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Report 7201

CONSTRUCTING FORMAL GROUPS. II:

OVER  $\mathbb{Z}$  - ALGEBRAS

by Michiel Hazewinkel

February 29, 1972

Preliminary and Confidential

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1. INTRODUCTION

In [1] we wrote down some explicit power series over  $\mathbb{Q}[\dots, T_{p^i}, \dots]$  which turned out to be the logarithms of a  $p$ -typically universal formal group and a formal group universal over  $\mathbb{Z}_{(p)}$ -algebras. Both formal groups are defined over  $\mathbb{Z}[\dots, T_{p^i}, \dots]$ . In this note we show how to fit together these formal groups for different  $p$  to get a universal formal group (over  $\mathbb{Z}[\mathbb{T}_2, \mathbb{T}_3, \dots]$ ). If  $f(X)$  is the logarithm of this universal group  $f(X) = \sum a_i X^i$ , then its  $p$ -typical part,  $X + \sum_{i=1}^{\infty} \frac{a_i}{p^i} X^{p^i}$ , is precisely the logarithm of the  $p$ -typically universal formal group constructed in [1].

It turns out that there many ways of fitting together the  $p$ -typical formal groups. Most of them do not give nice formulas for the  $T_{p^i}$  in terms of the  $a_i$ . One special choice gives inverse formulas comparable to formulas (8) of [1]. In [1] we used these formulas to get generators in dimensions  $2(p^n-1)$  of  $\Omega^{ev}(pt)$ , the complex cobordism ring modulo torsion. Using the universal formal group constructed in section 4 of this note we get a complete set of free generators of  $\Omega^{ev}(pt)$  over  $\mathbb{Z}$ .

Section 2 contains some preliminaries; section three gives the general construction of a universal formal group. In section 4 we discuss a special case with nice properties of the construction of section 3. Section 5 contains the application to complex cobordism theory alluded to above. In section 6 we discuss the more dimensional case which is completely analogical. In section 7 we discuss isomorphisms.

Section 8, finally, is independant of [1] and the rest of this paper. It is elementary (given the existence of universal formal groups and some more results of Lazard) and it would surprise me if it were not already known.

## 2. PRELIMINARIES.

Let  $F(X,Y)$  be a power series over  $Z[T_2, T_3, \dots]$  or  $Z_{(p)}[T_2, T_3, \dots]$  in  $X, Y$ . We denote (cf. also [1]) with  $F^{(p^k)}(X,Y)$  the power series obtained from  $F(X,Y)$  by replacing the parameters  $T_2, T_3, \dots$  with  $T_2^{p^k}, T_3^{p^k}, \dots$

2.1. Lemma. Let  $F(X,Y) \in Z_{(p)}[T_2, \dots][[X,Y]]$ . Then we have

$$\frac{1}{n}(F(X,Y)^{p^k})^n \equiv \frac{1}{n}(F^{(p^k)}(X^{p^k}, Y^{p^k}))^n \pmod{p}$$

The proof is completely elementary. For  $n = p^l$  it is contained in the proof of theorem (1.2) of [1].

Let  $f_T(X)$  be a power series in  $X$  over  $Q[T_2, T_3, \dots]$ . The power series  $f^{(p^k)}(X)$  is obtained by raising the parameters  $T_i$  to the  $p^k$ -th power.

2.2. Theorem. Let  $f_T(X)$  be a power series in  $X$  over  $Q[T_2, T_3, \dots]$  such that

$$(2.2.1) \quad f_T(X) - \sum \frac{T_i}{p} f_T^{(p^i)}(X^{p^i}) \in Z_{(p)}[T][[X]] \text{ for all primes } p$$

Let  $F_T(X,Y) = f_T^{-1}(f_T(X) + f_T(Y))$ . Then all coefficients of  $F_T(X,Y)$  are in  $Z[T]$ .

Proof. Fix a prime  $p$  for the moment. Then we have

$$f_T(X) = g_p(X) + \sum \frac{p^i}{p} f_T^{(p^i)}(X^{p^i})$$

for some power series  $g_p(X) \in Z_{(p)}[T][[X]]$ . Now repeat the proof of theorem (1.2) of [1], using lemma (2.1) instead of formula (11) of [1], to show that

$$F(X,Y) \in Z_{(p)}[T][[X,Y]]$$

This must hold for all  $p$ , which concludes the proof. *qed.*

2.3. Remark. Let  $f(X)$  be a power series over  $Q[T]$  such that

$$(2.3.1) \quad f(X) - \sum \frac{p^i}{p} f^{(p^i)}(X^{p^i}) \in Z_{(p)}[T][[X]]$$

and let  $u(X) = X + u_2 X^2 + \dots \in Z_{(p)}[T][[X]]$ ; let  $g(X) = f(u(X))$ . Then  $g(X)$  also satisfies

$$(2.3.2) \quad g(X) - \sum \frac{p^i}{p} g^{(p^i)}(X^{p^i}) \in Z_{(p)}[T][[X]]$$

This follows immediately from lemma (3.1) of [1]. Now let  $G(X,Y)$  be a universal formal group law over  $Z[T]$  and let  $g(X)$  be its logarithm. Consider  $G$  and  $g$  over  $Z_{(p)}[T]$ . Over  $Z_{(p)}[T]$ ,  $G(X,Y)$  must be isomorphic to the universal formal group law constructed in (1.3) of [1]. Combining this with (2.3.2) we see that a universal formal group must have a logarithm which satisfies (2.2.1) for all  $p$ .

In view of (2.2) it therefore only remains to construct reasonable power series  $f_T$  which satisfy (2.2.1) for all  $p$ . This is the subject matter of the next section.

2.4. Remark. If  $f_{T,S}(X)$  is a power series over  $Q[T,S]$ , where the  $S$  are additional variables, such that (2.2.1) holds for all  $p$ , then also

$$F_{T,S}(X,Y) \in Z[T,S][[X,Y]]$$

where  $F_{T,S}(X,Y) = f_{T,S}^{-1}(f_{T,S}(X) + f_{T,S}(Y))$ . Same proof.

### 3. CONSTRUCTION OF A UNIVERSAL FORMAL GROUP.

#### 3.1. The Induction Step.

Suppose we have constructed a power series  $f_T$  up to and including degree  $s-1$  such that (2.2.1) holds for all primes  $p$  mod degree  $s$ .

