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ON THE STABILITY OF DYNAMIC ECONOMIC PROCESSES

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Contents

1. INTRODUCTION

The first purpose of this note is to argue that the usual notions of stability for dynamic economic processes as found in e.g.[2], [6], [12], [13], [17], suffer from serious drawbacks. On two counts: they offer no guarantee that the behaviour of the process, disturbed by a whole sequence of small disturbances at times $t_1 < t_2 < \dots$, is even approximately the same as the behaviour of the original system, and they offer no guarantee that a process governed by almost the same function has approximately the same behaviour. Thus it is possible to have a process with, in the terminology of [2], [6], [12], [13], [17], a globally stable equilibrium point (the strongest notion of stability in [2], [6], [12], [13], [17]) such that the process has no chance at all of remaining near equilibrium for any appreciable amount of time when disturbed by a sequence of (random) disturbances at a sequence of (random) moments $t_1 < t_2 < \dots$ in time. And it is possible to have a process with a globally stable equilibrium point such that there are processes arbitrarily near to it with no equilibrium points at all, or only one nonstable equilibrium point.

These are (in my opinion) rather serious drawbacks, if one reflects that all processes in reality are subject to disturbances all the time, and that the precise functions governing the evolution of the system are rarely, if ever, precisely known.

In nr.2 we give and discuss various notions of stability; nr 3 contains some examples and nr. 4 a discussion of these examples. The remainder of this paper (nr.5 - nr.9) is devoted to properties of dynamical systems which are stable with respect to suitable perturbations, In nr. 5 we examine the number of equilibrium points of a given system, show that "most" systems have only finitely many equilibria and that a suitable refinement of this notion (having a finite set of equilibria) is stable under small enough perturbations. In nr. 6 we discuss (modified) Liapunov functions and their relation to stability. It turns out that e.g. in [17] more is proved than is actually stated in the theorems; i.e. the processes examined are much more stable than is indicated by the theorems. This is the subject matter of nr.7, which also gives some complements for these processes. In nr. 8 we analyse stability under sequences of disturbances; nr. 9, finally, gives an existence theorem for positive equilibrium points.

2. VARIOUS CONCEPTS OF STABILITY.

We study dynamic economic processes. In particular we study the problem whether certain quantities, prices or values, which evolve subject to certain economic laws approach equilibrium values. Generally speaking the processes according to which these quantities evolve are subject to sudden (small) disturbances; moreover the laws governing changes in these quantities are often not exactly known. This makes the study of the stability of the processes involved important. Typically, we shall have in mind a tâtonnement price adjustment proces.

$$\text{(T)} \qquad \dot{p}_i = f_i(p_1, \dots, p_n) \qquad i = 1, \dots, n$$

where p_i is the price of commodity i, and $f_i(p_1, \dots, p_n)$ is a function of the prices $p_1, \dots, p_n$, which has the same sign as $h_i(p_1, \dots, p_n)$, the excess demand for commodity i, if the prices of commodity 1, ..., n are $p_1, \ldots, p_n$. Thus (1) reflects: "prices rise if excess demand is positive", and nothing more. In this process no exchange of commodities is allowed at nonequilibrium prices. (p is an equilibrium price vector if $f_i(p) = 0$, $i = 1, \ldots, n$). Cf. [12], [14] and [17] for a discussion of this proces). More generally we also consider non tâtonnement processes (cf [12]):

(NT)
$$\dot{p}_i = f_i(p;s)$$
$$\dot{s}_{ij} = g_{ij}(p;s)$$

where p_i is the price of commodity i, and s_{ij} is the amount of the j-th commodity held by the i-th individual. Both the p_i and the s_{ij} in (T) and (NT) are usually supposed to be nonnegative and sometimes supposed to be positive.

One asks oneself whether prices according to (T) and (NT) approach equilibrium values, and whether these equilibrium values are stable. More generally one could also ask whether a given movement of prices is stable; this is, however, essentially the same problem, cf. 2.7.

Both (T) and (NT) are particular cases of a system of autonomous differential equations on a set $M \subset \mathbb{R}^n$ (usually M is a differentiable manifold and the inclusion is a differentiable embedding [1]))

(DS)
$$\dot{x} = f(x)$$

$x \in M$, f a continuous n-vector valued function on M.

