Stat.

ON FUNCTIONS DERIVED FROM REGULARLY VARYING FUNCTIONS

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Summary

Some theorems on integrals and integral transforms of regularly varying functions are proved.

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Introduction and notation

A generalization of Karamata's theorem on integrals of regularly varying functions is proved. Using Laplace-Stieltjes transforms it is shown that any regularly varying function with exponent α $(\alpha + 1 \notin \mathbb{N})$ is asymptotic to another regularly varying function all of whose derivations are regularly varying.

Suppose U is a positive function on $\mathbb{R}^+$. U is <u>regularly varying</u> at ∞ (or 0+) with exponent α, in short α-varying, notation $U \in R\,V_\alpha^{(\infty)}$ (or $R\,V_\alpha^{(0)}$ respectively), if for all $x > 0$

$$\frac{U(tx)}{U(t)} \to x^\alpha$$

as $t \to \infty$ (or $t \downarrow 0$ respectively); c.f. Karamata [5] and [6], Feller [2] VIII, 9 and XIII, 5.

If U is non-decreasing and if for suitable functions $a(t) > 0$ and $b(t)$ and all $x > 0$

$$\frac{U(tx) - b(t)}{a(t)} \to \log x$$

as $t \to \infty$ we say $U \in \Pi^{(\infty)}$. If U is non-increasing and if for suitable functions $a(t) > \infty$ and $b(t)$ and all $x > 0$

$$\frac{U(tx) - b(t)}{a(t)} \to \log x$$

as $t \to \infty$ we say $U \in \Pi^{(0)}$; c.f. de Haan [3], section I, 4. By abuse of language we shall also write $U(x) \in R\,V_\alpha^{(\infty)}$.

1. Integrals of regularly varying functions

We start from a well known result. Suppose F is a probability distribution function and $F(0+) = 0$. Then (Feller [2], VIII, 9 th. 2. cf. Pitman [7], lemma 3) for $\alpha > 0$, $\beta < 0$, $\alpha + \beta > 0$

$$\text{(P1)}\qquad \int_0^x t^{\alpha}dF(t) \in RV^{(\infty)}_{\alpha+\beta} \iff \int_0^x t^{\alpha-1}(1-F(t))dt \in RV^{(\infty)}_{\alpha+\beta}$$

$$\iff 1-F(x) \in RV^{(\infty)}_{\beta} \iff \lim_{x\to\infty} \frac{\int_0^x t^{\alpha-1}(1-F(t)dt}{x^{\alpha}(1-F(x))} = \frac{1}{\alpha+\beta}$$

$$\iff \lim_{x\to\infty} \frac{\int_0^x t^{\alpha}dF(t)}{x^{\alpha}(1-F(x))} = \frac{-\beta}{\alpha+\beta}$$

A variant is the following. Suppose U is non-decreasing, $U(0+) = 0$, then for $\alpha > 0$, $\beta > 0$

$$\text{(P2)}\qquad \int_0^x t^{\alpha}dU(t) \in RV^{(\infty)}_{\alpha+\beta} \iff \int_0^x t^{\alpha-1}U(t)dt \in RV^{(\infty)}_{\alpha+\beta}$$

$$\iff U(x) \in RV^{(\infty)}_{\beta} \iff \lim_{x\to\infty} \frac{\int_0^x t^{\alpha-1}U(t)dt}{x^{\alpha}\,U(x)} = \frac{1}{\alpha+\beta}$$

$$\iff \lim_{x\to\infty} \frac{\int_0^x t^{\alpha}dU(t)}{x^{\alpha}\,U(x)} = \frac{\beta}{\alpha+\beta}\,.$$

We want to present some analogous statements. Proofs are given in the next section. Firstly for probability distributions F

$$\text{(P3)}\qquad \int_0^x tdF(t) \in RV^{(\infty)}_{0} \iff \int_0^x (1-F(t))dt \in RV^{(\infty)}_{0}$$

$$\iff \lim_{x\to\infty} \frac{\int_0^x (1-F(t))dt}{x(1-F(x))} = \infty \iff \lim_{x\to\infty} \frac{\int_0^x tdF(t)}{x(1-F(x))} = \infty.$$

A sufficient (but not necessary) condition is $1 - F(x) \in RV^{(\infty)}_{-1}$.

