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ON AN ABEL-TAUBER THEOREM FOR LAPLACE TRANSFORMS

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1. INTRODUCTION

The well known Abel-Tauber theorems for Laplace transforms of probability distributions (given e.g. in Feller 1, XIII, 5 ex. c) for indices $-\alpha$ of regular variation with $0 \leq \alpha < 1$ can be complemented by a similar theorem for $\alpha = 1$ (cf. Wichura, 5). This result can be derived by comparing two sets of equivalent conditions on the distribution function and its transform respectively for the domain of attraction of a stable distribution of exponent $\alpha = 1$ (Nevels, 3). Here we present a direct approach to this result. In Section 1 we derive the theorem for $\alpha = 1$ using Karamata's Tauberian theorem and some simple considerations. Section 2 contains the corresponding result for $\alpha = 2, 3, \ldots$ All these results are known. In the final section we present a more general result.

2. $\alpha = 1$

Let F be a distribution function with $F(0-) = 0$ and let $\check{F}$ be its Laplace transform. Write $\Psi(t) = t^{-1}(1 - \check{F}(t))$. We shall prove

THEOREM 1. The following assertions are equivalent:

(1) $$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-1} \quad \text{for all } x > 0$$

and

(2) $$\lim_{t\downarrow 0} \frac{\Psi(tx) - \Psi(t)}{\Psi(te) - \Psi(t)} = \log x \quad \text{for all } x > 0 .$$

Both imply

(3) $$\lim_{t\to\infty} \frac{\int_0^t (1 - F(s))ds - \Psi(1/t)}{t(1 - F(t))} = \gamma ,$$

Euler's constant.

Remark. Relation (1) says that $1 - F$ is -1-varying at infinity and (2) says that $\Psi(1/t) \in \Pi$ (cf. 2 section 1.4).

For the proof we need the following

LEMMA. Suppose U has a negative non-increasing derivative $-u$. Then

$$\lim_{t \downarrow 0} \frac{U(tx) - U(t)}{U(te) - U(t)} = \log x \text{ for all } x > 0 \tag{4}$$

if and only if u is -1-varying at $0+$. Moreover then $-t.u(t) \sim \{U(te) - U(t)\}$ as $t \downarrow 0$.

Remark. "u is -1-varying at $0+$" $\lim_{t \downarrow 0} \frac{u(tx)}{u(t)} = x^{-1}$ for $x > 0$.

Proof

The method of proof is adapted from 2, section 2.7. If u is -1-varying, then

$$\{U(tx) - U(t)\}/tu(t) = -\int_1^x \frac{u(ts)}{u(t)}\, ds.$$

Since the integrand tends to s^{-1} uniformly, (4) follows.

Next suppose (4) holds. We write

$$\frac{U(tx) - U(t)}{U(te) - U(t)} = \frac{-t.u(t)}{U(te) - U(t)} \int_1^x \frac{u(ts)}{u(t)}\, ds.$$

The last integral by our assumption on u is at most $x - 1$ when $x > 1$. Hence

$$\limsup_{t \downarrow 0} \frac{-t.u(t)}{U(te) - U(t)} \leq \frac{\log x}{x - 1}$$

for all $x > 1$. Similarly

$$\liminf_{t \downarrow 0} \frac{-t.u(t)}{U(te) - U(t)} \geq \frac{\log x}{x - 1}$$

for all $0 < x < 1$. Hence $-t.u(t) \sim (U(te) - U(t))$ as $t \to \infty$. By (4) $U(t) - U(te)$ is a slowly varying function, so is $t.u(t)$.

Remark. A similar property is true at $t = \infty$.

<u>Proof</u> of theorem 1. Let $U(t) = \int_0^t (1 - F(s))ds$ and $V(t) = \int_0^t s(1 - F(s))ds$. Let $\check{U}$ and $\check{V}$ be the Laplace-Stieltjes transforms of U and V. Simple calculations show

$$\Psi(t) = \check{U}(t) = \int_t^\infty \check{V}(s)ds.$$

Now $1 - F$ is -1-varying at infinity if and only if V is 1-varying at infinity (this can be seen by the method of proof of the lemma above cf. Pitman, <u>4</u>). By a standard Abel-Tauber theorem (Feller <u>1</u>, XIII, 5 theorem 2) the 1-variation of V is equivalent to the -1-variation of $\check{V}$ at $0+$. This in turn by the lemma above is equivalent to (2).

