TRIANGULAR - SQUARE - PENTAGONAL NUMBERS

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ABSTRACT. In this note it is shown that no integer \( m > 1 \) is simultaneously triangular, square and pentagonal.

1. Polygonal numbers may be defined as follows: let \( n \) and \( k \) be positive integers. The \( (n+2) \)-gonal number of rank \( k \), denoted by \( p(n;k) \), is the sum of the first \( k \) natural numbers of the form \( 1 \pmod{n} \). Thus

\[
p(n;k) = \sum_{j=0}^{k-1} (nj+1) = \frac{1}{2}k(kn-n+2).
\]

The name "polygonal number" is adequately explained by the following picture:

![Polygonal numbers diagram](image)

1 3 6 10  
triangular numbers  
\((n=1)\)

1 4 9 16  
square numbers  
\((n=2)\)

1 5 12 22  
pentagonal numbers  
\((n=3)\)

Polygonal numbers have many interesting properties, most of which are at present looked upon as merely recreational.
A nice property, discovered by Fermat and proved by Cauchy, is the following one: every positive integer may be written as the sum of at most \( n \) \( n \)-gonal numbers.

For more information on polygonal numbers, the reader is referred to [2], pp. 185 - 199.

Our aim is to investigate whether polygonal numbers of a different kind (i.e. \( n \)-gonal numbers and \( m \)-gonal numbers with \( n \neq m \)) may be equal. As a matter of fact, it is easy to show that each of the following equations has infinitely many solutions \( (k,t) \):

\[
\begin{align*}
1) & \quad p(1;k) = p(2;t) \\
2) & \quad p(1;k) = p(3;t) \\
3) & \quad p(2;k) = p(3;t)
\end{align*}
\]

Thus there are infinitely many triangular-square numbers and the same is true for triangular-pentagonal numbers and square-pentagonal numbers. Indeed, solving (1) boils down to solving the Pellian equations

\[ x^2 - 2y^2 = \pm 1 \]

in integers \( x \) and \( y \). Similarly, the solutions of (2) may be derived from the solutions \( (x,y) \) of the Diophantine equation

\[ 3x^2 - y^2 = 2 \]

and finally, (3) may be solved by determining all solutions \( (x,y) \) of

\[ 3x^2 - 2y^2 = 1. \]
In [1], p. 740, A.J. Phelps poses the following problem: find all triangular-square-pentagonal numbers. The object of this note is to prove that 1 is the only such number.

2. This section is devoted to the proof of

**Theorem.** \( p(1;k) = p(2;\ell) = p(3;m) \) if and only if \( k = \ell = m = 1 \).

To this end we shall need two lemmas. Firstly,

**Lemma 1.** If \( p(1;k) = p(2;\ell) = p(3;m) \) then \( m = u^2v^2 \), where \( u \) and \( v \) are odd integers such that

\[
(4) 
\quad u^4 + 2u^2v^2 - 2v^4 = 1.
\]

The proof of the theorem is then completed by

**Lemma 2.** The only solution of equation (4) in odd positive integers \( u \) and \( v \) is \((u,v) = (1,1)\).

Indeed, if \( p(1;k) = p(2;\ell) = p(3;m) \) then \( m = u^2v^2 \) with \( u = v = 1 \). Hence \( p(3;m) = 1 \). The converse of the assertion is trivial.

**Proof of Lemma 1.**

From \( p(1;k) = p(2;\ell) = p(3;m) \) it follows that

\[
k(k+1) = 2\ell^2 = m(3m-1)
\]

and hence

\[
k = 2^r a^2, \quad k + 1 = 2^{1-r}b^2 \quad \text{with} \quad a, b \in \mathbb{N}, \quad (a, b) = 1 \quad \text{and} \quad r \in \{0, 1\},
\]
\[
m = 2^s c^2, \quad 3m - 1 = 2^{1-s}d^2 \quad \text{with} \quad c, d \in \mathbb{N}, \quad (c, d) = 1 \quad \text{and} \quad s \in \{0, 1\},
\]
\[
\ell = ab = cd.
\]

