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ERASMUS UNIVERSITY, ROTTERDAM

Netherlands School of Economics

ECONOMETRIC INSTITUTE

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# A NESTED CES-TYPE UTILITY FUNCTION AND ITS DEMAND AND PRICE-INDEX FUNCTIONS

by W.J. Keller

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CONTENTS



## SUMMARY

The Nested Modified CES-type (NEMCES) utility function is introduced here; it consists of an N-level utility tree of CES-type functions. Modifications to the CES-type utility function are made to enable more realistic descriptions in the limiting case where substitution is no longer possible. Attractive price- and quantity-indices are developed and an elegant system of demand functions, specified in terms of price-indices, results. Additionally, some suggestions for solving estimation problems in case of NEMCES and CES-type functions are made.

## 1. Introduction

In the theory of consumer demand the utility tree is a well-known concept. This concept was developed by STROTZ (1957), who also studied implications for problems like aggregation over commodities and corresponding price indices. The utility tree is, however, seldom used for the formulation of operational models. The reason is the same as for the limited use of consumer demand theory : only from highly simplified utility functions we are able to derive analytical results.

Leaving the quadratic utility functions aside, in view of the rather awkward demand functions which it yields, we can summarize the utility functions adopted most frequently [1] in the following function

$$(1.1) \qquad \omega = F\left(\sum_{j=1}^{J} \alpha_j (q_j - \bar{q}_j)^{\beta_j}\right) \qquad \text{with } \alpha_j > 0$$

where $\omega$ is utility, F a monotonic transformation, $q_j$ the quantity consumed of the j-th commodity [2] with price $p_j$; $\bar{q}_j$, $\alpha_j$ and $\beta_j$ are parameters. This function is based on the 'direct additive logarithmic' utility function of HOUTHAKKER (1960). It includes the Stone-Geary utility function (SAMUELSON (1948b), GEARY (1950), STONE (1954)) by letting all the $\beta_j$ approach to zero, provided that a relevant transformation of (1.1) has taken place; this function leads to the well-known Linear Expenditure System (LES); see KLEIN, RUBIN (1948), SAMUELSON (1948a), STONE (1954).

A utility function similar to the Constant Elasticity of Substitution (CES) production function (see ARROW et al. (1961)) is obtained if all the $\beta_j$ are put equal to each other. The resulting system of demand functions was called Generalized Linear Expenditure System (GLES) by GAMALETSOS (1973). Similar demand functions were earlier developed by BASMANN (1968) and BREMS (1968).

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1) Besides the demand functions mentioned here resulting from so-called 'direct' utility functions as (1.1), there exist also demand functions derived from 'indirect' utility functions. A review of the implications of the demand functions resulting from the indirect addilog utility function (see also LESER (1941) and HOUTHAKKER (1960))can be found in SOMERMEIJER and LANGHOUT (1972).

2) We shall use the term commodity in a broad sense : all quantifiable objects to which the individual can connect utility and a price are called commodities. An example of a 'commodity' defined in this way is leisure.

Unfortunately no explicit demand function has been derived for the above-mentioned utility function (1.1). The GLES demand function is the most general one that we are able to derive analytically from (1.1); the assumption $\beta_j = \beta$ (for all j) is indispensable in order to obtain an analytical solution. This assumption, however, makes the interpretation of (1.1) less general; the elasticities of substitution between all pairs of commodities are equal to each other, viz. to $(1-\beta)^{-1}$.

In this paper we shall show that a merger between the concept of the utility tree and the CES-type utility functions, results in a more general class of demand functions than the system of GLES demand function. The underlying utility function consists of nested CES-type functions [3]; they have the advantage that the resulting demand functions are not hard to derive analytically and easy to interpret. As a by-product we derive practical formulations of price-indices for groups of commodities at various levels of aggregation.

## 2. The implicit solution

We consider an individual faced with the problem of allocating his budget to a set of commodities, such as to maximize his utility.

The quantities are denoted by $q_j$, for j = 1, . ., J, corresponding to commodities with prices $p_j$, respectively. The utility function is defined as

(2.1) $$\omega = \omega(q_j;\ j = 1, . ., J)$$

The budget restriction is

(2.2) $$y = \sum_j p_j q_j$$

where y equals the individual's budget.

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3) In the theory of production functions the two-level CES-function is studied by SATO (1967). An interesting paper by BERNDT and CHRISTENSEN (1973) presents relationships between the concept of separability, elasticities of substitution and the CES and two-level CES production function. Their definition of strong separability of production functions equals the additivity definition of utility functions presented by HOUTHAKKER (1960).
Recently BROWN et al. (1972), have studied a two-level CES utility function. The nested CES utility function presented here can be considered as a generalization to N levels.

In order to find the maximum of (2.1) subject to (2.2), we consider the Lagrangean

(2.3) $$L = \omega + \lambda(y - \sum_j p_j q_j)$$ with $\lambda$ a Lagrange multiplier,

and differentiate it with respect to $q_j$ and $\lambda$ and equate the result to zero. This yields

(2.4) $$\frac{\partial \omega}{\partial q_j} = \lambda\, p_j \qquad \text{for } j = 1,.., J$$

(2.5) $$y = \sum_j p_j q_j$$

Multiplying (2.4) by $q_j$ and summing over j, yields, in combination with (2.5) :

(2.6) $$\lambda = \frac{W}{y}$$

where

(2.7) $$W = \sum_j \frac{\partial \omega}{\partial q_j} q_j$$

The implicit solution can be found by substituting (2.6) into (2.4) :

(2.8) $$\frac{\partial \omega}{\partial q_j} = \frac{W p_j}{y} \qquad \text{for } j = 1,.., J.$$

## 3. Nested utility functions

We now pay attention to the specific structure of the utility function. We assume that the utility function (2.1) is built up as a utility tree or, as we shall call it, a <u>nested utility function</u>. Therefore we introduce the concept of the <u>utility component</u>. The utility function is written as a function of the 'first level' utility components $\phi_{i_1}$, $i_1 = 1, . ., I$ :

(3.1) $$\omega = \phi(\phi_{i_1} ; i_1 = 1, . ., I)$$

and the utility components $\phi_{i_1}$ are written as functions of the second level utility components $\phi_{i_1 i_2}$ :

(3.2) $\phi_{i_1} = \phi_{i_1} (\phi_{i_1 i_2} ; i_2 = 1, . ., I_{i_1})$

and so on. Finally

(3.3) $\phi_{i_1 .. i_{N-1}} = \phi_{i_1 .. i_{N-1}} (\phi_{i_1 .. i_N} ; 1, .., I_{i_1 .. i_{N-1}})$

We shall call the N-th level utility components $\phi_{i_1 . . i_N}$ the elementary or basic utility components or in short utility elements. As the name says, they are the basic elements of the utility function. We assume that the utility elements $\phi_{i_1 . . i_N}$ can be quantified in well-defined units, and that prices per unit are known.