Let  $p$  be a prime dividing  $s$ , and let  $q$  be a power of  $p$  which divides  $s$ . Then according to (2.2.1) there must be a term

$$(3.1.1) \quad \frac{T}{p} a_d^{(q)}, \quad d = q^{-1}s, \quad a_d \text{ de coefficient of } X^d$$

in the coefficient of  $X^s$ . The coefficient  $a_d$  looks like  $d^{-1}c_d$ ,  $c_d \in Z[T]$ , which can be written as a sum

$$(3.1.2) \quad a_d = \sum_{q'|d} \frac{c_{d,q'}}{q'}, \quad q' \text{ a power of a prime,}$$

Substituting this in (3.1.1) we find a contribution

$$(3.1.3) \quad \sum_{q'|d} \frac{T}{p} \frac{c_{d,q'}^{(q)}}{q'}$$

to  $a_s$ . We get such a contribution for every prime power  $q$  dividing  $s$ . We find therefore that  $a_n$  must contain

$$(3.1.4) \quad \sum_{q|n} \sum_{q'|d} \frac{T}{p} \frac{c_{d,q'}^{(q)}}{q'}, \quad d = q^{-1}n, \quad q, q' \text{ prime powers.}$$

If we use (3.1.4) to define  $a_s$ , (2.2.1) is in general not satisfied. This can be repaired by adding to each summand  $p^{-1}(q')^{-1} \cdot T_q \cdot c_{d,q'}^{(q)}$  a term of the form  $m(q,q') p^{-1}(q')^{-1} T_q c_{d,q'}^{(q)}$ , where  $m(q,q') \in Z$  is such that

$$(3.1.5) \quad 1 + m(q,q') \equiv \begin{cases} 1 \pmod{p} \\ 0 \pmod{q'} \end{cases} \quad \text{if } (q,q') = 1$$

$$(3.1.6) \quad 1 + m(q,q') \equiv 1 \pmod{pq'} \quad \text{if } (q,q') = p^s$$

Let  $n(q,q') = 1 + m(q,q')$ , and define

$$(3.1.7) \quad a_s = \sum_{q|n} \sum_{q'|d} \frac{T_q}{p} \frac{c_{d,q'}^{(q)}}{q'} n(q,q') + \alpha_s T_s$$

where  $\alpha_s = 1$  if  $s$  is not a power of a prime and  $\alpha_s = 0$  otherwise.

We maintain that  $f_T(X) = X + a_2 X^2 + \dots + a_s X^s$  then satisfies (2.2.1) for all  $p \bmod$ . degree  $s + 1$ .

Indeed, fixe a prime  $p_0$ , then we must show that

$$(3.1.8) \quad a_s - \sum_{p_0|q|s} \frac{T_q}{p_0} \frac{a(q)}{q^{-1}s} \in Z_{(p_0)}[T]$$

The sum (3.1.8) is equal to

$$(3.1.9) \quad \sum_{p_0|q} \sum_{p_0|q'} \frac{n(q,q')-1}{p_0 q'} T_q c_{d,q'}^{(q)} + \sum_{p_0|q, p_0|q'} \frac{n(q,q')-1}{p_0 q'} T_q c_{d,q'}^{(q)} + \sum_{p_0|q, p_0|q'} \frac{n(q,q')}{p_0 q'} T_q c_{d,q'}^{(q)} + \alpha_n T_n$$

(where  $d = q^{-1}s$ ;  $q$  a power of  $p$  in the third and fourth terms).  
 The first term of (3.1.9) is in  $Z_{(p_0)}[T]$  because of (3.1.6); the second term of (3.1.9) is in  $Z_{(p_0)}[T]$  because  $(q')^{-1} \in Z_{(p_0)}$  and (3.1.5); the third term because  $p^{-1} \in Z_{(p_0)}$  and (3.1.5) and the fourth term because  $p^{-1}, (q')^{-1} \in Z_{(p_0)}$ ; finally:  $\alpha_n T_n \in Z[T]$ .

Note that we can choose for the  $n(q,q')$  any numbers in  $Z$  which have properties (3.1.5), (3.1.6); in particular we can, if we wish, let  $n(q,q')$  depend not only on  $q, q'$  but also on  $s$  and on the way in which the term  $c_{d,q'}$  arose.

### 3.2. Ordered Factorizations.

An ordered factorization of  $s \in N$  is a sequence of numbers  $(q_1, q_2, \dots, q_t, d)$  where the  $q_i$  are powers of primes and  $d \in N$  is

not a power of a prime (but  $d = 1$  is possible). Example: the different ordered factorizations of  $s = 12$  are

$(2, 2, 3, 1), (4, 3, 1), (2, 3, 2, 1), (3, 2, 2, 1), (3, 4, 1),$   
 $(2, 6), (12).$

3.3. Lemma. If we use the procedure of (3.1) to construct  $f_T$ , then the monomials in  $T$  occurring in  $a_s$  are of the form

$$T_{q_1}^{q_1} T_{q_2}^{q_1 q_2} \dots T_{q_t}^{q_1 q_2 \dots q_{t-1}} T_d^{q_1 q_2 \dots q_t}$$

where  $T_1 = 1$  and  $(q_1, q_2, \dots, q_t, d)$  is an ordered factorization of  $s$ .

Proof. By induction; elementary.

3.4. A Formula for  $a_s$

For every ordered factorization  $(p_1^{k_1}, \dots, p_t^{k_t}, d)$  of  $s$  let  $n(p_1^{k_1}, \dots, p_t^{k_t}, d)$  be a number  $\in \mathbb{Z}$  such that

$$(3.4.1) \quad n(p_1^{k_1}, p_2^{k_2}, \dots, p_t^{k_t}, d) \equiv \begin{cases} 1 \pmod{p_1} & \text{if } p_1 \neq p_2 & (t \geq 2) \\ 0 \pmod{p_2^r} & \text{if } p_1 \neq p_2 = \dots = p_{r+1} \neq p_{r+2} & (t \geq 2) \\ 1 \pmod{p_1^r} & \text{if } p_1 = p_2 = \dots = p_r \neq p_{r+1} & (t \geq 1) \end{cases}$$

We define  $a_s$  by the formula

$$(3.4.2) \quad a_s = \sum_{(p_1^{k_1}, \dots, p_t^{k_t}, d)} \frac{n(q_1, \dots, q_t, d)}{p_1} \frac{n(q_2, \dots, q_t, d)}{p_2} \dots \frac{n(q_t, d)}{p_t} T_{q_1}^{q_1} T_{q_2}^{q_1 q_2} \dots T_{q_t}^{q_1 \dots q_{t-1}} T_d^{q_1 \dots q_t}$$

where  $q_i = p_i^{k_i}$ , and  $(p_1^{k_1}, \dots, p_t^{k_t}, d)$  runs through all ordered factorizations of  $s_i$ ; as above we set  $T_1 = 1$ .