Next suppose U is as above. For $\alpha > 0$

$$\text{(P4)}\qquad U \in \Pi^{(\infty)} \iff \int_0^x t^{\alpha}dU(t) \in RV^{(\infty)}_{\alpha} \iff \int_x^{\infty} t^{-\alpha}dU(t) \in RV^{(\infty)}_{-\alpha}$$

Finally suppose U_1 is non-decreasing, U_2 is continuous and strictly increasing, $U_2(0+) = 0$. Suppose $\alpha > 0$, $\beta > 0$.

(P5) Any two of the following statements imply the others.

a. $U_1 \in RV_\alpha^{(\infty)}$

b. $U_2 \in RV_\beta^{(\infty)}$

c. $\int_0^x U_1(t)dU_2(t) \in RV_{\alpha+\beta}^{(\infty)}$

d. $$\lim_{x\to\infty} \frac{\int_0^x U_1(t)dU_2(t)}{U_1(x)\ U_2(x)} = \frac{\beta}{\alpha+\beta}$$

This generalizes (P2). Similarly for functions in $\Pi^{(\infty)}$ we have the following.

(P6) Suppose $U_1 \in RV_\alpha^{(\infty)}$, U_1 is continuous and strictly increasing, $U_1(0+) = 0$.

$$U_2(x) \in \Pi^{(\infty)} \iff \int_0^x U_1(t)dU_2(t) \in RV_\alpha^{(\infty)}$$

$$\iff \int_x^\infty \frac{dU_2(t)}{U_1(t)} \in RV_{-\alpha}^{(\infty)}.$$

This generalizes (P4).

2. <u>Proofs and remarks</u>

<u>Proof of (P3)</u>:

$\int_0^x tdf(t) \in RV_0^{(\infty)}$ if and only if $\lim_{x\to\infty} \int_0^x tdF(t)/\{x(1-F(x))\} = \infty$ by Feller [1], VIII, 9 th. 2. Now $\lim_{x\to\infty} \int_0^x tdF(t)/\{x(1-F(x)\} = \infty \iff \lim_{x\to\infty} \int_0^x(1-F(t)dt/\{x(1-F(x))\} = \infty$ is a matter of partial integration. If $\alpha(x) = \int_0^x(1-F(t))dt/\{x(1-F(x))\} \to \infty\ (x \to \infty)$ then $\int_0^x(1-F(t))dt = \{\int_0^1(1-F(t))dt\}$ exp $\int_1^x\{t\ \alpha(t)\}^{-1}dt$ and the latter is in $RV_0^{(\infty)}$ by the representation theorem for regularly varying functions. If $\int_0^x(1-F(t))dt \in RV_0^{(\infty)}$ then by property 8, p. 22 of de Haan [2] $\lim_{x\to\infty} \int_0^x(1-F(t))dt/\{x(1-F(x))\} = \infty$.

<u>Remark</u>: This is related to the weak law of large numbers (Feller [2], VII, 7, th. 2).

Remark: The statements of (P3) are implied by the set of equivalent statements ($\alpha > 0$)

$$1 - F(x) \in R\,V_{-1}^{(\infty)} \iff \int_0^x (1-F(t))\,dt \in \Pi^{(\infty)}$$

$$\iff \int_0^x t\,dF(t) \in \Pi^{(\infty)} \iff \int_0^x t^{\alpha}(1-F(t))\,dt \in R\,V_{\alpha}^{(\infty)}$$

(the equivalence of these statements follows from (P4)).

Proof of (P5):

U_2 has a proper inverse U_2^{-1}. So

$$\int_0^x U_1(t)\,dU_2(t) = \int_0^{U_2(x)} U_1(U_2^{-1}(s))\,ds.$$

We shall write $U_1 o U_2^{-1}(s)$ for the compound function $U_1(U_2^{-1}(s))$.

Assume a) and b). Then $U_1 o U_2^{-1} \in R\,V_{\alpha/\beta}^{(\infty)}$, and hence

$$\lim_{x\to\infty} \frac{\int_0^x U_1(t)\,dU_2(t)}{U_1(x)\;U_2(x)} = \lim_{x\to\infty} \frac{\int_0^{U_2^{-1}(x)} U_1(t)\,dU_2(t)}{x\;U_1 o U_2^{-1}(x)}$$

$$= \lim_{x\to\infty} \frac{\int_0^x U_1 o U_2^{-1}(s)\,ds}{x\;U_1 o U_2^{-1}(x)} = \frac{\beta}{\alpha+\beta}.$$