Following SAMUELSON (1948a) and FRISCH (1956) we introduce minimum quantities of each commodity $q_j$, say $\bar{q}_j$. Now we associate with each utility element the quantity of a commodity consumed in excess of the subsistence or minimum quantity of that commodity :

(3.4) $\phi_{i_1 .. i_N} \equiv \tilde{q}_{i_1 .. i_N} \equiv \tilde{q}_j$ for all $(i_1 .. i_N)$ and j

where

(3.5) $\tilde{q}_j = q_j - \bar{q}_j > 0$

and where there exists a one-to-one correspondence between the subscript $(i_1 .. i_N)$ and the subscript j, for j = 1, . ., J, where

(3.6) $J = \sum_{i_1} . . \sum_{i_{N-1}} I_{i_1 .. i_{N-1}}$

This completes the definition of the structure of the nested utility function. An example of the structure of such a function is given in fig. 1.

Subsequently we shall reduce the 'compounded' subscript $(i_1 .. i_n)$ to simple $i_n$, for ease of notation. The utility components are now written as

(3.7) $\phi_{i_{n-1}} = \phi_{i_{n-1}} (\phi_{i_n} ; i_n = 1, . ., I_{i_{n-1}})$ for n = 1, . ., N

with

(3.8) $\phi_{i_0} \equiv \phi(\phi_{i_1} ; i_1 = 1, . ., I) \equiv \omega$ and

$\phi_{i_N} \equiv \tilde{q}_{i_N}$

where we used the abbreviated notation

(3.9) $\tilde{q}_{i_N} \equiv \tilde{q}_{i_1 \ldots i_N}$

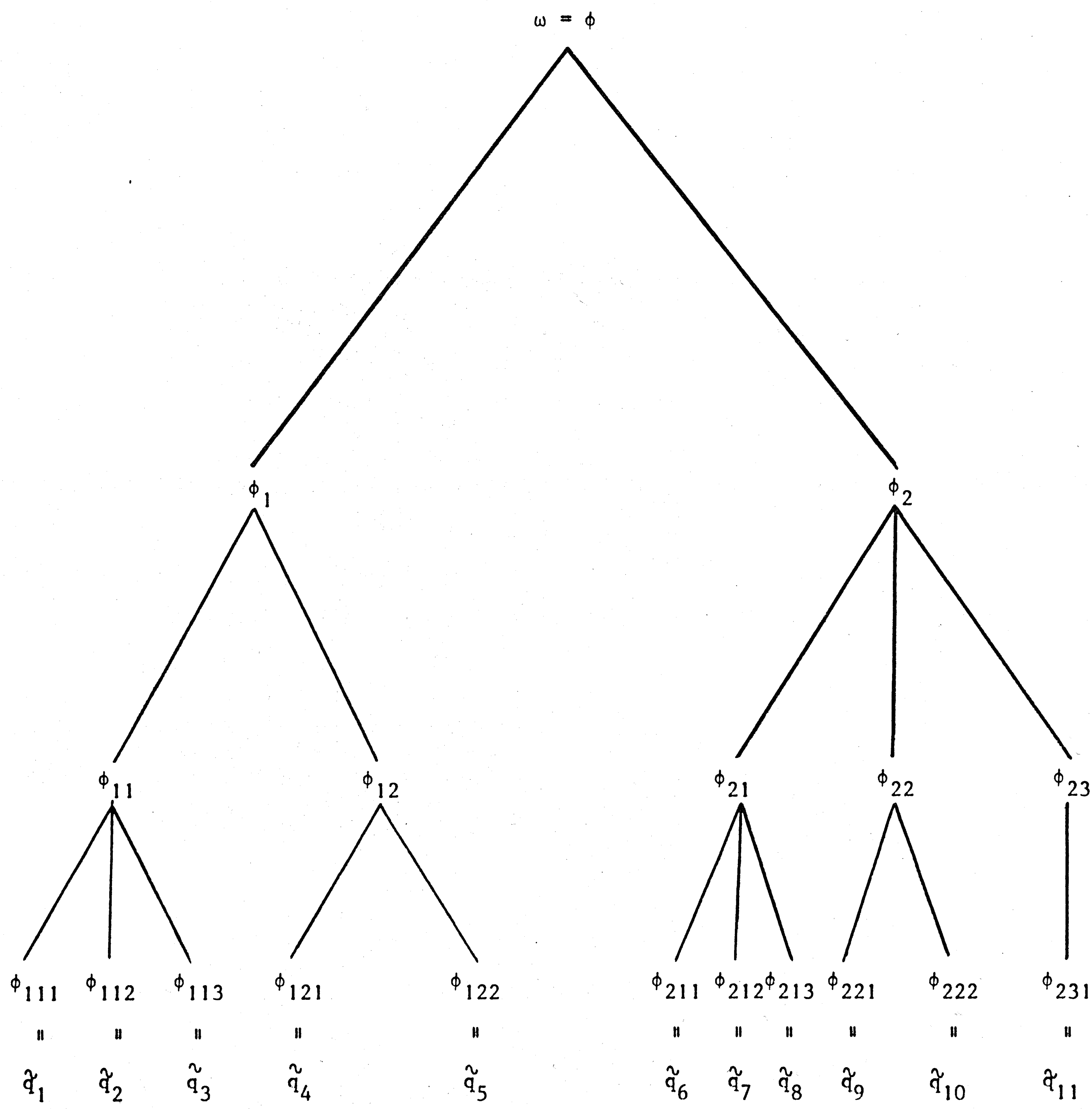


fig. 1

## 4. The implicit solution for the nested utility function

First we shall redefine the utility function in terms of the excess quantities $\tilde{q}_j$ (see (3.5)) :

$$(4.1) \qquad \omega = \omega(\tilde{q}_j; \; j = 1, \ldots, J)$$

Following the lines of (2.1) through (2.8) we find the conditions

$$(4.2) \qquad \frac{\partial \omega}{\partial \tilde{q}_j} = \frac{\tilde{W} p_j}{\tilde{y}} \qquad j = 1, \ldots, J$$

where

$$(4.3) \qquad \tilde{W} = \sum_j \frac{\partial \omega}{\partial \tilde{q}_j} \tilde{q}_j \qquad \text{and}$$

$$(4.4) \qquad \tilde{y} = y - \sum_j p_j \bar{q}_j$$

We assume that $\tilde{y}$ is positive ; i.e. that y exceeds minimum consumption :

$$(4.5) \qquad y > \sum_j p \bar{q}_j$$

We now state [4]

ASSUMPTION 1. The functions $\phi_{i_{n-1}}(\;)$, for n = 1,. ., N, are linear homogeneous (homogeneous of degree one) in their arguments.