3.5. Theorem.

Define  $f_T(X)$  as



$$(3.5.1) \quad f_T(X) = \sum_{s=1}^{\infty} a_s X^s$$

where  $a_s$  is given by formula (3.4.2). Then  $f_T(X)$  satisfies (2.2.1) for all  $p$ . Let  $F_T(X,Y) = f_T^{-1}(f_T(X) + f_T(Y))$ , then  $F_T(X,Y)$  is a universal formal group.

Proof. The product

$$\frac{n(q_1, \dots, q_t, d)}{p_1} \frac{n(q_2, \dots, q_t, d)}{p_2} \dots \frac{n(q_t, d)}{p_t}$$

where  $q_i$  is a power of  $p_i$  is of the form

$$\frac{c}{p_1^r}$$

with  $c \in \mathbb{Z}$ , if  $p_1 = p_2 = \dots = p_r \neq p_{r+1}$ . This follows immediately from (3.4.1) by induction. It follows from this and (3.3) that the  $a_s$  are related to each other in the manner discussed in (3.1). The power series  $f_T(X)$  therefore satisfies (2.2.1). Theorem (2.2) then shows that all coefficients of  $F_T(X,Y)$  are in  $\mathbb{Z}[T]$ . Finally writing  $F_{T(s)}$  for

$F_{(T_1, T_2, \dots, T_s, 0, 0, \dots)}$  we have

$$(3.5.2) \quad F_T(X,Y) = F_{T(s)}(X,Y) + \beta(s+1)T_{s+1} \pmod{\text{degree } s + 2}$$

where  $\beta(s+1) = 1$  if  $s+1$  is not a power of a prime and  $\beta(s+1) = \frac{1}{p}$  if  $s+1$  is a power of  $p$ . This follows immediately from (3.4.1). The relation (3.5.2) implies that  $F_T$  is a universal formal group, [2].

### 3.6. Examples.

The different ordered factorizations of 12 are

(2,2,3,1), (4,3,1), (3,2,2,1), (2,3,2,1), (3,4,1), (2,6), (12)

Let  $n(2, q_1, q_2, \dots, q_t, d) = n(q_1, \dots, q_t)$ ,  $t \geq 2$ ,  $n(q, d) = 1$

$n(2,2,3) = 1$ ,  $n(2,3) = 3$ ,  $n(4,3) = 3$ ,  $n(3,2,2) = 4$ ,  $n(2,2) = 1$ ,

$n(2,3,2) = 3$ ,  $n(3,2) = 4$ ,  $n(3,4) = 4$ . Then we find for  $a_{12}$

$$a_{12} = \frac{1}{4}T_2^2T_3^4 + \frac{1}{2}T_4T_3^4 + \frac{1}{3}T_3T_2^3T_6 + T_2T_3^2T_6 + \frac{2}{3}T_3T_4^3 + \frac{1}{2}T_2T_6^2 + T_{12}$$

The ordered factorizations of 6 are

$$(2,3,1), (3,2,1), (6)$$

Using the same n's we find for  $a_6$

$$a_6 = \frac{1}{2}T_2T_3^2 + \frac{2}{3}T_3T_2^3 + T_6$$

#### 4. INVERSE FORMULAE.

The formula (3.4.2) permits us of course to write  $T_s$  in terms of the  $a_d^{(q)}$ ,  $d|s$  and the  $T_{s'}$ ,  $s'|s$ ,  $s' < s$ . In analogy with formula (8) of [ ] however, we would like to find a formula for  $T_s$  in terms of the  $a_d$  and the  $T_{s'}$ , where  $d$  and  $s'$  divide  $s$ .

Note that this is not possible with the choices for the  $n(q_1, q_2, \dots, q_t, d)$  which we used in (3.6). (A redefinition of  $n(3,2,2)$  as  $n(3,2,2) = 16$  remedies this).

##### 4.1. Some Special $n(q_1, q_2, \dots, q_t, d)$

We define inductively

$$b(p_1^{k_1}, \dots, p_t^{k_t}, d) = b(p_1, \dots, p_t)$$

$$(4.1.1) \quad b(p_i) = 1, b(d) = 1$$

$$b(p_1, \dots, p_t) = \prod_{p \in J} c(p, p_t) b(p_1, \dots, p_{t-1}),$$

where  $J = \{p_1, \dots, p_t\}$  and

$$c(p, p') = 1 \quad \text{if } p = p'$$

$$(4.1.2) \quad c(p, p') \equiv 1 \pmod{p} \quad \text{if } p \neq p'$$

$$c(p, p') \equiv 0 \pmod{p'} \quad \text{if } p \neq p'$$

(One can e.g. take  $c(p,p') = (p')^{p-1}$ , if  $p' \neq p$ ).

Note that the factor  $c(p,p_t)$  occurs precisely once in  $\prod_{p \in J} c(p,p_t)$  if  $p \in J$ .

Now define  $n(q_1, \dots, q_t, d)$  by the formula

$$(4.1.3) \quad n(q_1, \dots, q_t, d) = \frac{b(q_1, \dots, q_t, d)}{b(q_2, \dots, q_t, d)}, \quad n(d) = 1$$

#### 4.2. Lemma.

The  $n(q_1, \dots, q_t, d)$  as defined by (4.1.3) satisfy the conditions

(3.4.1).