From \( 3m - 1 = 2^{1-s}d^2 \) we deduce that \( 2^{1-s}d^2 \equiv -1 \pmod{3} \) and consequently \( s = 0 \). Then \( m \) is odd and both \( c \) and \( d \) are odd.
This shows that \( l \) is odd and hence both \( a \) and \( b \) are odd. Now \( 2^r + 1 \equiv 2^{1-r} \) (mod 8) yields \( r = 0 \). So we have

\[
\begin{align*}
2b^2 - a^2 &= 1 \\
3c^2 - 2d^2 &= 1 \\
ab &= cd
\end{align*}
\]

(5)

We define \( v := (a,c) \), \( u := (b,c) \), \( v' := \frac{a}{v} \) and \( u' := \frac{b}{u} \). Then \( (u,v) = (u',v) = (u',v') = 1 \), since \( (a,b) = 1 \).

Clearly, \( c = uv \) and \( d = u'v' \). Indeed, \( cd = ab = uu'vv' \) and both \( c/uv \) and \( d/u'v' \) are positive integers.

Expressing (5) in terms of \( u, u', v \) and \( v' \) gives in particular

\[
2u^2v^2 = (u^2 + v'^2)(2u'^2 - v'^2).
\]

(6)

Now \( (u^2, u^2 + v'^2) = (2v^2, 2u'^2 - v'^2) = (u^2 + v'^2, 2u'^2 - v'^2) = 1 \). This follows easily from \( (u,v) = (u',v) = (u,v') = 1 \) and the fact that \( u, u', v \) and \( v' \) are odd.

From (6) we deduce that

\[
2v^2 = u^2 + v'^2 \quad \text{and} \quad u^2 = 2u'^2 - v'^2.
\]

Hence \( a^2 = 2v^4 - u^2v^2 \) and \( 2b^2 = u^4 + u^2v^2 \). Since \( 2b^2 - a^2 = 1 \), according to (5), this leads immediately to the quartic equation (4). Also note that \( m = c^2 = u^2v^2 \).

Proof of Lemma 2.

Let \( u \) and \( v \) be odd positive integers, satisfying (4).

Further, consider the irreducible polynomial
\[ f(x) := x^4 + 2x^2 - 2 \in \mathbb{Q}[x]. \]

The discriminant of \( f \) equals \(-2^93^3\). Hence, the equation \( f(x) = 0 \) has two real roots and one pair of complex conjugate ones. Let \( \theta \) be one of the real roots of \( f(x) = 0 \).

The quartic extension \( K := \mathbb{Q}(\theta) \) of \( \mathbb{Q} \) has the sub-field \( \mathbb{Q}(\sqrt{3}) \). It is not difficult to show that \( \{1, \theta, \theta^2, \theta^3\} \) is an integer basis of \( K \) and that \( \{1+\theta, 1-\theta\} \) is a fundamental set of units (cf. [3]).

Now (4) may be written as

\[ \text{Norm}_{K/\mathbb{Q}}(u-v\theta) = 1, \]

and hence

\[ (7) \quad u - v\theta = \pm(1+\theta)^p(1-\theta)^q \]

with \( p, q \in \mathbb{Z} \) (the only cyclotomic units of \( K \) are \( \pm 1 \)). If we disregard the sign of \( u \) and that of \( v \), we may neglect the \( \pm \) sign in (7). Moreover, there is no loss of generality in assuming that \( p > q \). Then

\[ (8) \quad u - v\theta = (1+\theta)^p-q(1-\theta^2)^q. \]

Now \( p - q \) is odd, for otherwise \( v \) could not be odd, as can be seen from

\[ u^2 - v^2\theta^2 = (1-\theta^2)^p+q = \sum_{j=0}^{\infty} (p+q)_j (-\theta^2)^j = 1 - (p+q)\theta^2 + 2(...). \]

Set \( 2n + 1 := p - q \). Then \( n \in \mathbb{Z}, n \geq 0 \). We intend to show that \( n = 0 \). To this end we define rational integers \( a_i, b_i, c_i \) and \( d_i \) for each \( i = 0, 1, 2, \ldots \) by

\[ (1+\theta)^{2i+1} = a_i + b_i\theta + c_i\theta^2 + d_i\theta^3. \]
This gives
\[
\begin{align*}
  a_i + c_i \theta^2 &= \sum_{j=0}^{i} (2i+1) \theta^{2j}, \\
  b_i + d_i \theta^2 &= \sum_{j=0}^{i} (2i+1) \theta^{2j}.
\end{align*}
\]