Now, by Euler's theorem, the following holds true, for all n, $i_n$ [5]

$$(4.6) \qquad \sum_{i_n} \frac{\partial \phi_{i_{n-1}}}{\partial \phi_{i_n}} \phi_{i_n} = \phi_{i_{n-1}}$$

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4) This assumption implies that $\phi(\;)$ is linear homogeneous. The resulting demand functions are, however, invariant against any monotonic transformation of $\omega(\;)$ and therefore of $\phi(\;)$. The assumption that $\phi(\;)$ is linear homogeneous is therefore superfluous. For reasons of symmetry we state this assumption and suppose that a monotonic transformation has taken place in cases where the function $\phi(\;)$ is homogeneous of degree $\mu \neq 1$.

5) Below we write 'all n, $i_n$' in order to indicate all possible values of the compounded index $(i_1 .. i_n)$

Repeated application of (4.6) yields for (4.3)

$$(4.7)\qquad \tilde{W} = \sum_j \frac{\partial\omega}{\partial\tilde{q}_j}\,\tilde{q}_j = \sum_{i_1} \frac{\partial\phi}{\partial\phi_{i_1}} \sum_{i_2} \frac{\partial\phi_{i_1}}{\partial\phi_{i_2}} \,..\, \sum_{i_N} \frac{\partial\phi_{i_{N-1}}}{\tilde{q}_{i_N}}\,\tilde{q}_{i_N} = \omega$$

The set of first order conditions can now be rewritten;

$$(4.8)\qquad \frac{\partial\phi}{\partial\phi_{i_1}}\,\frac{\partial\phi_{i_1}}{\partial\phi_{i_2}}\,..\,\frac{\partial\phi_{i_{N-1}}}{\partial\tilde{q}_{i_N}} = \frac{\omega\, p_j}{\tilde{y}} = \frac{\omega\, p_{i_N}}{\tilde{y}}$$

where we introduce $p_{i_1..i_N}$ ( in short notation $p_{i_N}$ ) equivalent to $p_j$, according to the correspondence between $\tilde{q}_{i_N}$ and $\tilde{q}_j$. Multiplying (4.8) by $\tilde{q}_{i_N}$ and summing up over all the (n-1)th level higher indices (i.e. over $i_n$, $i_{n+1}$ and so on until $i_N$), using (4.6) we arrive at the following set of equations:

$$(4.9)\qquad \frac{\partial\phi}{\partial\phi_{i_1}} = \frac{\omega}{\tilde{y}}\, p_{i_1} \qquad\qquad \text{all } i_1$$

$$(4.10)\qquad \frac{\partial\phi_{i_{n-1}}}{\partial\phi_{i_n}} = \frac{p_{i_n}}{p_{i_{n-1}}} \qquad\qquad \text{all } n = 2,..,N;\ i_n$$

where the price-indices of the (n-1)th level utility component are defined as

$$(4.11)\qquad p_{i_{n-1}} = \frac{1}{\phi_{i_{n-1}}} \sum_{i_n} p_{i_n}\phi_{i_n}$$

or alternatively

$$(4.12)\qquad p_{i_{n-1}} = \frac{1}{\phi_{i_{N-1}}} \sum_{i_n} \,..\, \sum_{i_n} p_{i_N}\,\tilde{q}_{i_N} \qquad\qquad \text{all } n = 2,..,N;\ i_n$$

According to (4.12) the price-indices of the utility components are weighted averages of the 'elementary' prices $p_{i_N}$, with the ratio of the corresponding utility element and the utility component as weights. In (4.11) we have defined the price-indices recursively.

## 5. The nested CES-type utility function

We now specify the utility function further by adopting additionally

ASSUMPTION 2. Each function $\phi_{i_{n-1}}(\ )$, for all $n$, $i_n$, has a constant elasticity of substitution [6] between all its arguments $\phi_{i_n}$.

Assumptions 1 and 2 imply utility components of the CES-type [7]

$$(5.1)\qquad \phi_{i_{n-1}} = A_{i_{n-1}} \left[ \sum_{i_n} \alpha_{i_n} \phi_{i_n}^{-\rho_{i_{n-1}}} \right]^{-1/\rho_{i_{n-1}}} \qquad \text{for all } n,\ i_n$$

where we used the shorthand notation

$$(5.2)\qquad A_{i_{n-1}} \equiv A_{i_1 \ldots i_{n-1}}\ ;\ \alpha_{i_n} \equiv \alpha_{i_1 \ldots i_n}\ ;$$

$$(5.3)\qquad \rho_{i_{n-1}} \equiv \rho_{i_1 \ldots i_{n-1}}$$

and define

$$(5.4)\qquad A_{i_0} = A \quad \text{and} \quad \rho_{i_0} = \rho$$

The elasticity of substitution parameter $\sigma_{i_{n-1}}$ is related to $\rho_{i_{n-1}}$ by

$$(5.5)\qquad \sigma_{i_{n-1}} = \frac{1}{1+\rho_{i_{n-1}}} \qquad \text{all } n,\ i_n$$

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6) We use the concept of the direct partial elasticity of substitution of a function $f(x_1,..x_M)$, which is defined as

$$\sigma_{m_1 m_2} = \frac{\partial \log(x_{m_1}/x_{m_2})}{\partial \log\left(-\dfrac{\partial x_{m_1}}{\partial x_{m_2}}\right)}$$

where the function value $f(\ )$ and all arguments except $x_{m_1}$ and $x_{m_2}$ are held constant.

7) See, e.g., ALLEN (1967) p.52.