Proof. One checks directly that  $n(p_1^{t_1}, d) = 1$ ,  $n(p_1^{t_1}, p_2^{t_2}) = n(p_1, p_2) = c(p_1, p_2)$ ; further  $n(q_1, \dots, q_t, d) = n(p_1, \dots, p_t)$  if  $q_i$  is a power of  $p_i$ . By induction we get from (4.1.1) that

$$(4.2.1) \quad b(p_1, \dots, p_t) = \prod_{p \in J_t} c(p, p_t) \dots \prod_{p \in J_2} c(p, p_2), \quad J_i = \{p_1, \dots, p_i\}$$

Let  $I_t = \{p_2, \dots, p_t\}$ ,  $I_i = \{p_2, \dots, p_i\}$ ,  $i = 2, \dots, t$ . The numbers

$$\prod_{p \in J_i} c(p, p_i) \quad \text{and} \quad \prod_{p \in I_i} c(p, p_i)$$

are either equal or differ by a factor  $c(p_1, p_i)$  depending on whether  $p_1$  is in  $I_i$  or not. It follows that

$$(4.2.2) \quad n(p_1, \dots, p_t) = \prod_{p_1 \notin I_i} c(p_1, p_i),$$

The first congruence of (3.4.1) follows immediately from this.

Moreover if  $p_1 = p_2$  then  $p_1 \in I_i$  for all  $i = 2, \dots, t$  so that

$$(4.2.3) \quad n(p_1, p_2, \dots, p_t) = 1 \quad \text{if } p_1 = p_2$$

Finally, suppose that  $p_1 \neq p_2 = p_3 = \dots = p_{r+1}$ . Then for

$i = 2, \dots, r+1$  we have  $p_i = p_2$  and  $I_i = \{p_2\}$ ,  $p_1 \notin I_i$ , so that

$n(p_1, p_2, \dots, p_t)$  contains  $r$  factors  $c(p_1, p_2)$  which proves the

second congruence of (3.4.1).

q.e.d.

Remark. The formula (4.2.2) can be rephrased as

$$n(p_1, p_2, \dots, p_t) = c(p_1, p_2) \dots c(p_1, p_t) \text{ if } p_i \neq p_1, i = 2, \dots, t,$$

$$(4.2.4) \quad n(p_1, p_2, \dots, p_t) = 1 \quad \text{if } p_1 = p_2$$

$$n(p_1, p_2, \dots, p_t) = n(p_1, \dots, p_r) \text{ if } p_i \neq p_1, i = 2, \dots, r, \\ p_{r+1} = p_1$$

Let  $a'_s$  be the element of  $Q[T]$  obtained by setting  $T_d = 0$  for all  $d \neq 1$  which are not a power of a prime. Let  $NP = \{n \in \mathbb{N} \mid n \neq 1, n \text{ not a power of a prime}\}$ . We have

$$(4.2.5) \quad a'_s = \sum_{\substack{k_1 \dots k_t \\ (p_1, \dots, p_t, 1)}} \frac{n(q_1, \dots, q_t, 1)}{p_1} \dots \frac{n(q_t, 1)}{p_t} \\ T_{q_1}^{q_1} T_{q_2}^{q_2} \dots T_{q_t}^{q_1 \dots q_{t-1}}$$

#### 4.3. Proposition.

Let the  $n(q_1, \dots, q_t, d)$  be defined by (4.1.3). Then we have

$$(4.3.1) \quad a_s = \sum_{\substack{d|s \\ d \neq 1}} \frac{m(s, d)}{\mu(d)} a'_{s/d} T_d^{s/d}$$

where  $m(s, d) = 1$  if  $d \in NP$ ,  $m(s, p^t) = \prod_{p' \in J} c(p', p)$ , where  $J$  is the set

of primes occurring in  $s$ ; and  $\mu(d) = 1$  if  $d \in NP$ ,  $\mu(p^t) = p$ .

Further we have

$$(4.3.2) \quad a'_s = a_s - \sum_{\substack{d|s, d \in NP \\ d \neq 1}} a'_{s/d} T_d^{s/d}$$

Proof. Both these formulas are proved by looking at the formula (3.4.2) for  $a_s$ . Take a fixed  $d$ , and consider all ordered factorizations

$(p_1^{k_1}, \dots, p_t^{k_t}, d)$  of  $s$ . First suppose that  $d$  is not a power of a prime,  $d \neq 1$ . The part of  $a_s$  consisting of terms involving  $T_d$  is then

$$(4.3.3) \quad \sum_{\substack{k_1 \quad k_t \\ (p_1, \dots, p_t, 1)}} \frac{n(q_1, q_2, \dots, q_t, d)}{p_1} \cdot \dots \cdot \frac{n(q_t, d)}{p_t} T_{q_1}^{q_1} T_{q_2}^{q_2} \dots T_{q_t}^{q_t} T_d^{q_1 \dots q_t}$$

where the sum is over all ordered factorizations of  $s/d$  ending in 1. Combining this with (4.1.3) and (4.1.1) proves formula (4.3.2). Cf. (4.2.5). Now let  $d = q$  be a power of a prime, and consider the coefficient of  $T_q^{s/q}$  in  $a_s$ . This is equal to

$$(4.3.4) \quad \sum_{(q_1, \dots, q_t, q, 1)} \frac{n(q_1, \dots, q_t, q, 1)}{p_1} \cdot \dots \cdot \frac{n(q_t, q, 1)}{p_t} \cdot \frac{n(q, t)}{p} T_{q_1}^{q_1} \dots T_{q_t}^{q_t} T_d^{q_1 \dots q_t}$$

where the sum is over all factorizations ending in  $(\dots, q, 1)$ , and these correspond bijectively to all factorizations ending in 1 of  $q^{-1}$ 's.

According to (4.1.3) and the first two formulas of (4.1.1) we have

$$(4.3.5) \quad n(q_1, \dots, q_t, q, 1) \cdot \dots \cdot n(q_t, q, 1) \cdot n(q, 1) = b(p_1, \dots, p_t, p)$$

and using the third formula of (4.1.1) and again (4.1.3) and the first two formulas of (4.1.1) we see that

$$(4.3.6) \quad n(q_1, \dots, q_t, q, 1) \cdot \dots \cdot n(q_t, q, 1) \cdot n(q, 1) = \\ = m(s, q) n(q_1, \dots, q_t, 1) \dots n(q_t, 1)$$

This in combination with (4.3.4) and the argument used to establish (4.3.2) proves (4.3.1).

4.4. Remark. The formulae (4.3.1) and (4.3.2) permit one to write  $T_s$  as an expression in the  $T_d$ ,  $d < s$ ,  $d|s$  and the  $a_d$ ,  $d|s$ . This is the reason why this section is headed "inverse formulae".

## 5. GENERATORS FOR THE COMPLEX COBORDISM RING.

Let  $\Omega^{ev}(pt)$  denote the complex cobordism ring modulo torsion.