To show (3) we write

$$\begin{aligned}\int_0^t (1 - F(s))ds - \Psi(1/t) &= \int_0^1 t(1 - F(st))ds - \int_0^\infty e^{-s/t}(1 - F(s))ds \\ &= \int_0^1 t(1 - F(st))ds - \int_0^\infty te^{-s}(1 - F(st))ds \\ &= \int_0^1 \frac{1 - e^{-s}}{s} st(1 - F(st))ds - \int_1^\infty \frac{e^{-s}}{s} st(1 - F(st))ds.\end{aligned}$$

Now $(st)^\rho(1 - F(st))/t^\rho(1 - F(t))$ tends to $s^{\rho-1}$ as $t \to \infty$ uniformly on $(0, 1]$ when $\rho > 1$ and uniformly on $[1, \infty)$ when $\rho < 1$ hence

$$\begin{aligned}\lim_{t\to\infty} \{\int_0^1 \frac{1 - e^{-s}}{s} \cdot \frac{st(1 - F(st))}{t(1 - F(t))} ds - \int_1^\infty \frac{e^{-s}}{s} \cdot \frac{st(1 - F(st))}{t(1 - F(t))} dt\} = \\ = \int_0^1 \frac{1 - e^{-s}}{s} ds - \int_1^\infty \frac{e^{-s}}{s} ds = \gamma.\end{aligned}$$

<u>Corollary</u>. Under the conditions of the theorem

$$\lim_{t\downarrow 0} \frac{t^2 \frac{d}{dt} \left(\frac{1 - \check{F}(t)}{t}\right)}{1 - F(1/t)} = -1$$

<u>Proof</u>. Use the final conclusion of the lemma and a similar property with respect to $\int_0^x (1 - F(t))dt$.

3. $\alpha > 1$

For completeness we mention an analogous result when $1 - F$ is $-n$-varying with $n = 2, 3, \ldots$, given also by Nevels. Define $F_1(t) = F(t)$ and $1 - F_{n+1}(t) = \int_t^\infty (1 - F_n(s))ds \leq \infty$ for $n = 1, 2, \ldots$ and $t > 0$. When finite $1 - F_n$ is a distribution tail. Now $1 - F$ is $-n$-varying at infinity if and only if $1 - F_n$ is finite and -1-verying at infinity. Define $U_n(t) = \int_0^t (1 - F_n(s))ds$. Its Laplace-Stieltjes transform is

$$\check{U}_n(t) = \left(\frac{-1}{t}\right)^n \{\check{F}(t) - \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} \int_0^\infty x^k dF(x)\}.$$

Application of theorem 1 yields

THEOREM 2. Let n be a positive integer. The following two statements are equivalent: $\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-n}$ for all $x > 0$ and

$$\lim_{t\downarrow 0} \frac{\check{U}_n(tx) - \check{U}_n(t)}{\check{U}_n(te) - \check{U}_n(t)} = \log x \text{ for all } x > 0.$$

Both imply

$$\lim_{t\to\infty} \frac{U_n(t) - \check{U}_n(1/t)}{t(1 - F_n(t))} = \gamma.$$

4. A MORE GENERAL RESULT

The approach to theorem 1 is through the measure corresponding to $U(t) = \int_0^t (1 - F(s))ds$ which has a monotone derivative. The latter property is not necessary as the following result shows.

THEOREM 3. Suppose U is non-decreasing and right-continuous; furthermore $U(0-) = 0$. Suppose $\check{U}(t) = \int_0^\infty e^{-ts}dU(s)$ is finite for all $t > 0$. The following assertions are equivalent:

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{U(te) - U(t)} = \log x \quad \text{for all } x > 0 \tag{5}$$

and

$$(6) \qquad \lim_{t\downarrow 0} \frac{\check{U}(tx) - \check{U}(t)}{\check{U}(te) - \check{U}(t)} = \log x \quad \text{for all } x > 0.$$

Both imply

$$(7) \qquad \lim_{t\to\infty} \frac{U(t) - \check{U}(1/t)}{t^{-1} \int_0^t s\,dU(s)} = \gamma.$$

Remark. In the formulation of (5) we tacitly assume

$$(8) \qquad U(te) - U(t) > 0 \quad \text{for sufficiently large } t.$$

This is implied by (6).

Proof. Define $Q(t) = \int_0^t s\,dU(s)$. By theorem 1.4.1, a/b of 2 relation (5) holds if and only if Q is 1-varying at infinity. So (5) is equivalent to -1-variation of $\check{Q}(t)$ at 0+. Now $\check{U}(t) = \int_t^\infty \check{Q}(s)ds$ hence by the lemma the equivalence of (5) and (6) is established.

To show (7) we use the representation

$$U(t) = g(t) + \int_0^t \frac{g(s)}{s}\,ds \quad \text{with } g(t) = t^{-1} \int_0^t s\,dU(s)$$

(cf. proof of theorem 1.4.1 in 1, part b. => d.). Then

$$\check{U}(1/t) = t^{-1} \int_0^\infty e^{-s/t}\, U(s)ds = \int_0^\infty e^{-s} g(ts)ds + \int_0^\infty \frac{e^{-s}}{s}\, g(ts)ds$$

so that (analogous to the proof of theorem 1)

$$U(t) - \check{U}(1/t) = g(t) - \int_0^\infty e^{-s} g(ts)ds + \int_0^1 \frac{1 - e^{-s}}{s}\, g(ts)ds - \int_1^\infty \frac{e^{-s}}{s}\, g(ts)ds .$$

Hence

$$\lim_{t\to\infty} \frac{U(t) - \check{U}(1/t)}{g(t)} = \gamma.$$

<u>Corollary</u>. Under the conditions of the theorem

$$\lim_{t\downarrow 0} \frac{\frac{d}{dt}\check{U}(t)}{\int_0^{1/t} s\,dU(s)} = -1.$$

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