ii), (iii), (iv)

(viii) hold follows from the constructions above. Uniqueness follows because (i), (ii), (iii), (iv), (viii) determine the  $A$ -algebra structure on  $B^{\text{NU}\{0\}}$ ,  $\underline{V}$ ,  $\underline{f}$ ,  $E$  uniquely for  $A$ -torsion free  $A$ -algebras  $B$ , and then these structure elements are uniquely determined by (i) - (iv), (viii) for all  $A$ -algebras, by the functoriality requirement (because for every  $A$ -algebra  $B$  there exists an  $A$ -torsion free  $A$ -algebra  $B'$  together with a surjective  $A$ -algebra homomorphism  $B' \rightarrow B$ . Of the remaining identities some have already been proved in the  $C_q(F; -)$ -setting ((v) and (vi)). They can all be proved by checking that they give the right answers whenever composed with the  $w_{q,n}^F$ . This proves that they hold over  $A$ -torsion free algebras  $B$ , and then they hold in general by functoriality. So to prove (vii) we calculate

$$w_{q,0}^F(V(\underline{b}(\underline{f}\underline{c}))) = 0$$

$$\begin{aligned} w_{q,n}^F(V(\underline{b}(\underline{f}\underline{c}))) &= \sigma^{n-1}(\omega) w_{q,n-1}^F(\underline{b}(\underline{f}\underline{c})) = \sigma^{n-1}(\omega) w_{q,n-1}^F(\underline{b}) w_{q,n-1}^F(\underline{f}\underline{c}) \\ &= \sigma^{n-1}(\omega) w_{q,n-1}^F(\underline{b}) w_{q,n}^F(\underline{c}) \end{aligned}$$

and, on the other hand

$$w_{q,0}^F((\underline{V}\underline{b})\underline{c}) = w_{q,0}^F(\underline{V}\underline{b}) w_{q,0}^F(\underline{c}) = 0 \cdot w_{q,0}^F(\underline{c}) = 0$$

$$w_{q,n}^F((\underline{V}\underline{b})\underline{c}) = w_{q,n}^F(\underline{V}\underline{b}) w_{q,n}^F(\underline{c}) = \sigma^{n-1}(\omega) w_{q,n-1}^F(\underline{b}) w_{q,n}^F(\underline{c})$$

This proves (vii). To prove (ix) we proceed similarly

$$\begin{aligned} w_{q,m}^F \circ w_{q,\infty}^F(w_{q,n}^F) \circ E &= w_{q,n}^F \circ w_{q,m}^F \circ E = w_{q,n}^F \circ \underline{f}^m \\ &= w_{q,n+m}^F = w_{q,m}^F \circ \underline{f}^n \end{aligned}$$

(Here the first equality follows from the functoriality of the morphisms  $w_{q,m}^F$  which says that for all  $\phi: B' \rightarrow B \in \underline{\text{Alg}}_A$  we have  $w_{q,m}^F \circ w_{q,\infty}^F(\phi) = \phi \circ w_{q,m}^F$ ; now substitute  $w_{q,n}^F$  for  $\phi$ ).

6.18. Remark.  $\underline{V}\underline{f} = \underline{f}\underline{V}$  does of course not hold in general (also not in the case of the usual Witt vectors). It is however, true in  $w_{q,\infty}^F(B)$  if  $\omega B = 0$ , as easily follows from (6.11), which implies that  $f(b_0, b_1, \dots) = (b_0^q, b_1^q, \dots)$  if  $\omega B = 0$ .

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