It is freely generated by countably many generators over  $Z$ . There is also a canonically defined formal group over it. Cf. [3]. The logarithm of this formal group law is equal to

$$(5.1) \quad \ell(X) = \sum_{n=0}^{\infty} \frac{P_n}{n+1} X^{n+1}$$

where  $P_n \in \Omega^{-2n}(pt)$  is the cobordism class of  $CP^n$ . Cf [3]. Quillen [3] has shown that this formal group law is universal. It is therefore isomorphic to the formal group law constructed above, in particular to the one which uses the  $n(q_1, \dots, q_t, d)$  defined and used in section 4. We can therefore use proposition (4.3) to find a set of generators for the complex cobordism ring

5.2. Theorem.

The following inductively defined elements,  $s = 2, 3, \dots$ , constitute a set of free generators over  $Z$  of the complex cobordism ring  $\Omega^{ev}(pt)$ .

$$\begin{aligned} t_s = & \mu(s) \frac{P_{s-1}}{s} - \mu(s) \sum_{dd_1=s, d_1 \neq 1, s} \frac{m(s, d_1)}{\mu(d_1)} \frac{P_{d_1-1}}{d_1} t_{d_1}^d + \dots \\ & + \mu(s) \sum_{\substack{dd_2 d_1=s, d_2 \in NP \\ d_1 \neq 1, s}} \frac{m(s, d_1)}{\mu(d_1)} \frac{P_{d_1-1}}{d_1} t_{d_2}^d t_{d_1}^{dd_2} + \dots \\ & + (-1)^i \mu(s) \sum_{\substack{dd_1 \dots d_2 d_1=s \\ d_1, \dots, d_2 \in NP, d_1 \neq 1, s}} \frac{m(s, d_1)}{\mu(d_1)} \frac{P_{d_1-1}}{d_1} t_{d_1}^d t_{d_{i-1}}^{dd_i} \dots t_{d_1}^{dd_1 \dots d_2} \end{aligned}$$

(We take  $P_0 = 1$ )

Proof. This follows immediately from proposition (4.3). Use formula (4.3.1) and then eliminate the  $a'_{s/d}$  inductively by means of (4.3.2).

(If  $d$  is a prime power  $a_d = a_d'$ ). Note that  $\mu(s)\mu(d_1)^{-1}m(s,d_1)$  is always an integer.

### 5.3. Some Examples.

We take  $c(3,2) = 4$ ,  $c(2,3) = 3$ . Using (5.2) one then easily calculates

$$\begin{aligned} t_2 &= P_1 \\ t_3 &= P_2 \\ t_4 &= \frac{1}{2}P_3 - \frac{1}{2}P_1^3 \\ t_6 &= \frac{P_5}{6} - \frac{2P_2P_1^3}{3} - \frac{P_1P_2^2}{2} \\ t_9 &= \frac{P_8}{3} - \frac{P_2^4}{3} \\ t_{18} &= \frac{P_{17}}{18} - \frac{2P_8P_1^9}{9} - \frac{P_5P_2^6}{6} - \frac{P_2}{3} \left( \frac{P_5}{6} - \frac{2P_2P_1^3}{3} - \frac{P_1P_2^2}{2} \right)^3 \\ &\quad + \left( \frac{P_5}{6} - \frac{2P_2P_1^3}{3} - \frac{P_1P_2^2}{2} \right) P_2^6 \end{aligned}$$

## 6. MORE DIMENSIONAL UNIVERSAL FORMAL GROUPS.

In this section we study higher dimensional formal groups. All formal groups considered will be commutative. To get a universal  $n$ -dimensional formal group, we work over the ring  $Q[\dots, T_q(i,j), \dots; \dots, S_d(i), \dots]$  where the  $T_q(i,j)$  and  $S_d(i)$  are indeterminates, one for each prime power  $q$  and  $1 \leq i, j \leq n$ ; and one for each  $1 \leq i \leq n$  and multiindex

$d = (d_1, \dots, d_n)$ ,  $d_i \geq 0$ ,  $d \neq (0,0,\dots,0)$  which is not of the form  $p^r e_j$  where  $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ , the 1 in the  $j$ -th place,

$j = 1, \dots, n$ ;  $p$  prime;  $r = 0, 1, 2, \dots$ . Let  $T_q$  denote the  $n \times n$  matrix  $(T_q(i,j))$  and  $S_d$  the column vector

$$S_d = \begin{pmatrix} S_d(1) \\ \vdots \\ S_d(n) \end{pmatrix}$$

If  $d$  is a multiindex  $d = (d_1, \dots, d_n)$  then  $X^d$  denotes  $X^d = X_1^{d_1} \dots X_n^{d_n}$ .

Our first result is completely analogous to theorem (2.2).

### 6.1. Theorem.

Let  $f(X)$  be an  $n$ -dimensional column vector of power series in the  $n$ -variables  $X_1, \dots, X_n$  over  $Q[\dots, T_q(i,j), \dots; \dots, S_d(i), \dots]$

such that

$$(6.1.1) \quad f(X) - X - \sum_{i=1}^{\infty} \frac{T_q^i}{p^i} f^{(p^i)}(X^{p^i}) \in Z_{(p)}[T,S]$$

for all primes  $p$ . (Here  $X$  is the column vector of the  $X_1, \dots, X_n$  and  $X^{p^i}$  is short for the column vector of the  $X_1^{p^i}, \dots, X_n^{p^i}$ ;  $f^{(p^i)}$  denotes (as usual) the power series obtained from  $f$  by raising all the parameters  $T_q(i,j), S_d(i)$  to the power  $p^i$ ). Let  $F(X,Y) = f^{-1}(f(X) + f(Y))$ , then all the coefficients of  $F(X,Y)$  are in  $Z[\dots, T_q(i,j), \dots; \dots, S_d(i), \dots]$ )

Proof. Same proof as of theorem (2.2).

As in the one dimensional case it remains to construct power series such that (6.1.1) holds for all primes  $p$ . We also know that there exist such power series.. This is exactly the same problem as we encountered in sections 3,4. We recall and introduce some notation.

### 6.2. Ordered Factorizations, etc.

Let  $s$  be a multiindex  $s = (s_1, \dots, s_n)$ . We write NPM for the set of all multiindices which are not of the form  $p^r e_j$ ,  $j = 1, \dots, n$ ;  $p$  prime;  $r = 1, 2, \dots$ , (Note that we start with  $r = 1$  here).

