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ON THE KALMAN FILTER AND THE ECONOMETRIC GENERAL LINEAR MODEL

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In this note we show how a particular econometric estimator for the general linear model (the one best adapted to the noise present in the observations) arises as the limit of Kalman state estimators of a discrete dynamical system with trivial dynamics.

1. First consider a discrete dynamical system given by the equations

$$(1)\qquad \begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + f_k + C_k w_k \\ y_k &= H_k x_k \\ z_k &= y_k + v_k \end{aligned}$$

where x_k is an n-dimensional state vector; u_k are deterministic controls; f_k a deterministic forcing vector; w_k is random noise in the system; y_k is the p-dimensional output vector; z_k are the observations of the y_k corrupted by random noise v_k; A_k, B_k, C_k, H_k are matrices of appropriate sizes, which are assumed to be known. One further makes the assumptions

$$(2)\qquad E(w_k) = 0 = E(v_k),\ E(w_k w_l^T) = \delta_{kl} Q_k,\ E(v_k v_l^T) = \delta_{kl} R_k,\ E(w_k v_l^T) = 0$$

where E denotes expectation and T denotes transposition. The covariance matrices Q_k and R_k are also assumed to be known.
Given all this, the Kalman filter with starting values $\hat{x}_o = 0$, P_o is given by the equations (cf. e.g. [2]).

$$(3)\qquad \begin{aligned} P'_k &= A_{k-1} P_{k-1} A^T_{k-1} + C_{k-1} Q_{k-1} C^T_{k-1} \\ P_k &= P'_k - K_k H_k P'_k \\ K_k &= P'_k H^T_k (H_k P'_k H^T_k + R_k)^{-1} \\ \hat{x}'_k &= A_{k-1} \hat{x}_{k-1} + f_{k-1} + B_{k-1} u_{k-1} \\ \hat{x}_k &= \hat{x}'_k + K_k (z_k - H_k \hat{x}'_k) \end{aligned}$$

2. Now consider the general linear model as it is often used in econometrics

(4) $$z = Hx + v$$

(H is usually written X, and x is usually written β; cf. [1] for a discussion of the linear model), where x is to be estimated from the observations z and v is random noise with $E(v) = 0$, $E(vv^T) = R$, where R is nonnegative definite.
We write the model as (a rather trivial) discrete dynamical system

(5) $$x_{k+1} = x_k,\ y_k = Hx_k,\ z_k = y_k + v$$

Now we are going to apply the discrete Kalman filter (3) to it starting with $\hat{x}_o = 0$ and an arbitrary initial covariance matrix P_o. Using a double induction one finds

(6) $$K_1 = P_oH^T(HP_oH^T+R)^{-1},\ K_n = P_oH^T(nHP_oH^T+R)^{-1}$$

It follows from this that

(7) $$K_n = K_{n-1} - K_nHK_{n-1}$$

and hence, using induction again, that the estimate for $\hat{x}_n$ is equal to

(8) $$\hat{x}_n = nP_oH^T(nHP_oH^T+R)^{-1}z$$

3. We are interested in what happens as n goes to infinity. There are (at least) two interesting cases.

Case A: HP_oH^T is nonsingular. (I.e. in any case less outputs than the dimension of the system). In this case the R in equation (8) can be neglected as n goes to infinity and we find the estimator

(9) $$\hat{x} = P_oH^T(HP_oH^T)^{-1}z$$

Case B. R is nonsingular, $p \geq n$, rank(H) = n. This is what is usually assumed in the econometric general linear model.
Premultiplication with $H^TR^{-1}H$ of the matrix in (8) gives

$$(H^TR^{-1}H)(nP_oH^T)(nHP_oH^T+R)^{-1} = H^TR^{-1} - H^T(nHP_oH^T+R)^{-1} \tag{10}$$

Now

$$\lim_{n\to\infty} H^T(nHP_oH^T+R)^{-1} = 0 \tag{11}$$

which is seen as follows. Because $\text{rank}(H^T) = \text{rank}(HP_oH^T)$ it suffices to prove that $HP_oH^T(nHP_oH^T+R)^{-1}$ goes to zero. But

$$HP_oH^T(nHP_oH^T+R)^{-1} = n^{-1}I - n^{-1}R(nHP_oH^T+R)^{-1}, \tag{12}$$

and one easily sees that the terms in $(nHP_oH^T+R)^{-1}$ are bounded independantly of n (e.g. by diagonalizing HP_oH^T). This proves (11).
Using (10) and (11) in (8) it follows that in case B one obtains the limit estimator

$$\hat{x} = (H^TR^{-1}H)^{-1}H^TR^{-1}z \tag{13}$$

which is the econometric generalized least squares estimator with weighting matrix equal to $R = E(vv^T)$.

REFERENCES

1. J. Koerts, A.P.J. Abrahamse. On the Theory and Applications of the General Linear Model, Rotterdam Univ. Pr., 1969.
2. H.W. Sorenson. Kalman Filtering Techniques. In: C.T. Leondes (ed). Advances in Control Systems. Vol. 3, Acad. Pr., 1966, 219-292.