Under the conditions of positive derivatives and convex-to-the-origin isoquants of the function (5.1) we have [8)]

(5.6) $A_{i_{n-1}}, \alpha_{i_n} > 0$ — all $n$, $i_n$

(5.7) $\rho_{i_{n-1}} > -1$ or $\sigma_{i_{n-1}} > 0$

Without loss of generality we assume [9)]

(5.8) $A_{i_{n-1}} = 1$ and $\sum_{i_n} \alpha_{i_n} = 1$ for all $n$, $i_n$

The CES-function (5.1) encompasses utility functions of the linear-, the Cobb-Douglas and the Leontief-type as limiting cases [10)]; in a simplified notation; if

(5.9) $$\phi = \left(\sum_i \alpha_i \phi_i^{-\rho}\right)^{-1/\rho}$$

then

(5.10) $$\lim_{\sigma \to \infty} \phi = \sum_i \alpha_i \phi_i \qquad \text{(linear)}$$

(5.11) $$\lim_{\sigma \to 1} \phi = \prod_i \phi_i^{\alpha_i} \qquad \text{(Cobb-Douglas)}$$

(5.12) $$\lim_{\sigma \to 0} \phi = \min(\phi_i;\ \text{all } i) \qquad \text{(Leontief)}$$

The latter relation shows that for $\sigma$ converging to zero the CES-type utility function passes into a Leontief type function with all the so-called 'input-output'

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8) Conditions (5.7) ensure a unique maximum of a one-level CES-type utility function, subject to the budget constraint. The derivation of the demand equations presented in this paper resembles a step-by-step maximalization procedure (see GREEN(1964) p.25, for the conditions for a two-stage maximalization). Consequently conditions (5.7) are sufficient to garantee a unique maximum of our nested utility function, subject to the budget constraint.

9) In cases in which (5.8) do not hold, we can always transform the utility function in such a manner that the transformed one satisfies our assumptions and conditions (5.8).

10) See, for example, HENDERSON and QUANDT (1971) p.85.

coëfficients equal to one. In order to avoid this unrealistic consequence, we introduce the Modified CES (MCES) function by redefining the distribution parameter $\alpha_i$ as a function of $\rho$ ; in simplified notation the function can be written as

$$(5.13) \qquad \phi = \left[\sum_i \alpha_i^2 \left(\frac{\phi_i}{\alpha_i}\right)^{-\rho}\right]^{-1/\rho}$$

now

$$(5.14) \qquad \lim_{\sigma \to \infty} \phi = \Sigma\, \alpha_i \phi_i$$

$$(5.15) \qquad \lim_{\sigma \to 1} \phi = \prod_i \left(\frac{\phi_i}{\alpha_i}\right)^{\alpha_i^2}$$

$$(5.16) \qquad \lim_{\sigma \to 0} \phi = \min \left(\frac{\phi_i}{\alpha_i} \;;\ \text{all } i\right)$$

Thus, the parameters $\alpha_i$ remain effective even in the limit if $\sigma \to 0$,[11)] and also if $\sigma \to \infty$. As we shall see, an other advantage of this formulation is that the price-index functions are similar to the corresponding utility component functions except for the elasticity of substitution parameter.

As an illustration we have graphed isoquants (or indifference curves) corresponding to the utility functions (5.14) through (5.16) in fig. 2, 3 and 4, for a 2-dimensional case .

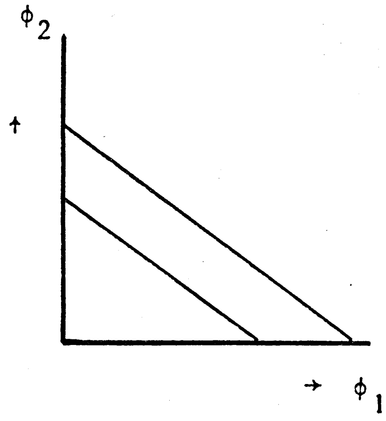


fig. 2

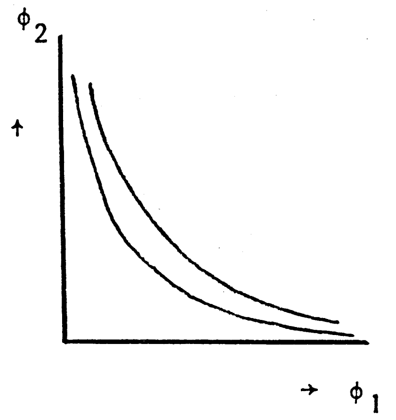


fig. 3

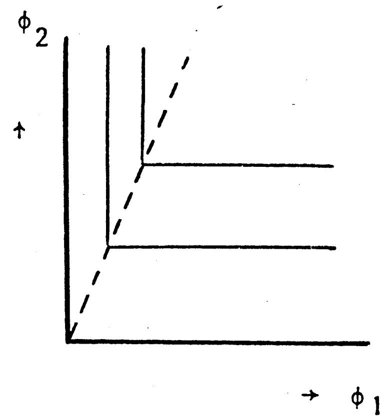


fig. 4

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11) This reformulation could be important for parameter estimation ; likelihood functions based on the CES type (5.9) become independent of the $\alpha_i$'s if $\sigma$ tends to zero. This explains the statistically insignificant estimates in a paper by BROWN et al. (1972). They estimated the parameters of a 2-level CES-type utility function and found insignificant $\alpha_i$'s in case of near-zero values of the elasticity of substitution parameter $\sigma$.

The utility components can now be formulated as

$$(5.17) \qquad \phi_{i_{n-1}} = \left[ \sum_{i_n} \alpha_{i_n}^2 \left(\frac{\phi_{i_n}}{\alpha_{i_n}}\right)^{-\rho_{i_{n-1}}} \right]^{-1/\rho_{i_{n-1}}}$$

where [11a)]

$$(5.18) \qquad \alpha_{i_n} > 0 \; ; \; \sum_{i_n} \alpha_{i_n}^2 = 1; \; \rho_{i_{n-1}} > -1, \qquad \text{for all } n, i_n$$

## 6. The solution for a nested MCES-utility function

We now try to find explicit solutions to the allocation problem, in the case of a nested utility function, where the utility components $\phi_{i_{n-1}}$ for $n = 1,..,N$ have the MCES form as defined in (5.17). In this case we shall call the utility function (2.1) a Nested Modified CES-type utility function, or in short, a NEMCES utility function.