An ordered factorization of  $s = (s_1, \dots, s_n)$  is a sequence

$$(q_1, \dots, q_t, d)$$

where  $q_i$  is a prime power and  $d = (d_1, \dots, d_n)$  is a multiindex which is in NPM such that  $q_1, \dots, q_t d_i = s_i$ .

We also introduce the symbols  $S_{e_j}(i)$  as  $S_{e_j}(i) = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker index;  $S_{e_j}$  is the column vector of the  $S_{e_j}(i)$ .

For every ordered factorization  $(q_1, \dots, q_t, d)$  of a multiindex  $s$  we define  $n(q_1, \dots, q_t, d) = n(q_1, \dots, q_t) = n(p_1, \dots, p_t) =$  the number defined in section 4. Now let the column vector  $a_s$ ,  $s$  a multiindex, be defined by



$$(6.2.1) \quad a_s = \sum_{(q_1, \dots, q_t, d)} \frac{n(q_1, \dots, q_t, d)}{p_1} \cdots \frac{n(q_t, d)}{p_t},$$

$$T_{q_1}^{(q_1)} T_{q_2}^{(q_1)} \cdots T_{q_t}^{(q_1 \dots q_{t-1})} S_d^{(q_1 \dots q_t)}$$

where  $(q_1, \dots, q_t, d)$  runs through all ordered factorizations of  $s$ ;  $T_q^{(n)}$  is the matrix  $(T_q^n(i, j))$  and  $S_d^{(n)}$  is the column vector consisting of the  $S_d^n(i)$ .

### 6.3. Theorem.

Let  $f(X)$  be the  $n$ -dimensional vector of power series defined by

$$f(X) = \sum_s a_s X_1^{s_1} \cdots X_n^{s_n}$$

where  $s$  runs through all multiindices  $s = (s_1, \dots, s_n)$ ,  $s_i \geq 0$ ,  $s \neq (0, 0, \dots, 0)$ . Let

$$F(X, Y) = f^{-1}(f(X) + f(Y))$$

Then we have

- (i)  $f(X)$  satisfies (6.1.1) for all primes  $p$ .
- (ii) The coefficients of  $F(X, Y)$  are in  $Z[\dots, T_q(i, j), \dots; S_d(i), \dots]$
- (iii)  $F(X, Y)$  is a universal commutative  $n$ -dimensional formal group.

Proof. (i) follows directly from the definition of  $a_s$  and the properties of  $n(q_1, \dots, q_t, d)$ , cf. section 3. (ii) follows from (i) in virtue of theorem 1.1. As to (iii), this follows from (ii) because we have enough free parameters. More precisely one uses the result of Lazard cited as proposition (4.1) in [1].  
q.e.d.

6.4. Remark. As in the one dimensional case one has formulae like those of proposition (4.3.) which can be used to write the  $T_q(i, j)$  and  $S_d(i)$  inductively in terms of the  $a_s$ .

## 7. ISOMORPHISMS.

In sections 3, 4 we constructed certain power series  $f_T(X)$

over  $Q [T_2, T_3, \dots]$  such that

$$f_T(X) - \sum \frac{p^i}{p} f_T^{(p^i)}(X^{p^i}) \in Z_{(p)}[T][[X]]$$

for all primes  $p$ . In a certain sense the construction used there is the only one possible.

7.1. Lemma. Let  $f_{T,S}(X) \in Q[T_2, T_3, \dots; S_2, S_3, \dots][[X]]$  be a power series such that

$$(7.1.1) \quad f_{T,S}(X) - \sum \frac{p^i}{p} f_{T,S}^{(p^i)}(X^{p^i}) \in Z_{(p)}[T,S][[X]]$$

for all  $p$ . Then if  $a_n$  denotes the coefficient of  $X^n$  we have

$$(7.1.2) \quad a_n = \sum_{q|s} \sum_{q'|d} \frac{p^q}{p} \frac{c_{d,q'}}{q'} \cdot n(q,q') + b_n(S,T)$$

where  $d = q^{-1}n$ ,  $a_d = \sum (q')^{-1} c_{d,q'}$ ,  $c_{d,q'} \in Z[T,S][[X]]$ ,

$b_n \in Z[T,S][[X]]$  and  $n(q,q')$  any numbers such that  $n(q,q') \equiv 1 \pmod{p}$ ,

$n(q,q') \equiv 0 \pmod{q'}$  if  $(q,q') = 1$  and  $n(q,q') \equiv 1 \pmod{pq'}$  if

$(q,q') = p^s$ .

Proof. It follows immediately from (7.1.1) that  $a_n$  must be of the form given by (7.1.2). Assume for the moment that there are no monomials in  $S, T$  which occur both in  $b_n(S, T)$  and in the double sum part of  $a_n$ . It then immediately follows from (7.1.1) that  $b_n(S, T) \in Z[T, S][[X]]$ . Necessary and sufficient for (7.1.1) to hold is then that the expression (3.1.9) be in  $Z_{(p_0)}[S, T]$  for every  $p_0$

(with  $a_n T_n$  left out). First let  $(q, q') = p^s$ . The necessary and

sufficient condition on  $n(q, q')$  is that

$(pq')^{-1} \{n(q, q') - 1\} c_{d, q'} \in Z_{(p)}[S, T]$ . Any  $n(q, q') \equiv 1 \pmod{pq'}$  works.

It may of course happen that  $c_{d, q'}$  contains a few factors  $p$  so that a

$\bar{n}(q, q') \equiv 1$  modulo a smaller power of  $p$  than the exponent of  $pq'$  also

works. The difference  $\{n(q, q') - \bar{n}(q, q')\} (pq')^{-1} c_{d, q'}$  is then in

$Z[S, T]$  and can be absorbed in  $b_n(S, T)$ . Now let  $(q, q') = 1$ . The

necessary and sufficient conditions on  $n(q, q')$  are (cf. (3.1.9)).

$$(pq')^{-1}(n(q,q') - 1)c_{d,q'}^{(q)} \in Z_{(p)}[S,T]$$

$$(pq')^{-1}(n(q,q')c_{d,q'}^{(q)}) \in Z_{(p')}[S,T]$$

Any  $n(q,q')$  such that  $n(q,q') \equiv 1 \pmod{p}$  and  $n(q,q') \equiv 0 \pmod{q'}$  works. It may of course happen that  $c_{d,q'}^{(q)}$  is divisible by say  $p^a p'^b$ , which case we must have  $\bar{n}(q,q') \equiv 1 \pmod{p^{1-a}}$ ,  $\bar{n}(q,q') \equiv 0 \pmod{p'^{-b}q'}$ . The difference  $(pq')^{-1}\{n(q,q') - \bar{n}(q,q')\}c_{d,q'}^{(q)}$  is in  $Z[S,T]$  and can be absorbed into  $b_n(S,T)$ . q.e.d

### 7.2. Corollary.

Let  $f_T(X)$  be the power series  $f_T(X) = \sum a_s X^s$ , where  $a_s$  is given by (3.4.2). Substitute  $X + S_2 X^2 + \dots$  for  $X$  in  $f_T(X)$  and let the resulting series be  $g(X) = \sum d_s X^s$ .

