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ON AN ABEL-TAUBER THEOREM FOR LAPLACE TRANSFORMS

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# ON AN ABEL-TAUBER THEOREM FOR LAPLACE TRANSFORMS

by Laurens de Haan

## 1. INTRODUCTION

The well known Abel-Tauber theorems for Laplace transforms of probability distributions (given e.g. in Feller 1, XIII, 5 ex. c) for indices  $-\alpha$  of regular variation with  $0 \leq \alpha < 1$  can be complemented by a similar theorem for  $\alpha = 1$  (cf. Wichura, 5). This result can be derived by comparing two sets of equivalent conditions on the distribution function and its transform respectively for the domain of attraction of a stable distribution of exponent  $\alpha = 1$  (Nevels, 3). Here we present a direct approach to this result. In Section 1 we derive the theorem for  $\alpha = 1$  using Karamata's Tauberian theorem and some simple considerations. Section 2 contains the corresponding result for  $\alpha = 2, 3, \dots$ . All these results are known. In the final section we present a more general result.

## 2. $\alpha = 1$

Let  $F$  be a distribution function with  $F(0-) = 0$  and let  $\tilde{F}$  be its Laplace transform. Write  $\psi(t) = t^{-1}(1 - \tilde{F}(t))$ . We shall prove

THEOREM 1. The following assertions are equivalent:

$$(1) \quad \lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-1} \quad \text{for all } x > 0$$

and

$$(2) \quad \lim_{t \downarrow 0} \frac{\psi(tx) - \psi(t)}{\psi(te) - \psi(t)} = \log x \quad \text{for all } x > 0.$$

Both imply

$$(3) \quad \lim_{t \rightarrow \infty} \frac{\int_0^t (1 - F(s)) ds - \psi(1/t)}{t(1 - F(t))} = \gamma,$$

Euler's constant.

Remark. Relation (1) says that  $1 - F$  is  $-1$ -varying at infinity and (2) says that  $\Psi(1/t) \in \Pi$  (cf. 2 section 1.4).

For the proof we need the following

LEMMA. Suppose  $U$  has a negative non-increasing derivative  $-u$ . Then

$$(4) \quad \lim_{t \downarrow 0} \frac{U(tx) - U(t)}{U(te) - U(t)} = \log x \text{ for all } x > 0$$

if and only if  $u$  is  $-1$ -varying at  $0+$ . Moreover then  $-t \cdot u(t) \sim \{U(te) - U(t)\}$  as  $t \downarrow 0$ .

Remark. " $u$  is  $-1$ -varying at  $0+$ "  $\lim_{t \downarrow 0} \frac{u(tx)}{u(t)} = x^{-1}$  for  $x > 0$ .

#### Proof

The method of proof is adapted from 2, section 2.7.

If  $u$  is  $-1$ -varying, then

$$\{U(tx) - U(t)\}/tu(t) = - \int_1^x \frac{u(ts)}{u(t)} ds.$$

Since the integrand tends to  $s^{-1}$  uniformly, (4) follows.

Next suppose (4) holds. We write

$$\frac{U(tx) - U(t)}{U(te) - U(t)} = \frac{-t \cdot u(t)}{U(te) - U(t)} \int_1^x \frac{u(ts)}{u(t)} ds.$$

The last integral by our assumption on  $u$  is at most  $x - 1$  when  $x > 1$ .

Hence

$$\limsup_{t \downarrow 0} \frac{-t \cdot u(t)}{U(te) - U(t)} \leq \frac{\log x}{x - 1}$$

for all  $x > 1$ . Similarly

$$\liminf_{t \downarrow 0} \frac{-t \cdot u(t)}{U(te) - U(t)} \geq \frac{\log x}{x - 1}$$

for all  $0 < x < 1$ . Hence  $-t \cdot u(t) \sim (U(te) - U(t))$  as  $t \rightarrow \infty$ . By (4)  $U(t) - U(te)$  is a slowly varying function, so is  $t \cdot u(t)$ .

Remark. A similar property is true at  $t = \infty$ .

Proof of theorem 1. Let  $U(t) = \int_0^t (1 - F(s))ds$  and  $V(t) = \int_0^t s(1 - F(s))ds$ . Let  $\check{U}$  and  $\check{V}$  be the Laplace-Stieltjes transforms of  $U$  and  $V$ . Simple calculations show

$$\Psi(t) = \check{U}(t) = \int_t^\infty \check{V}(s)ds.$$

Now  $1 - F$  is  $-1$ -varying at infinity if and only if  $V$  is  $1$ -varying at infinity (this can be seen by the method of proof of the lemma above cf. Pitman, 4). By a standard Abel-Tauber theorem (Feller 1, XIII, 5 theorem 2) the  $1$ -variation of  $V$  is equivalent to the  $-1$ -variation of  $\check{V}$  at  $0+$ . This in turn by the lemma above is equivalent to (2).

