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ON THE RELATIONSHIP BETWEEN PRODUCTION FUNCTIONS AND INPUT-OUTPUT ANALYSIS WITH FIXED VALUE SHARES

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1. INTRODUCTION AND SUMMARY

In this paper we examine the necessary and sufficient conditions for constancy of the value input-output coefficients. In his well-known article published in the Review of Economic Studies Klein [1952/1953] studied the same problem. Nevertheless, there is a striking difference between his approach and ours. Klein assumes that all input- and output prices are known, either as fixed or as given functions of input demand. He subsequently shows that every production function which is zero-homogeneous in all inputs and outputs yields constant value-shares of inputs in the outputs, assuming optimal allocation of both. This is true whenever one specific vector of input- and output prices is considered.

We, however, require constant value shares for <u>any</u> exogenously determined variable vector of input prices and output prices. This additional requirement restricts the class of admissible production functions.

First, we consider the case of a single homogeneous output per sector. We show that constancy of value input-output coefficients holds good <u>if</u> <u>and</u> <u>only</u> <u>if</u> the production function is of the linear-homogeneous Cobb-Douglas type.

Secondly, we consider the case of multi-output production functions that are <u>separable</u> into an input part and an output part. In this case, the optimal allocation of outputs as well as inputs requires the additional assumption of profit maximization. In the latter case we

[1] The author is indebted to Prof.Dr. W.H. Somermeyer for his valuable comments and to Dr. J. van Daal for his constructive criticism.

show that the necessary and sufficient condition for constancy of the value input-output coefficients is precisely the production function that Klein advances as "an obvious example of a production function ..." (c.f. Klein, p. 133, formulae (11)). He then adds: "Other functions satisfying the system of partial differential equations can easily be constructed". We prove that if the constancy is to hold for any vector of prices, this latter assertion cannot be upheld.

2. CONSTANCY OF INPUT-OUTPUT RATIOS IN REAL TERMS

In input-output analysis traditionally two variants are distinguished.

The first variant assumes constant ratios between the inputs in the production process and the output in physical quantities. In economic applications this is represented by variables in real terms. The second variant assumes constant ratios between inputs and outputs in nominal terms:

$$\frac{p_{mt}x_{mt}}{\pi_t y_t} = c_m \text{ is constant for } m = 1, \ldots, M \text{ and for all observations } t^{(2)} \qquad (1)$$

in which: y_t = output (assuming a single homogeneous output per sector)

π_t = the price of output

x_{mt} = input of production factor m (= 1, ..., M)

p_{mt} = price of factor m.

The first variant is based on the so-called Leontief production function:

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2 From now on, we shall drop the index "t".

$$y = \min \left(\frac{x_1}{a_1}, \ldots, \frac{x_m}{a_m}, \ldots, \frac{x_M}{a_M}\right) \qquad (2)$$

with a_m the technical coefficients ($m = 1, \ldots, M$), supposed to be constant.

Assuming that the output y is produced at minimum cost, so that no waste occurs, we can rewrite (2) as:

$$y = \frac{x_1}{a_1} = \ldots = \frac{x_m}{a_m} = \ldots = \frac{x_M}{a_M} \qquad (3)$$

Equations (3) express the familiar input-output relations.

3. CONSTANCY IN RATIOS BETWEEN INPUTS AND SINGLE OUTPUT

We now prove that the second variant results from minimization of costs $C = \sum_{m=1}^{M} p_m x_m$ if and only if the constraint

$$y = g(x_1, \ldots, x_M) \qquad (4)$$

is the linear-homogeneous Cobb-Douglas production function.

Proof:

If all inputs x_m ($m = 1, \ldots, M$) increase in the same proportion μ, with input prices p_m independent of x_m ($m = 1, \ldots, M$) and π independent of y, the value input-output ratios c_m (c.f. (1)) can remain constant if and only if y is increased in the same proportion μ; hence, the production function should be homogeneous of degree one. Consequently, the Euler relations:

$$y = \sum_{m=1}^{M} x_m \frac{\partial g}{\partial x_m} \qquad (5)$$

hold good.

The first-order conditions for minimization of costs (c.f. (4)) are:

$$p_m = \lambda \frac{\partial g}{\partial x_m} \quad \text{for } m = 1, \ldots, M \qquad (6)$$

in which λ is the Lagrange multiplier.

Substitution of (6) into (1) yields:

$$\frac{\lambda}{\pi} \cdot \frac{x_m g_m}{y} = c_m \quad \text{for } m = 1, \ldots, M \qquad (7)$$

Summation of (7) over m followed by substitution of (5) leads to:

$$\frac{\lambda}{\pi} = \sum_{m=1}^{M} c_m \qquad (8)$$

Equations (7) and (8) together imply:

$$x_m \frac{\partial g}{\partial x_m} = d_m y \quad \text{for } m = 1, \ldots, M \qquad (9)$$

in which $d_m = \dfrac{c_m}{\sum_{m'=1}^{M} c_m}$. Obviously, $\sum_{m=1}^{M} d_m = 1$.