Then we have

$$(7.2.1) \quad d_s = \sum_{(q_1, \dots, q_t, d)} \frac{n(q_1, \dots, q_t, d)}{p_1} \dots \frac{n(q_t, d)}{p_t} T_{q_1}^{q_1} T_{q_2}^{q_1 \dots q_2} \dots T_{q_t}^{q_1 \dots q_{t-1}} \cdot (T_d^{q_1 \dots q_t} + b_d^{(q_1 \dots q_t)}(S,T))$$

$$+ \sum_{(q_1, \dots, q_t, 1)} \frac{n(q_1, \dots, q_t, 1)}{p_1} \dots \frac{n(q_t, 1)}{p_t} T_{q_1}^{q_1} \dots T_{q_{t-1}}^{q_1 \dots q_{t-2}} p_t \cdot b_{q_t}^{(q_1, \dots, q_{t-1})}(S,T)$$

Proof. This follows from (7.1) because  $g(X)$  satisfies (7.1.1) if  $f_T(X)$  satisfies (7.1.1). Cf. [1] (3.1) and (3.2).

### 7.3. Corollary.

Let  $b_d(S,T)$  be any polynomial in  $S,T$ ;  $d = 2, 3, \dots$ . Let  $\mathcal{E}_{T,S}(X) = \sum d_s X^s$  be the power series given by (7.2.1). The formal groups  $F_T(X,Y)$  and  $G_{T,S}(X,Y)$  are then isomorphic over  $Z[T,S]$ .

Proof. Suppose we have proved this already mod degree  $n$  for all series of polynomials  $b_d(S,T)$ . Let  $\psi$  be the power series over  $Z[T,S]$

establishing the isomorphism mod  $n$ . The power series  $f(\phi(X))$  and  $g(X)$  both have coefficients of the form (7.2.1) and they coincide mod degree  $n$ . It follows that their polynomials ( $b'_d(S,T)$  and  $b_d(S,T)$  resp.) coincide for  $d < n$ . It follows that we can find a  $u(S,T) \in \mathbb{Z}[S,T]$  such that  $f(\phi(X) + u(S,T)X^n)$  and  $g(X)$  coincide mod degree  $n + 1$ .

q.e.d.

Now let  $h_{S,T}(X)$  be the power series  $h_{S,T}(X) = \sum b_s X^s$ ,  $b_s$  given by (7.2.1) with  $b_d(S,T) = S_d$ . Let  $t = (t_2, \dots, t_n, \dots)$ ,  $s = (s_2, s_3, \dots)$  be two sequences of elements from a characteristic zero ring  $A$ . Let  $h_{t,s}(X)$  and  $f_t(X)$  be the power series obtained from  $h_{T,S}(X)$  and  $f_T(X)$  by substituting  $t_i$  and  $s_i$  for  $T_i$  and  $S_i$ . Let  $H_{t,s}(X,Y)$  and  $F_t(X,Y)$  be the formal groups belonging to  $h_{t,s}(X)$  and  $f_t(X)$ .

#### 7.4. Corollary.

*if  $A$  is a characteristic zero ring.*

The formal groups  $H_{t,s}(X,Y)$  and  $F_t(X,Y)$  are isomorphic. Inversely, and  $H(X,Y)$  is isomorphic over  $A$  to  $F_t(X,Y)$  then there exist  $(s_2, s_3, \dots)$  such that the logarithm of  $H(X,Y)$  is equal to  $h_{t,s}(X)$ .

Proof. The first part follows from (7.3). As to the second part, suppose we have already found  $s_2, \dots, s_{n-1}$  such that

$$h(X) \equiv h_{t,s}(X) \pmod{\text{degree } n}$$

The formal groups  $H(X,Y)$  and  $H_{t,s}(X,Y)$  are isomorphic and congruent mod degree  $n$ . It follows that there exists an  $s_n$  such that

$$h(X) \equiv h_{t,s}(X) \pmod{\text{degree } n + 1}$$

#### 7.5. Remarks.

1. Corollary (7.4) can of course be used as a criterium for testing whether two formal groups over a characteristic zero ring are isomorphic.
2. Similar results can be obtained for more dimensional formal groups.

## 8. A LOCAL GLOBAL RESULT.

Let  $K$  be an algebraic number field,  $A$  denotes its ring of integers. If  $\mathfrak{p}$  is a prime ideal,  $A_{(\mathfrak{p})}$  is the localization of  $A$  at  $\mathfrak{p}$ , and  $A_{\mathfrak{p}}$  is the completion of  $A_{(\mathfrak{p})}$ . We shall view  $A_{(\mathfrak{p})}$  as a subring of  $K$ ;  $v_{\mathfrak{p}}$  is the valuation on  $A_{\mathfrak{p}}$  and  $K$  belonging to the prime ideal  $\mathfrak{p}$ .

## 8.1. Lemma.

Let the prime  $p$  decompose as  $p = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_n^{e_n}$  in  $A$ . For every prime  $\mathfrak{p}$  dividing  $p$  let there be given a number  $a_{\mathfrak{p}} \in A_{\mathfrak{p}}$ . Then there exists an  $a \in A$  such that  $a - a_{\mathfrak{p}} \in \mathfrak{p}A_{\mathfrak{p}}$  for all the primes  $\mathfrak{p}$  dividing  $p$ .