From (9) we obtain the relations
\[
\begin{align*}
  a_n &= 2 \sum_{j=0}^{n} (2n+1) \beta_{j-1}, \\
  b_n &= 2 \sum_{j=0}^{n} (2n+1) \beta_{j-1}, \\
  c_n &= \sum_{j=0}^{n} (2n+1) \beta_j \\
  d_n &= \sum_{j=0}^{n} (2n+1) \beta_j.
\end{align*}
\]

Inserting these values for \(a_n, b_n, c_n\) and \(d_n\) into (11) yields
\[
0 = \sum_{i,j=0}^{n} \left\{ (2n+1)(2n+1) - (2n+1)(2n+1) \right\} \beta_{i-1} \beta_j =
\]
\[ \begin{align*}
\sum_{i,j=0}^{n} r_{ij}(n) \binom{2n}{2i} \binom{2n}{2j} \beta_{i-1} \beta_{j},
\end{align*} \]

with
\[ r_{ij}(n) := \frac{j-i}{(2i+1)(2j+1)(2n-2i+1)(2n-2j+1)}, \quad i, j = 0, 1, 2, \ldots, n \]

Dividing through by \(2(n+1)(2n+1)^2\) and making use of (12), we deduce that (the sums considered are empty in case \(n = 0\) or \(1\))
\[ (13) \quad 0 = \sum_{j=1}^{n} r_{0j}(n) \binom{2n}{2j} \beta_j + 2 \sum_{i=2}^{n} \sum_{j=1}^{n} r_{ij}(n) \binom{2n}{2i} \binom{2n}{2j} \beta_{i-1} \beta_j. \]

Let \(v_2\) be the 2-adic valuation, defined on \(\mathbb{Q}\), written additively. We have for \(i, j = 1, 2, \ldots, n\)
\[ v_2(r_{0j}(n)) = v_2(j), \]
\[ v_2(r_{ij}(n)) = v_2(j-i) \geq 0, \]
\[ v_2(\frac{2n}{2j}) \geq v_2(n) - v_2(j) \quad \text{and} \]
\[ v_2(\beta_j) \geq [\frac{1}{j}]. \]

Clearly, (14)\(_1\) and (14)\(_2\) follow immediately from the definition of \(r_{ij}(n)\); (14)\(_3\) is a consequence of the fact that for \(j = 1, \ldots, n\)
\[ \binom{2n}{2j} = \frac{n(n-1)}{j(2j-1)} \quad \text{and} \quad (14)\(_4\) \text{ may be derived from (12) by means of induction. Consequently,} \]
\[ v_2(r_{0j}(n) \binom{2n}{2j} \beta_j) \begin{cases} = v_2(n) & \text{if } j = 1 \\ \geq v_2(n) + [\frac{1}{j}] \geq v_2(n) + 1 & \text{if } j \geq 2, \end{cases} \]

and
\[ v_2(2r_{ij}(n) \binom{2n}{2i} \binom{2n}{2j} \beta_{i-1} \beta_j) \geq 1 + v_2(\binom{2n}{2j} \beta_j) \geq 1 + v_2(n) - v_2(j) + [\frac{1}{j}] \geq 1 + v_2(n) \quad \text{for } i \geq 2, \ j \geq 1. \]
Applying this to (13), we see that $v_2(n) = -\infty$ and hence $n = 0$. Having established this, it follows from (8) that

$$ (15) \quad u - v^2 = (1+\theta)(1-\theta^2)q. $$

Since $(1-\theta^2)q$ is a unit of the form $A + B\theta^2$ with $A, B \in \mathbb{Z}$, (15) can only be true when $(1-\theta^2)q = 1$, i.e. $q = 0$. This proves the lemma.

For the $p$-adic method used in the proof of LEMMA 2, we refer to [6] and [5], chapter 23.

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