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A characterization of multidimensional extreme-value distributions

Laurens de Haan

1. Introduction. One can ask to what extent there is an analogue in extreme-value theory of the preponderant role which the normal distribution plays in the limit theory for partial sums of i.i.d. summands. In this note we prove a characterization of multidimensional extreme-value distributions similar to a well known characterization of the multivariate normal law. The characterization of the normal law using the independence of certain linear combinations of independent random variables fails to have an analogue for sample maxima.

2. The result. It is well known (Lukacs and Laha [8] th. 2.1.4) that every linear combination of the components of a p-dimensional random vector has a univariate normal distribution if and only if the random vector has a p-variate normal df. We prove the following analogue for extreme-value distributions (for information about multidimensional extreme-value distributions we refer to [3] and [5]).

Theorem. Let $X = (X_1, X_2, \ldots, X_p)$ be a random vector with non-negative components and $\alpha > 0$. X has a p-variate extreme value distribution with for some $b_1, \ldots, b_p > 0$, $P\{X_k \le x\} = \exp -(b_k x)^{-\alpha} = \Phi_\alpha(b_k x)$ for $k = 1, 2, \ldots, p$ if and only if every random variable Y of the form

$$Y = \max (a_1X_1, a_2X_2, \ldots, a_pX_p)$$

with $a_k > 0$ $(k = 1, 2, \ldots, p)$ is a multiple of a random variable with df Φ_α.

Proof. By considering $(X_1^{1/\alpha}, X_2^{1/\alpha}, \ldots, X_p^{1/\alpha})$ it is clear that it is sufficient to prove the case $\alpha = 1$.

First suppose that X has a p-variate extreme value distribution with the mentioned marginals. Denoting the df of X by $G(x_1, x_2, \ldots, x_p)$ one then has (Geffroy [3] Ch. VI, p. 167 sqq)

(1) $G(ax_1, ax_2, \ldots, ax_p) = G^a(x_1, x_2, \ldots, x_p)$

for all

$a, x_1, x_2, \ldots, x_p > 0$. Now $P\{Y \leq x\} = P\{a_1X_1 \leq x, \ldots, a_pX_p \leq x\} = G(a_1^{-1}x, \ldots, a_p^{-1}x)$. By (1) the righthand side equals

$$G^{1/x}(a_1^{-1}, a_2^{-1}, \ldots, a_p^{-1}) = \exp - \frac{-\log G(a_1^{-1}, a_2^{-1}, \ldots, a_p^{-1})}{x}.$$

As $0 < G(a_1^{-1}, \ldots, a_p^{-1}) < 1$ for all $a_1 > 0, \ldots, a_p > 0$ (Balkema and Resnick [1] th. 3 and its cor. 2), the distribution of Y is of type Φ_1.

Conversely suppose Y satisfies the conditions of the theorem for all positive $a_1, a_2, \ldots, a_p$. Then G satisfies

$$G(a_1^{-1}x, \ldots, a_p^{-1}x) = \exp - \frac{c(a_1, \ldots, a_p)}{x}$$

for $x > 0$ where $c(a_1, \ldots, a_p)$ is positive. It follows that

$$G(a_1^{-1}x, a_2^{-1}x, \ldots, a_p^{-1}x) = G^{1/x}(a_1^{-1}, a_2^{-1}, \ldots, a_p^{-1})$$

which (Geffroy, loc. cit.) implies that G is a p-variate extreme value distribution with all marginals of type Φ_1. □

Remark. By applying the transformations

$$(T_1, T_2, \ldots, T_p) = (\alpha^{-1} \log X_1, \alpha^{-1} \log X_2, \ldots, \alpha^{-1} \log X_p)$$

and

$$(S_1, S_2, \ldots, S_p) = (-X_1^{-1}, -X_2^{-1}, \ldots, -X_p^{-1})$$

one gets from the theorem characterizations of the extreme value distributions with marginals $\exp(-e^{-x})$ and $\exp -(-x)^{\alpha}$ respectively, which are the two other classes of one-dimensional extreme value distributions given by Gnedenko [4].