Proof. First we show that for every  $\mathfrak{p}_i$  there is a  $b_{\mathfrak{p}_i} \in A_{\mathfrak{p}_i}$  such that  $a_{\mathfrak{p}_i} + \mathfrak{p}b_{\mathfrak{p}_i} \in \bigcap_{i=1}^n A_{(\mathfrak{p}_i)}$ . We can in any case assume that  $a_{\mathfrak{p}_i} \in A_{(\mathfrak{p}_i)}$

for  $i = 1, \dots, n$ . Let  $\pi_1, \dots, \pi_n$  be elements of  $A$  such that  $v_{\mathfrak{p}_i}(\pi_j) = \delta_{ij}$ ,  $i, j = 1, \dots, n$ . Then we can write

$$a_{\mathfrak{p}_i} = \frac{c_{\mathfrak{p}_i}}{\pi_1^{t_1} \dots \pi_{i-1}^{t_{i-1}} \pi_{i+1}^{t_{i+1}} \dots \pi_n^{t_n}}, \quad c_{\mathfrak{p}_i} \in B = \bigcap_{i=1}^n A_{(\mathfrak{p}_i)}$$

Let  $b_{\mathfrak{p}_i}$  be of the form  $b_{\mathfrak{p}_i} = (\pi_1^{t_1+e_1} \dots \pi_{i-1}^{t_{i-1}+e_{i-1}} \pi_{i+1}^{t_{i+1}+e_{i+1}} \dots \pi_n^{t_n+e_n})^{-1} d_{\mathfrak{p}_i}$ ,

$d_{\mathfrak{p}_i} \in B$ .

The problem is then to choose  $d_{\mathfrak{p}_i}$  such that

$$c_{\mathfrak{p}_i} + \pi_i^{e_i} d_{\mathfrak{p}_i} \equiv 0 \pmod{\prod_{j \neq i} \pi_j^{t_j+e_j}}$$

which can be done because the  $\pi_i$  are prime to each other. We can therefore

assume that the  $a_{\mathfrak{p}}$  are all in  $B$ . Now for each  $i$  let  $e_{\mathfrak{p}_i}$  be of the form

$$e_{\mathfrak{p}_i} = \prod_{j \neq i} \pi_j^{-e_j} f_{\mathfrak{p}_i}. \quad \text{Then } a_{\mathfrak{p}_i} + \mathfrak{p}e_{\mathfrak{p}_i} \text{ is of the form } a_{\mathfrak{p}_i} + \pi_i^{e_i} f_{\mathfrak{p}_i}$$

with  $f_{\mathfrak{p}_i} \in B$

And the next problem is therefore to find an  $a' \in B$  such that  $a' \equiv a_{\mathfrak{p}_i}$

$\text{mod } \pi_i^e$  which can be done by the Chinese remainder theorem.

We have now found an  $a' \in B$  which satisfies the requirements of the lemma. It now suffices to show that there is an  $b \in B$  such that  $a = a' + pb$  is in  $A$ . Let  $\sigma_{l_1}, \dots, \sigma_{l_m}$  be the prime ideals of  $A$  such that  $v_{\sigma_{l_j}}(a') < 0$ . Choose elements  $\rho_j$  of  $A$  such that  $v_{\sigma_{l_j}}(\rho_j) = \delta_{ij}$ ,  $v_{\sigma_{l_i}}(\rho_j) = 0$ . Then we can write

$$a' = \frac{c'}{\rho_1^{r_1} \dots \rho_m^{r_m}}$$

with  $c' \in A$ .

let  $b'$  be of the form  $b' = \rho_1^{-r_1} \dots \rho_m^{-r_m} d'$ ,  $d' \in A$ . The problem is then to find a  $d' \in A$  such that  $c' + pd' \equiv 0 \text{ mod } \rho_1^{r_1} \dots \rho_m^{r_m}$  which can be done because  $p$  and  $\prod_i \rho_i^{r_i}$  are prime to each other.

### 8.2. Proposition.

Let  $F$  and  $G$  be two formal groups over  $A$ . Then  $F$  and  $G$  are isomorphic over  $A$  if and only if they are isomorphic over all  $A_p$ .

Proof. The isomorphism between  $F$  and  $G$ , if it exists, is equal to  $g^{-1}(f(X))$ , where  $f, g$  are the logarithms of  $F$  and  $G$ . The coefficients of  $g^{-1}(f(X))$  are in  $A$  iff they are in  $A_p$  for all  $p$ .

### 8.3. Proposition.

Suppose we have a formal group  $F_p$  over  $A_p$  for all prime divisors  $p$  of  $A$ . Then there exists a formal group  $F$  over  $A$  such that  $F$  is isomorphic to  $F_p$  for all  $p$  over  $A_p$ .

Proof. Suppose we have already constructed  $F$  up to and including degree  $n$ . If  $n+1$  is not a prime power  $F$  and  $F_p$  are also isomorphic mod degree  $n+2$ , for all  $p$ , and we can extend  $F$  to degree  $n+1$  arbitrarily. Now suppose that  $n+1$  is a power of the prime  $p$ . For each prime  $\varphi$  dividing  $p$ , let  $\phi_\varphi$  be a power series over  $A_\varphi$  establishing the isomorphism between  $F$  and  $F_\varphi$  mod degree  $n+1$ .

and let  $F'_p(X,Y) = \phi_p^{-1} F(\phi_p(X), \phi_p(Y))$  and let  $f'_p(X)$  be the logarithm of  $F'_p$ . Let  $f_T(X)$  be the power series over  $Z[T]$  given by (3.4.2)

and  $t = (t_2, \dots, t_n)$ , be such that  $f_t(X) \equiv f(X) \pmod{\text{degree } n+1}$

where  $f$  is the logarithm of  $F$ . For each  $p$  dividing  $p$  let  $t(p) = (t_2(p), \dots)$ .

be such that  $f_{t(p)}(X) = f'_p(X)$  then  $t_i(p) = t_i$  for  $i = 2, \dots, n$

Now choose  $t_{n+1} \in A$  such that

$$t_{n+1} - t_{n+1}(p) \in pA_p \text{ for all } p.$$

Every formal group  $F_s(X,Y)$  with  $s_i = t_i$  for  $i < n+1$ ,  $s_i$  arbitrary for  $i > n+1$  is then isomorphic to  $F_p(X,Y)$  modulo degree  $n+2$ , for all primes  $p$  dividing  $p$ . As to the primes  $q$  not dividing  $p$ ,  $F_s(X,Y)$  and  $F_q(X,Y)$  are isomorphic mod degree  $n+2$  if they are isomorphic mod degree  $n+1$  because  $q$  is prime to  $n+1$ .

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