The solution of the system of partial differential equations (9) is:

$$y = A \prod_{m=1}^{M} x_m^{d_m} \qquad (10)$$

in which A is a constant of integration and $\sum_{m=1}^{M} d_m = 1$, i.e. the linear-homogeneous Cobb-Douglas production function.

4. CONSTANCY IN RATIOS BETWEEN INPUTS AND JOINT OUTPUT

Above, we assumed that any sector produces a single output only. Within an input-output framework, however, we can treat every delivery by a sector as a separate output.

Therefore, we now suppose that a particular sector produces N outputs, denoted by $y_1, \ldots, y_N$, with prices $\pi_1, \ldots, \pi_N$ independent of the outputs.

Then, (1) is rewritten as:

$$\frac{p_m x_m}{\sum_{n=1}^{N} \pi_n \cdot y_n} = c_m = \text{constant for } m = 1, \ldots, M \qquad (11)$$

Since in this case we deal with combinations of outputs instead of a single output, we replace the production function (4) by:

$$h_y(y_1, \ldots, y_N) = h_x(x_1, \ldots, x_M), \qquad (12)$$

with h_x and h_y the input part and the output part of the production function, respectively. Since multiplication of all y_n in (11) by the same factor (say μ) ensures constancy of c_m only if all x_m too are multiplied by the same factor, h_x and h_y have to be homogeneous of the same degree, say h. Evidently, however, we may as well assume - by raising both h_x and h_y to the same power h^{-1} - that both h_x and h_y are linear-homogeneous.

According to the theory of market behaviour of the firm any combination $(y_1, \ldots, y_N)$ is produced at minimum cost. Consequently, our problem becomes:

minimize $C = \sum_{m=1}^{M} p_m x_m$ subject to (12).

Define Y to be equal to $h_y(y_1, \ldots, y_N)$ and define an appropriate price index π such that:

$\pi = \sum_{n=1}^{N} \pi_n y_n / Y$ and (11) becomes:

$$\frac{p_m x_m}{\sum_{n=1}^{N} \pi_n y_n} \equiv \frac{p_m x_m}{\pi . Y} = c_m = \text{constant} \qquad (13)$$

Then, we can apply the foregoing theorem to conclude that the input part of (12) is the linear-homogeneous Cobb-Douglas production function:

$$h_x(x_1, \ldots, x_M) = A \sum_{m=1}^{M} x_m^{d_m} \qquad (14)$$

(compare (10)).

Since the prices of the outputs are assumed to be exogenous, the first-order conditions for profit maximization are:

$$\pi_n = \frac{\partial \gamma}{\partial y_n} \quad n = 1, \ldots, N \qquad (15)$$

where γ denotes the (minimum) <u>cost function</u> associated with (14).

By virtue of the assumptions the only arguments of the cost function are the quantities produced $y_1, \ldots, y_N$.

It is easily proved that the optimum quantities of inputs for cost minimization subject to (14) are:

$$x_m = h_y(y_1, \ldots, y_N).(\frac{p_m}{d_m})^{-1} \prod_{m'=1}^{M} (\frac{p_{m'}}{d_{m'}})^{d_{m'}} \qquad (16)$$

entailing the cost function:

$$\gamma = kh_y(y_1, \ldots, y_N) \qquad (17)$$

where $k = \prod_{m'=1}^{M} (\frac{p_{m'}}{d_{m'}})^{d_{m'}}$ a constant since the $p_m (m = 1, \ldots, M)$ are given, taking into account that $\sum_{m=1}^{M} d_m = 1$.

Hence, the first-order conditions (16) can be rewritten as:

$$\pi_n = k \;.\; \frac{\partial h_y(y_1, \ldots, y_N)}{\partial y_n} \qquad (18)$$

Since the π_n are also given, we have:

$$\frac{\partial h_y(y_1, \ldots, y_N)}{\partial y_n} = b_n \text{ for } n = 1, \ldots, N \qquad (19)$$

with $b_n = \pi_n/k$.

Multiplying both members of (19) by y_n and summing over n yields:

$$h_y(y_1, \ldots, y_N) = \sum_{n=1}^{N} b_n y_n \qquad (20)$$

by virtue of the linear-homogeneity of h_y.

Substitution of (20) into (15) yields:

$$\sum_{n=1}^{N} b_n y_n = A \prod_{m=1}^{M} x_m^{d_m} \qquad (21)$$

c.f. eq. (11) of Klein.[3]

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3 In Klein's equation (11) a constant $\beta = \sum_{m=1}^{M} d_m$ appears. However, under the assumptions introduced in this paper $\beta = 1$.

Under the assumption of separability of the production function into an input part and an output part we have proved that the production function of the type (21) is the necessary and sufficient condition for constancy of the value input-output coefficients, if all prices are exogenous. It implies infinite substitutability between outputs and an elasticity of input substitution equalling 1.

REFERENCE

Klein, L.R. [1952/1953], "On the Interpretation of Professor Leontief's System; The Review of Economic Studies; Vol. XX, pp. 131-136.