To show (3) we write

$$\begin{aligned} \int_0^t (1 - F(s))ds - \Psi(1/t) &= \int_0^1 t(1 - F(st))ds - \int_0^\infty e^{-s/t}(1 - F(s))ds \\ &= \int_0^1 t(1 - F(st))ds - \int_0^\infty te^{-s}(1 - F(st))ds \\ &= \int_0^1 \frac{1 - e^{-s}}{s} st(1 - F(st))ds - \int_1^\infty \frac{e^{-s}}{s} st(1 - F(st))ds. \end{aligned}$$

Now  $(st)^\rho(1 - F(st))/t^\rho(1 - F(t))$  tends to  $s^{\rho-1}$  as  $t \rightarrow \infty$  uniformly on  $(0, 1]$  when  $\rho > 1$  and uniformly on  $[1, \infty)$  when  $\rho < 1$  hence

$$\begin{aligned} \lim_{t \rightarrow \infty} \left\{ \int_0^1 \frac{1 - e^{-s}}{s} \cdot \frac{st(1 - F(st))}{t(1 - F(t))} ds - \int_1^\infty \frac{e^{-s}}{s} \cdot \frac{st(1 - F(st))}{t(1 - F(t))} dt \right\} &= \\ &= \int_0^1 \frac{1 - e^{-s}}{s} ds - \int_1^\infty \frac{e^{-s}}{s} ds = \gamma. \end{aligned}$$

Corollary. Under the conditions of the theorem

$$\lim_{t \downarrow 0} \frac{t^2 \frac{d}{dt} \left( \frac{1 - \check{F}(t)}{t} \right)}{1 - F(1/t)} = -1$$

Proof. Use the final conclusion of the lemma and a similar property with respect to  $\int_0^x (1 - F(t))dt$ .

3.  $\alpha > 1$ 

For completeness we mention an analogous result when  $1 - F$  is  $-n$ -varying with  $n = 2, 3, \dots$ , given also by Nevels. Define  $F_1(t) = F(t)$  and  $1 - F_{n+1}(t) = \int_0^t (1 - F_n(s)) ds \leq \infty$  for  $n = 1, 2, \dots$  and  $t > 0$ . When finite  $1 - F_n$  is a distribution tail. Now  $1 - F$  is  $-n$ -varying at infinity if and only if  $1 - F_n$  is finite and  $-1$ -varying at infinity. Define  $U_n(t) = \int_0^t (1 - F_n(s)) ds$ . Its Laplace-Stieltjes transform is

$$\check{U}_n(t) = \left(\frac{-1}{t}\right)^n \{F(t) - \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} \int_0^\infty x^k dF(x)\}.$$

Application of theorem 1 yields

THEOREM 2. Let  $n$  be a positive integer. The following two statements are equivalent:  $\lim_{t \rightarrow \infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-n}$  for all  $x > 0$

and

$$\lim_{t \downarrow 0} \frac{\check{U}_n(tx) - \check{U}_n(t)}{\check{U}_n(te) - \check{U}_n(t)} = \log x \text{ for all } x > 0.$$

Both imply

$$\lim_{t \rightarrow \infty} \frac{U_n(t) - \check{U}_n(1/t)}{t(1 - F_n(t))} = \gamma.$$

## 4. A MORE GENERAL RESULT

The approach to theorem 1 is through the measure corresponding to  $U(t) = \int_0^t (1 - F(s)) ds$  which has a monotone derivative. The latter property is not necessary as the following result shows.

THEOREM 3. Suppose  $U$  is non-decreasing and right-continuous; furthermore  $U(0-) = 0$ . Suppose  $\check{U}(t) = \int_0^\infty e^{-ts} dU(s)$  is finite for all  $t > 0$ . The following assertions are equivalent:

$$(5) \quad \lim_{t \rightarrow \infty} \frac{U(tx) - U(t)}{U(te) - U(t)} = \log x \quad \text{for all } x > 0$$

and

$$(6) \quad \lim_{t \downarrow 0} \frac{\check{U}(tx) - \check{U}(t)}{\check{U}(te) - \check{U}(t)} = \log x \quad \text{for all } x > 0.$$

Both imply

$$(7) \quad \lim_{t \rightarrow \infty} \frac{U(t) - \check{U}(1/t)}{t^{-1} \int_0^t s dU(s)} = \gamma.$$

Remark. In the formulation of (5) we tacitly assume

$$(8) \quad U(te) - U(t) > 0 \quad \text{for sufficiently large } t.$$

This is implied by (6).

Proof. Define  $Q(t) = \int_0^t s dU(s)$ . By theorem 1.4.1, a/b of 2 relation (5) holds if and only if  $\check{Q}$  is 1-varying at infinity. So (5) is equivalent to -1-variation of  $\check{Q}(t)$  at  $0+$ . Now  $\check{U}(t) = \int_0^\infty \check{Q}(s) ds$  hence by the lemma the equivalence of (5) and (6) is established.

To show (7) we use the representation

$$U(t) = g(t) + \int_0^t \frac{g(s)}{s} ds \quad \text{with } g(t) = t^{-1} \int_0^t s dU(s)$$

(cf. proof of theorem 1.4.1 in 1, part b.  $\Rightarrow$  d.). Then

$$\check{U}(1/t) = t^{-1} \int_0^\infty e^{-s/t} U(s) ds = \int_0^\infty e^{-s} g(ts) ds + \int_0^\infty \frac{e^{-s}}{s} g(ts) ds$$

so that (analogous to the proof of theorem 1)

$$U(t) - \check{U}(1/t) = g(t) - \int_0^\infty e^{-s} g(ts) ds + \int_0^1 \frac{1 - e^{-s}}{s} g(ts) ds - \int_1^\infty \frac{e^{-s}}{s} g(ts) ds.$$

Hence

$$\lim_{t \rightarrow \infty} \frac{U(t) - \check{U}(1/t)}{g(t)} = \gamma.$$

Corollary. Under the conditions of the theorem

$$\lim_{t \downarrow 0} \frac{\frac{d}{dt} \check{U}(t)}{\frac{1}{t} \int_0^t s dU(s)} = -1.$$

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