For the solution of the system of equations (4.9) through (4.11) we use the derivative of the MCES function :

$$(6.1) \qquad \frac{\partial \phi_{i_{n-1}}}{\partial \phi_{i_n}} = \alpha_{i_n}^{2+\rho_{i_{n-1}}} \left(\frac{\phi_{i_{n-1}}}{\phi_{i_n}}\right)^{1+\rho_{i_{n-1}}} \qquad \text{all } n, i_n$$

Conditions (4.9) and (4.10) become

$$(6.2) \qquad \alpha_{i_1}^{2+\rho} \left(\frac{\phi}{\phi_{i_1}}\right)^{1+\rho} = \frac{\omega p_{i_1}}{\tilde{y}} \qquad \text{all } i_1$$

$$(6.3) \qquad \alpha_{i_n}^{2+\rho_{i_{n-1}}} \left(\frac{\phi_{i_{n-1}}}{\phi_{i_n}}\right)^{1+\rho_{i_{n-1}}} = \left(\frac{p_{i_n}}{p_{i_{n-1}}}\right) \qquad \text{all } n = 2,..,N; \; i_n$$

The equation (6.3) enables us to find solutions for the price-indices. Substituting (6.3) into (4.11) yields

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11a) After finishing this report, I found that $\sum_{i_n} \alpha_{i_n}^{1+\sigma_{i_{n-1}}} = 1$ would be a more appropriate normalization. Then the utility entropy (see (6.8)) is zero, except for the Cobb Douglas case. The other results remain unchanged.

$$(6.4)\qquad p_{i_{n-1}} = \left[\sum_{i_n} \alpha_{i_n}^2 \left(\frac{p_{i_n}}{\alpha_{i_n}}\right)^{-\rho'_{i_{n-1}}}\right]^{-1/\rho'_{i_{n-1}}} \quad \text{all } n = 2,..,N;\ i_n$$

where

$$(6.5)\qquad \rho'_{i_{n-1}} = -\frac{\rho_{i_{n-1}}}{1+\rho_{i_{n-1}}}, \text{ hence } \sigma'_{i_{n-1}} = \frac{1}{1+\rho'_{i_{n-1}}} = \frac{1}{\sigma_{i_{n-1}}}$$

With the knowledge of the 'elementary' prices $p_{i_N}$ we can find the price-indices at all stages of aggregation.

Equations (6.4) show that the (n-1)th level price-indices can be expressed as MCES functions of the n-th level price-indices, with parameters equal to the (n-1)th level utility component function, except for the elasticity of substitution parameter, which is the reciprocal of the one belonging to the corresponding utility component function.

We shall illustrate the properties of the price-index by means of a one-level model; in simplified notation (see (5.13)) :

$$(6.6)\qquad \phi = \left[\sum_i \alpha_i^2 \left(\frac{\phi_i}{\alpha_i}\right)^{-\rho}\right]^{-1/\rho}$$

$$(6.7)\qquad p = \left[\sum_i \alpha_i^2 \left(\frac{p_i}{\alpha_i}\right)^{-\rho'}\right]^{-1/\rho'} \quad \text{with } \rho' = \frac{-\rho}{1+\rho}$$

If all prices are equal ($p_i = \bar{p}$, say) the price-indices differ among the individuals because of different tastes. Now we introduce h; for $p_i = \bar{p}$ :

$$p = \left[\sum_i \alpha_i^2 \left(\frac{\bar{p}}{\alpha_i}\right)^{-\rho'}\right]^{-1/\rho'} = \bar{p}\ e^{h/2} \quad \text{where we define}$$

$$(6.8)\qquad h = \log\left[\sum_i \alpha_i^2 \left(\frac{1}{\alpha_i}\right)^{-\rho'}\right]^{-2/\rho'}$$

as the <u>utility entropy</u>. The parameter h measures the entropy of $(\alpha_i;\ i=1,..,I)$ : if all $\alpha_i$ are zero except one which is equal to 1, the entropy h has the minimum value zero in the limit; if all the $\alpha_i$ are equal (namely $1/\sqrt{I}$) the entropy h equals log I, which can be shown to be the maximum value, given the constraint $\Sigma\alpha_i^2 = 1$.

As a further illustration we consider the three particular cases mentioned in (5.14) through (5.16).

a. $\sigma = \infty$ ; $\sigma' = 0$

The utility function, price-index function and the utility entropy become (in the limit)

$$\phi = \sum_i \alpha_i \phi_i$$

$$p = \min \left(\frac{p_i}{\alpha_i} ;\text{all } i\right)$$

$$h = \log \{\min \left( \frac{1}{\alpha_i^2} ; \text{ all } i\right)\}$$

b. $\sigma = 1$ ; $\sigma' = 1$

Now

$$\phi = \prod_i \left( \frac{\phi_i}{\alpha_i}\right)^{\alpha_i^2}$$

$$p = \prod_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i^2} = e^{h/2} \prod_i p_i^{\alpha_i^2}$$

$$h = \sum_i \alpha_i^2 \log ( 1/\alpha_i^2)$$

In this case we recognize in h the formulation by THEIL (1967) for his concept of entropy. Our definition (6.8) appears to be a generalization.

c. $\sigma = 0$ ; $\sigma' = \infty$

Now

$$\phi = \min \left(\frac{\phi_i}{\alpha_i} ; \text{ all } i\right)$$

$$p = \sum_i \alpha_i p_i$$

$$h = \log (\Sigma\alpha_i)^2$$

Here we find the well-known linear price-index which is similar to the one often used in practice. Notice that this price-index is only 'consistent' with L-shaped isoquants of the utility function (the Leontief case); this implies fixed ratio's of the optimum quantities.

We shall now try to solve for the utility components. Solving $\phi_{i_1}$ from (6.2) using $\omega = \phi$, and substitution of this result in (5.17) with n = 1 yields the indirect utility function [12]

$$(6.9) \qquad \phi = \omega = \frac{\tilde{y}}{p} \quad \text{where} \quad p = p_{i_0}$$

and $p_{i_0}$ is the overall price index, equivalent to (6.4) for n = 1.
Equations (6.3) imply recursive relations for the utility components :

$$(6.10) \qquad \phi_{i_n} = \left[ \frac{\alpha_{i_n}^{2+\rho_{i_{n-1}}} \; p_{i_{n-1}}}{p_{i_n}} \right]^{\sigma_{i_{n-1}}} \phi_{i_{n-1}} \qquad \text{for } n = 1,..,N$$

including $\phi_{i_1}$ as a function of $\omega$. By means of (6.10) and (6.9), together with the recursive definition of the price indices in (6.4) and the knowledge of the elementary prices $p_{i_N}$, we are able to solve for the optimal utility elements $\tilde{q}_{i_N}$. Before doing so, we derive from (6.10) that

$$(6.11) \qquad p_{i_{n-1}} \phi_{i_{n-1}} = \sum_{i_n} p_{i_n} \phi_{i_n} \qquad \text{for all } n, i_n$$

from which we conclude that the utility components $\phi_{i_n}$ serve as consistent quantity indices [13]. In empirical studies this enables us to estimate the parameters of the utility function step by step: first estimate the parameters of the $(N-1)^{th}$ level utility components, given the prices $p_{i_N}$, the quantities $q_{i_N}$ and the budget $\sum_{i_N} p_{i_N} q_{i_N}$. From these results we construct price and quantity indices for the (N-1)th utility components and repeat the procedure, untill all the parameters are estimated.