Assume b) and c). The compound function

$$\int_0^{U_2^{-1}(x)} U_1(t)\,dU_2(t) = \int_0^x U_1 o U_2^{-1}(s)\,ds$$

then belongs to $R\,V_{\beta^{-1}(\alpha+\beta)}^{(\infty)}$. Since $U_1 o U_2^{-1}$ is monotone, it follows

$U_1 o U_2^{-1} \in R\,V_{\beta^{-1}\alpha}^{(\infty)}$. Hence

$$U_1 = U_1 o U_2^{-1} o U_2 \in R\,V_{\alpha}^{(\infty)}.$$

Assume a) and c). It is well known that there is a continuous and strictly increasing function U_3 such that $U_3(0+) = 0$ and $U_1(x) \sim U_3(x)$ as $x \to \infty$. Then $\int_0^x U_3(t)dU_2(t) \sim \int_0^x U_1(t)dU_2(t)$ as $x \to \infty$. The compound function

$$\int_0^{U_3^{-1}(x)} U_3(t)dU_2(t) = \int_0^x s\,dU_2 o U_3^{-1}(s)$$

is in $R\,V^{(\infty)}_{\alpha^{-1}(\alpha+\beta)}$. By (P2) then $U_2 o U_3^{-1} \in R\,V^{(\infty)}_{\alpha^{-1}\beta}$. The rest is as before.

Assume d) then

$$\lim_{x\to\infty} \frac{\int_0^x U_1 o U_2^{-1}(s)ds}{x\, U_1 o U_2^{-1}(x)} = \lim_{x\to\infty} \frac{\int_0^{U_2(x)} U_1 o U_2^{-1}(s)ds}{U_2(x)\; U_1 o U_2^{-1}(U_2(x))}$$

$$= \lim_{x\to\infty} \frac{\int_0^x U_1(t)dU_2(t)}{U_1(x)\; U_2(x)} = \frac{\beta}{\alpha+\beta}\,.$$

Hence $U_1 o U_2^{-1} \in R\,V^{(\infty)}_{\beta^{-1}\alpha}$. Once we know this a) and b) are equivalent. If in addition to d) we assume c) then $U_1(x)\; U_2(x) = U_2(x)\; U_1 o U_2^{-1}(U_2(x)) =$ $U_4 o U_2(x) \in R\,V^{(\infty)}_{\alpha+\beta}$ where $U_4(x) = x\; U_1 o U_2^{-1}(x)$. Clearly $U_4 \in R\,V^{(\infty)}_{\beta^{-1}(\alpha+\beta)}$ hence $U_2 \in R\,V^{(\infty)}_{\beta}$.

<u>Proof of (P6)</u>:

$$U_2 \in \Pi^{(\infty)} \iff U_2 o U_1^{-1} \in \Pi^{(\infty)} \iff \int_0^x t\,dU_2 o U_1^{-1}(t) \in R\,V_1^{(\infty)}$$

$$\iff \int_0^x U_1(t)dU_2(t) = \int_0^{U_1(x)} t\,dU_2 o U_1^{-1}(t) \in R\,V_\alpha^{(\infty)}$$

and similarly for the third statement of (P6). The second equivalence above follows from [3], theorem 1.4.1.b.

3. Derivatives

We prove the following:

Theorem 1. Any α-varying function U with $\alpha + 1 \notin \mathbb{N}$ is asymptotic to a function U_1 all whose derivatives are regularly varying provided they are given the correct sign.

Proof. First let $\alpha < 0$. There is a decreasing function U_2 such that $U(x) \sim U_2(x)$ as $x \to \infty$. Define $U_3(x) = U_2(\frac{1}{x})$ then $U_3 \in R\,V_{-\alpha}^{(0)}$. Denote its Laplace-Stieltjes transform by $\check{U}_3$. Then $U_3(x) \sim \{\Gamma(1-\alpha)\}^{-1}\,\check{U}_3(\frac{1}{x})$ as as $x \downarrow 0$. So $U(x) \sim \{\Gamma(1-\alpha)\}^{-1}\,\check{U}_3(x)$ as $x \to \infty$ and the latter function satisfies the requirements (property 8 p.22 de Haan [3]).