Example. Let E_n $(n = 0, \pm 1, \pm 2, \ldots)$ be independent random variables with df Φ_1. Suppose that the sequence $\{X_n\}_{n=-\infty}^{+\infty}$ of random variables satisfies

$$X_{n+1} = \max(\rho X_n, E_n)$$

for some ρ $(0 < \rho < 1)$ and $n = 0, \pm 1, \pm 2, \ldots$ (cf. Helland and Nilsen [6]). It is easily checked that then $X_n = \max_{j=0,1,2,\ldots} \rho^j E_{n-j}$ and hence $Y = \max(a_1 X_{n+1}, a_2 X_{n+2}, \ldots, a_p X_{n+p})$ satisfies the conditions of the theorem for any n and positive p. One could thus call the distribution of $\{X_n\}_{n=-\infty}^{\infty}$ an infinite-dimensional extreme value distribution.

The statement of the theorem can be extended to the domain of attraction of the extreme-value distributions: the random vector $X = (X_1, X_2, \ldots, X_p)$ is in the domain of symmetric attraction of a simple extreme value distribution (for explanation of the terms "symmetric" and "simple" see de Haan and Resnick [5] section 3) if and only if every Y of the form given in the theorem is in the domain of attraction of Φ_1.+)

3. Other characterizations. We wish to add some remarks about possible other characterizations. There are two famous characterizations of the normal distribution by means of properties of linear combinations of independent random variables: one involves the existence of two stochastically independent linear combinations and the other the existence of two different linear combinations with the same probability distribution (Chapters 5 and

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+) These properties can be formulated as properties of multidimensional regularly varying functions i.e. measurable positive functions $h(x_1, \ldots, x_p)$ such that $\{h(t, t, \ldots, t)\}^{-1} h(tx_1, tx_2, \ldots, tx_p)$ converges to a positive limit as $t \to \infty$ for any positive $x_1, x_2, \ldots, x_p$. The limit function $g(x_1, x_2, \ldots, x_p)$ then satisfies for $a, x_1, x_2, \ldots, x_p > 0$ $g(ax_1, ax_2, \ldots, ax_p) = a^\rho g(x_1, x_2, \ldots, x_p)$ where ρ is a real constant. It follows from the above that h is regularly varying as a multivariate function if and only if for any positive $a_1, a_2, \ldots, a_p$ the function $h(ta_1, ta_2, \ldots, ta_p)$ is regularly varying as a univariate function of t.

8 of Lukacs and Laha [8] respectively). The second characterization (due mainly to Marcinkiewicz and Linnik) has an analogue in extreme value theory due also to Linnik (see [7] section 2.6): the property characterizes a much wider class than the extreme value distributions.

We shall now consider a possible analogue of the first characterization of the normal distribution. A simple argument shows that a characterization of this sort is impossible in extreme value theory:
Let $X_1, X_2, \ldots, X_p$ be independent positive random variables and suppose that for non-negative numbers $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ the random variables

$$Y_1 = \max (a_1X_1, a_2X_2, \ldots, a_pX_p)$$

and

$$Y_2 = \max (b_1X_1, b_2X_2, \ldots, b_pX_p)$$

are independent. Let G_i be the distribution function of X_i for $i = 1, 2, \ldots, p$. Then for all $x,y > 0$

$$H(x,y) = P\{Y_1 \leq x, Y_2 \leq y\} = G_1(\min (\frac{x}{a_1}, \frac{y}{b_1})) \ldots G_p(\min (\frac{x}{a_p}, \frac{y}{b_p}))$$

so that for $y^{-1}x < \min (b_1^{-1}a_1, b_2^{-1}a_2, \ldots, b_p^{-1}a_p)$ the df $H(x,y)$ does not depend on y. So independence of Y_1 and Y_2 is possible only if the latter minimum is zero i.e. only in the trivial case that Y_1 and Y_2 are based on disjoint subsets of the X's.

We finish with the following remark. Let again $X_1, X_2, \ldots, X_p$ be independent, but let now $P\{X_i \leq x\} = \exp(-1/x)$ for $x > 0$. Let $A = (a_{ij})_{i,j=1}^{p}$ be a matrix with non-negative elements and define $Y_i = \max (a_{i1} X_1, a_{i2} X_2, \ldots, a_{ip} X_p)$ for $i = 1, 2, \ldots, p$. The joint distribution of $(Y_1, Y_2, \ldots, Y_p)$ then is a p-dimensional extreme-value distribution but one of special type, namely the measure occurring in the standard representation for such functions (see [5] section 2) is concentrated in p points. These points are taken as the column vectors of a conjugate matrix A* of A in an interesting paper on the kind of matrix algebra that is relevant here by Cuninghame-Green [2].

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