12) Notice that the indirect utility function is of course not invariant against monotonic transformations of the utility function. The same holds for the zero-th level price- and quantity-indices (see below).

13) See for the concept of consistent aggregates and step-by-step maximalization : GREEN (1964), Chapter 3.

The optimal solution can be formulated elegantly by introducing (see (6.10))

$$(6.12)\qquad V_{i_1..i_n}^{i_1..i_{n-1}} \equiv V_{i_n}^{i_{n-1}} = \frac{\phi_{i_n}}{\phi_{i_{n-1}}} = \alpha_{i_n}^{\sigma_{i_{n-1}}+1}\left(\frac{p_{i_{n-1}}}{p_{i_n}}\right)^{\sigma_{i_{n-1}}} \quad \text{all } n, i_n$$

the quantity shares, and

$$W_{i_1..i_n}^{i_1..i_{n-1}} \equiv W_{i_n}^{i_{n-1}} = \frac{p_{i_n}\phi_{i_n}}{p_{i_{n-1}}\phi_{i_{n-1}}} = \alpha_{i_n}^{\sigma_{i_{n-1}}+1}\left(\frac{p_{i_{n-1}}}{p_{i_n}}\right)^{\sigma_{i_{n-1}}-1} \quad \text{all } n, i_n$$

the value shares of the component at level n in terms of those at level n-1. Now

$$(6.14)\qquad \tilde{q}_{i_N} = \frac{\tilde{y}}{p}\prod_{n=1}^{N} V_{i_n}^{i_{n-1}}$$

or alternatively

$$(6.15)\qquad p_{i_N}\tilde{q}_{i_N} = \tilde{y}\prod_{n=1}^{N} W_{i_n}^{i_{n-1}}$$

This formula shows that the excess income $\tilde{y}$ is allocated to the different utility component-budgets by fractions of fractions, of fractions etc.

In the exponent of $\alpha_{i_n}$ in $V_{i_n}^{i_{n-1}}$ and $W_{i_n}^{i_{n-1}}$ we find the results of the modifications of the CES functions; in the case of the traditional CES function the exponent would have been $\sigma_{i_{n-1}}$ instead of $\sigma_{i_{n-1}}+1$. Therefore, if $\sigma \to 0$, the MCES function enables us to allocate the budgets in not necessarily equal quantities, by choosing a variety of values for $\alpha_{i_n}$.

We now summarize our results in the following theorem, using the reduced notation (i.e. the index $i_n$ stands for $i_1..i_n$) :

THEOREM 1

If the utility function $\omega(q_j\ ;\ j = 1,.., J)$ is a NEMCES function, i.e. for all $n = 1,.., N$ and $i_n$

$$\phi_{i_{n-1}} = \left[ \sum_{i_n=1}^{I_{i_{n-1}}} \alpha_{i_n}^2 \left( \frac{\phi_{i_n}}{\alpha_{i_n}} \right)^{-\rho_{i_{n-1}}} \right]^{-1/\rho_{i_{n-1}}}$$

with

$$\phi_{i_0} \equiv \omega$$

$$\phi_{i_N} \equiv \tilde{q}_{i_N} \quad \text{where} \quad \tilde{q}_{i_N} = q_{i_N} - \bar{q}_{i_N} > 0$$

$$q_{i_N} \equiv q_j \;;$$

utility is maximized subject to the budget constraint

$$y = \sum_j p_j q_j$$

where $p_j$ represents prices and y the budget,

by

$$q_{i_N} = \bar{q}_{i_N} + \frac{\tilde{y}}{p} \left( \prod_{n=1}^{N} V_{i_n}^{i_{n-1}} \right) \qquad \text{or alternatively}$$

$$p_{i_N} q_{i_N} = p_{i_N} \bar{q}_{i_N} + \tilde{y} \left( \prod_{n=1}^{N} W_{i_n}^{i_{n-1}} \right)$$

where the price-index of the (n-1)th level utility component is defined by

$$p_{i_{n-1}} = \left[ \sum_{i_n} \alpha_{i_n}^2 \left( \frac{p_{i_n}}{\alpha_{i_n}} \right)^{-\rho'_{i_{n-1}}} \right]^{-1/\rho'_{i_{n-1}}} \qquad \text{with}$$

$p_{i_0} \equiv p$ the overall price-index,

$p_{i_N} \equiv p_j$ the elementary prices, and

$$\rho'_{i_n} = - \frac{\rho_{i_n}}{1+\rho_{i_n}}$$

implying a price-index function similar to the utility component function, except for a reciprocal elasticity of substitution :

$$\sigma'_{i_n} = \frac{1}{\sigma_{i_n}} \qquad \underline{\text{where}} \quad \sigma_{i_n} = \frac{1}{1+\rho_{i_n}} \quad \underline{\text{and}} \quad \sigma'_{i_n} = \frac{1}{1+\rho'_{i_n}} \quad ;$$

<u>the excess income is</u>

$$\tilde{y} = y - \sum_{i_1} .. \sum_{i_N} p_{i_N} \bar{q}_{i_N}$$

<u>the quantity shares are</u>

$$V_{i_n}^{i_{n-1}} = \frac{\phi_{i_n}}{\phi_{i_{n-1}}} = \alpha_{i_n}^{\sigma_{i_{n-1}}+1} \left( \frac{p_{i_{n-1}}}{p_{i_n}} \right)^{\sigma_{i_{n-1}}}$$

<u>and the value shares</u>

$$W_{i_n}^{i_{n-1}} = \frac{p_{i_n} \phi_{i_n}}{p_{i_{n-1}} \phi_{i_{n-1}}} = \alpha_{i_n}^{\sigma_{i_{n-1}}+1} \left( \frac{p_{i_{n-1}}}{p_{i_n}} \right)^{\sigma_{i_{n-1}}-1} .$$

In the appendix we derive some elasticities for the NEMCES-case. The results are summarized in the following theorem (in reduced notation, i.e. the index $i_n$ stands for $i_1..i_n$).