Next let $\alpha > 0$ ($\alpha \notin \mathbb{N}$). There is an increasing function U_2 such that $U_2(0+) = 0$ and $U(x) \sim U_2(x)$ as $x \to \infty$. Denote its Laplace-Stieltjes transform by $\check{U}_2$. Then

$$U(x) \sim U_2(x) \sim \frac{1}{\Gamma(1+\alpha)}\,\check{U}_2(\frac{1}{x}) \quad \text{as } x \to \infty.$$

We shall prove that $U_1(x) = (\Gamma(1+\alpha))^{-1}\,\check{U}_2(\frac{1}{x})$ satisfies the requirements.

We have (Abramowitz and Stegun [1] Ch. 24, 1.2.I.c.)

$$\frac{d^n}{dx^n}\check{U}_2(\frac{1}{x}) = \sum_{m=1}^{n} \frac{n!}{m!}\binom{n-1}{m-1}(-1)^n\, x^{-n-m}\,\check{U}_2^{(m)}(\frac{1}{x}).$$

By property 8 of de Haan ([3], p.22) for m = 1, 2, ...

$$x^{-m}\,\check{U}_2^{(m)}(\frac{1}{x}) \sim (-\alpha)(-\alpha-1)\ \dots\ (-\alpha-m+1)\,\check{U}_2(\frac{1}{x})$$

as $x \to \infty$. Hence as $x \to \infty$

$$\frac{d^n}{dx^n}\check{U}_2(\frac{1}{x}) \sim n!\,(-1)^n\,x^{-n}\,\check{U}_2(\frac{1}{x}) \sum_{m=1}^{n}\binom{n-1}{n-m}\binom{-\alpha}{m}$$

$$= \binom{-\alpha+n-1}{n}\,n!\,(-1)^n\,x^{-n}\,\check{U}_2(\frac{1}{x})$$

$$= \alpha(\alpha-1)\ \dots\ (\alpha-n+1)\,x^{-n}\,\check{U}_2(\frac{1}{x}).$$

The theorem says that within each equivalence class of asymptotically equivalent regularly varying functions there is at least one function satisfying the requirements. A similar statement with a different definition of the equivalent class holds for functions in Π. This is the analogue of the previous theorem for $\alpha = 0$.

Theorem 2. Any function $U \in \Pi^{(\infty)}$, i.e. any non-decreasing function satisfying

$$\lim_{t\to\infty} \frac{U(tx) - b(t)}{a(t)} = \log x$$

for all $x > 0$ and suitably chosen functions $a(t) > 0$ and $b(t)$ has a companion function U_1 such that $(-1)^{n+1}\, U_1^{(n)}(x) \in R\, V_{-n}^{(\infty)}$ for $n = 1, 2, \ldots$ and

$$\lim_{t\to\infty} \frac{U(t) - U_1(t)}{a(t)} = 0.$$

Proof. The Laplace-Stieltjes transform $\check{U}(t)$ of U exists for all $t > 0$. We shall prove that $U_1(x) = \check{U}(x^{-1}\, e^{-\gamma})$ satisfies the requirements; here γ is Euler's constant. By de Haan [4] $\check{U} \in \Pi^{(0)}$ and

$$\lim_{t\to\infty} \frac{U(t) - \check{U}(\frac{1}{t})}{a(t)} = \gamma.$$

As in the previous proof we have

$$\frac{d^n}{dx^n}\, \check{U}(\tfrac{1}{x}) = \sum_{m=1}^{n} \frac{n!}{m!} \binom{n-1}{m-1} (-1)^n\, x^{-n-m}\, \check{U}^{(m)}(\tfrac{1}{x}).$$

By the lemma in [4] $\quad -\check{U}^{(1)}(\frac{1}{x}) \in R\, V_{-1}^{(0)}$ and by property 8, p.22 of [3] for $m = 1, 2, \ldots$ as $x \to \infty$

$$x^{-m}\, \check{U}^{(m)}(\tfrac{1}{x}) \sim (-1)^{m+1}\, (m-1)!\; x^{-1}\, \check{U}^{(1)}\, (\tfrac{1}{x}).$$

Hence as $x \to \infty$

$$\frac{d^n}{dx^n}\, \check{U}(\tfrac{1}{x}) \sim (-1)^n\, x^{-n-1}\, \check{U}^{(1)}(\tfrac{1}{x}) \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{(m-1)!\,n!}{m!} (-1)^{m+1}$$

$$= (-1)^n\, x^{-n-1} (n-1)!\; \check{U}^{(1)}(\tfrac{1}{x}).$$

References

[1] Abramowitz, M. and I.A. Stegun, ed. (1970). Handbook of mathematical functions. National bureau of standards, Washington.

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