<u>THEOREM 2</u>

<u>For the NEMCES utility function as described in theorem 1, the price and income elasticities and the elasticities of substitution,</u>
<u>expressed in terms of the excess quantities $\tilde{q}_j$ and the excess budget $\tilde{y}$, are</u> :

<u>a</u>.

$$\frac{\partial \log \tilde{q}_{i_N}}{\partial \log p_{j_N}} = - \bar{W}_{j_N} - W_{j_N} + \sum_{n=1}^{N} \sigma_{i_{n-1}} (W_{j_N}^{i_{n-1}} - W_{j_N}^{i_n})$$

b.

$$\frac{\partial \log \tilde{q}_{i_N}}{\partial \log \tilde{y}} = 1$$

c.

$$\sigma_{i_N j_N} = \frac{\partial \log (\tilde{q}_{i_N}/\tilde{q}_{j_N})}{\partial \log (-\partial \tilde{q}_{i_N}/\partial \tilde{q}_{j_N})} = \left[ \sum_{n=1}^{N} \left( A_{n-1}\, \sigma_{i_{n-1}}^{-1} + B_{n-1}\, \sigma_{j_{n-1}}^{-1} \right) \right]^{-1} \qquad (i \neq j)$$

where we define the following value shares, for $m \geq n$ :

$$W_{i_m}^{i_n} = \prod_{k=n+1}^{m} W_{i_k}^{i_{k-1}} = \frac{p_{i_m}\, \phi_{i_m}}{p_{i_n}\, \phi_{i_n}}$$

$$W_{j_m}^{i_n} = W_{j_m}^{j_n} \quad \text{if} \quad (j_1 .. j_n) = (i_1 .. i_n)$$

$$= 0 \quad \text{otherwise}$$

and

$$W_{i_N} \equiv W_{i_N}^{i_o} = \frac{p_{i_N}\, \tilde{q}_{i_N}}{\tilde{y}} \; ; \quad \bar{W}_{i_N} = \frac{p_{i_N}\, \bar{q}_{i_N}}{\tilde{y}} \; ;$$

and define

$$A_{n-1} = \frac{\left( \dfrac{W_{i_N}^{i_n} - W_{i_N}^{i_{n-1}}}{W_{i_N}} + \dfrac{W_{j_N}^{i_{n-1}} - W_{j_N}^{i_n}}{W_{j_N}} \right)}{\left( \dfrac{1}{W_{i_N}} + \dfrac{1}{W_{j_N}} \right)}$$

$$B_{n-1} = \frac{\left( \dfrac{W_{j_N}^{j_n} - W_{j_N}^{j_{n-1}}}{W_{j_N}} + \dfrac{W_{i_N}^{j_{n-1}} - W_{i_N}^{j_n}}{W_{i_N}} \right)}{\left( \dfrac{1}{W_{i_N}} + \dfrac{1}{W_{j_N}} \right)} .$$

In the price elasticities we can recognize three terms. The first two terms represent the so-called income effect; the last term the substitution effect. This part of the price elasticity is called the income compensated or Slutsky price elasticity. From

(6.16) $\quad W_{i_N}^{i_{n-1}} < W_{i_N}^{i_n}$

we conclude that the compensated own price elasticity is negative, as well as the uncompensated own-price elasticity.

The compensated cross-price elasticities are positive if

(6.17) $\quad i_1 \neq j_1$

In this case the relevant utility components are functions of either $\tilde{q}_{i_N}$ or $\tilde{q}_{j_N}$, except for the zero-level component $\phi(\ )$ which is a function of both. We say that the zero level component is a 'common' component of $\tilde{q}_{i_N}$ and $\tilde{q}_{j_N}$. If $i_1 = j_1$, the compensated cross-price elasticities could be positive as well as negative, allowing complementarity between elements.

From the expression for the elasticity of substitution we see that $\sigma_{i_N j_N}$ is a CES-type function (with $\rho = 1$) in its arguments $\sigma_{i_{n-1}}$ an $\sigma_{j_{n-1}}$, for n = 1,.., N.

We arrive at less complicated expressions for $A_{n-1}$ and $B_{n-1}$ by introducing the index M. M is such that

(6.18) $\quad i_1 .. i_{M-1} = j_1 .. j_{M-1}$

$\qquad\qquad i_M \neq j_M$

This implies that only the components at levels lower than M (i.e. M-1, M-2,.., 0) are 'common' components of $\tilde{q}_{i_N}$ and $\tilde{q}_{j_N}$. We find then

(6.19) 1. $A_{n-1},\ B_{n-1} = 0$ for $n = 1,.., M-1$

2. $A_{M-1} = \frac{1}{W_{i_M}} \left(\frac{1}{W_{i_N}} + \frac{1}{W_{j_N}}\right)^{-1}$ and $B_{M-1} = \frac{1}{W_{j_M}} \left(\frac{1}{W_{i_N}} + \frac{1}{W_{j_N}}\right)^{-1}$

and $\sigma_{i_{M-1}} = \sigma_{j_{M-1}}$

3. $A_{n-1} = \left(\frac{1}{W_{i_n}} - \frac{1}{W_{i_{n-1}}}\right) \left(\frac{1}{W_{i_N}} + \frac{1}{W_{j_N}}\right)^{-1}$

if $n = M+1,.., N$

$B_{n-1} = \left(\frac{1}{W_{j_n}} - \frac{1}{W_{j_{n-1}}}\right) \left(\frac{1}{W_{i_N}} + \frac{1}{W_{j_N}}\right)^{-1}$

A special case occurs if $M = N$ ; the (N-1)th level component is then a common component :

(6.20) $i_1 .. i_{N-1} = j_1 .. j_{N-1}$

then we find

(6.21) $\sigma_{i_N j_N} = \sigma_{i_{N-1}} = \sigma_{j_{N-1}}$ .

APPENDIX

By virtue of (6.1), (6.10) and (6.4), for m ≥ n, we find

$$\frac{\partial \phi_{i_n}}{\partial \phi_{j_m}} \cdot \frac{\phi_{j_m}}{\phi_{i_n}} = \frac{\partial p_{i_n}}{\partial p_{j_m}} \cdot \frac{p_{j_m}}{p_{i_n}} = W_{j_m}^{i_n}$$

where $W_{j_m}^{i_n}$ is defined as in theorem 2. From (6.14)

$$\frac{\partial \log \tilde{q}_{i_N}}{\partial \log p_{j_N}} = \frac{\partial \log \tilde{y}}{\partial \log p_{j_N}} - \frac{\partial \log p}{\partial \log p_{j_N}} + \sum_{n=1}^{N} \frac{\partial \log V_{i_n}^{i_{n-1}}}{\partial \log p_{j_N}}$$

We derive

$$\frac{\partial \log \tilde{y}}{\partial \log p_{j_N}} = - \frac{p_{j_N} \bar{q}_{j_N}}{\tilde{y}} = - \bar{W}_{j_N}$$

$$\frac{\partial \log p}{\partial \log p_{j_N}} = W_{j_N}$$

$$\frac{\partial \log V_{i_n}^{i_{n-1}}}{\partial \log p_{j_N}} = \sigma_{i_{n-1}} \left( \frac{\partial \log p_{i_{n-1}}}{\partial \log p_{j_N}} - \frac{\partial \log p_{i_n}}{\partial \log p_{j_N}} \right)$$

$$= \sigma_{i_{n-1}} \left( W_{j_N}^{i_{n-1}} - W_{j_N}^{i_n} \right) ,$$

Consequently

$$\frac{\partial \log \tilde{q}_{i_N}}{\partial \log p_{j_N}} = - \bar{W}_{j_N} - W_{j_N} + \sum_{n=1}^{N} \sigma_{i_{n-1}} \left( W_{j_N}^{i_{n-1}} - W_{j_N}^{i_n} \right)$$

For the income elasticity we find from (6.14)

$$\frac{\partial \log \tilde{q}_{i_N}}{\partial \log \tilde{y}} = 1$$

The direct partial elasticity of substitution between $\tilde{q}_{i_N}$ and $\tilde{q}_{j_N}$ is defined by

$$\sigma_{i_N j_N} = \frac{\partial \log (\tilde{q}_{i_N} / \tilde{q}_{j_N})}{\partial \log (- \partial \tilde{q}_{i_N} / \partial \tilde{q}_{j_N})} \qquad (i_1 .. i_N) \neq (j_1 .. j_N)$$

where $\omega$ and all utility elements other than $\tilde{q}_{i_N}$ and $\tilde{q}_{j_N}$ are held constant. Working out $\sigma_{i_N j_N}$ we find

$$\sigma_{i_N j_N} = \frac{\dfrac{1}{\tilde{q}_{i_N} \ \partial\omega/\partial\tilde{q}_{i_N}} + \dfrac{1}{\tilde{q}_{j_N} \ \partial\omega/\partial\tilde{q}_{j_N}}}{- \dfrac{\partial^2\omega/\partial\tilde{q}^2_{i_N}}{(\partial\omega/\partial\tilde{q}_{i_N})^2} - \dfrac{\partial^2\omega/\partial\tilde{q}^2_{j_N}}{(\partial\omega/\partial\tilde{q}_{j_N})^2} + 2 \ \dfrac{\partial^2\omega/\partial\tilde{q}_{i_N} \partial\tilde{q}_{j_N}}{(\partial\omega/\partial\tilde{q}_{i_N})(\partial\omega/\partial\tilde{q}_{j_N})}}$$

We find

$$\frac{\partial\omega}{\partial\tilde{q}_{i_N}} = \frac{\omega}{\tilde{q}_{i_N}} \cdot W_{i_N} \quad \text{and} \quad \frac{\partial\omega}{\partial\tilde{q}_{j_N}} = \frac{\omega}{\tilde{q}_{j_N}} \cdot W_{j_N}$$

and (also if $i_1 .. i_N = j_1 .. j_N$) :

$$\frac{\partial^2\omega}{\partial\tilde{q}_{i_N} \partial\tilde{q}_{j_N}} = \frac{\partial}{\partial\tilde{q}_{i_N}} \left( \frac{\partial\omega}{\partial\tilde{q}_{j_N}} \right) = \frac{\partial\omega}{\partial\tilde{q}_{j_N}} \cdot \frac{\partial \log (\partial\omega/\partial\tilde{q}_{j_N})}{\partial\tilde{q}_{i_N}}$$

$$= \frac{\omega}{\tilde{q}_{j_N}} \cdot W_{j_N} \sum_{n=1}^{N} \frac{\partial}{\partial \tilde{q}_{i_N}} \log \frac{\partial \phi_{j_{n-1}}}{\partial \phi_{j_n}}$$

$$= \frac{\omega}{\tilde{q}_{j_N}} W_{j_N} \sum_{n=1}^{N} \frac{\partial}{\partial \tilde{q}_{i_N}} \log \left[ \alpha_{j_n}^{2+\rho_{j_{n-1}}} \left( \frac{\phi_{j_{n-1}}}{\phi_{j_n}} \right)^{1+\rho_{j_{n-1}}} \right]$$

$$= \frac{\omega}{\tilde{q}_{j_N}} W_{j_N} \sum_{n=1}^{N} \frac{1}{\sigma_{j_{n-1}}} \left[ W_{i_N}^{j_{n-1}} - W_{i_N}^{j_n} \right]$$

$$= \frac{\omega}{\tilde{q}_{i_N}} W_{i_N} \sum_{n=1}^{N} \frac{1}{\sigma_{i_{n-1}}} \left[ W_{j_N}^{i_{n-1}} - W_{j_N}^{i_n} \right]$$

The last equality follows from the symmetry with respect to $\tilde{q}_{i_N}$ and $\tilde{q}_{j_N}$. Substitution of the results in $\sigma_{i_N j_N}$ yields

$$\sigma_{i_N j_N} = \left[ \sum_{n=1}^{N} \left( A_{n-1} \sigma_{i_{n-1}}^{-1} + B_{n-1} \sigma_{j_{n-1}}^{-1} \right) \right]^{-1}$$

**where**

$$A_{n-1} = \frac{\left( \frac{W_{i_N}^{i_n} - W_{i_N}^{i_{n-1}}}{W_{i_N}} + \frac{W_{j_N}^{i_{n-1}} - W_{j_N}^{i_n}}{W_{j_N}} \right)}{\left( \frac{1}{W_{i_N}} + \frac{1}{W_{j_N}} \right)}$$

$$B_{n-1} = \frac{\left( \frac{W_{j_N}^{j_n} - W_{j_N}^{j_{n-1}}}{W_{j_N}} + \frac{W_{i_N}^{j_{n-1}} - W_{i_N}^{j_n}}{W_{i_N}} \right)}{\left( \frac{1}{W_{i_N}} + \frac{1}{W_{j_N}} \right)}$$

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