We shall always assume that there exist unique solutions to the dynamical system (DS); i.e. we shall assume

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1) For a definition of a differentiable manifold cf [11]. If $M \subset \mathbb{R}^n$ is a k-dimensional differentiable manifold embedded in $\mathbb{R}^n$, then for each $x \in M$ there exist n differentiable functions $g_1, \ldots, g_n$ defined in an open neighbourhood of x in $\mathbb{R}^n$, such that $g(x) = 0$, $\frac{dg}{dy}(x)$ is nonsingular, and M is defined by $g_{k+1}(y) = \ldots = g_n(y) = 0$ in a neighbourhood of x. Thus $S^{n-1} = \{x \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 = 1\}$ is an (n-1)-dimensional differentiable manifold, differentiably embedded in $\mathbb{R}^n$.

(B) For every $x^o \in M$, there exists a unique solution $x(t;x^o)$, of (DS), $x(t;x^o) \in M$ for all $t \geq 0$, such that $x(0;x^o) = x^o$ For a fixed $t \geq 0$, $x(t;x_o)$ is a continuous function of x^o.

A solution $x(t,x^o)$ is sometimes called a motion of the dynamical system (DS). Condition B is e.g. satisfied if $M = \mathbb{R}^n$ and f satisfies a global Lipschitz condition [2]).

A set $\{x(t;x^o) \mid t \geq 0\}$ is called a (positive semi-)trajectory of (DS).

Sometimes we shall also assume the somewhat stronger existence condition

(B') For every $x^o \in M$, there exists a unique solution $x(t;x^o)$ of (DS) defined for all $t \in R$, passing through x^o at $t = 0$; i.e. such that $x(0;x^o) = x^o$. For a fixed $t \geq 0$, $x(t;x_o)$ is a continuous function of x^o.

One then calls a set $\{x(t;x^o) \mid t \in R\}$ a trajectory of (DS)

2.1. Definition of Equilibrium

A point $e \in M$ is said to be an equilibrium of (DS) if $f(e) = 0$. The motion of (DS) starting in e is then $x(t;e) = e$ for all $t \geq 0$.

2.2. The Reason for Stability Analysis. (cf. [12] section 2.2)

Processes like (T) and (NT) have the property that prices rise for those commodities whose demand exceeds supply, and fall for those commodities where the reverse holds. Negishi [12], 2.2 argues:

"We know from experience that under this process prices usually do not explode towards infinity or con tract to zero, but

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2) The function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies a global Lipschitz condition, if there exists a constant $K > 0$ such that

$$||f(x) - f(y)|| \leq K||x - y||$$

for all $x, y \in \mathbb{R}^n$.

converge to an equilibrium such that the supply of and demand for commodities are equal. Hence, the process which we choose to represent reality must display the same stability".

and

"The equilibrium once established in this way is continuously subject to changes and disturbances, such as of taste, technology resources and weather. Suppose the system, which has been in equilibrium is thrown out of it by some of those changes or disturbances. It is known empirically that the economy is in fact fairly shock-proof. Dynamic market forces are generated which bring the economy back to equilibrium when it is perturbed, i.e. there exists a stable adjustment process when the economy is out of equilibrium. Realistic economic models should contain such a dynamic equilibrating process".

This suggests the following

2.3. Provisional Definitions.

(i) If for any $x^o \in M$, $\lim_{t\to\infty} x(t;x^o) = e$ for some equilibrium point $e \in M$, then we say that the system (DS) is stable.

(ii) A particular equilibrium position e is said to be globally stable if for every x^o, $\lim_{t\to\infty} x(t;x^o) = e$, and

(iii) A particular equilibrium position e is said to be locally stable if $\lim_{t\to\infty} x(t,x^o) = e$ for all x^o in a sufficiently small neighbourhood of e.

These seem to be quite generally accepted notions of stability in economics. Cf. [13] p. 162, [2], [6], [12], [14], [17].

Thus examples (2.1) and (2.2) below have one globally stable equilibrium position e, according to this terminology. Suppose, however, that in (2.2) the system is disturbed slightly out of equilibrium along the trajectory m; then it might very well take a very long time before the system is again in the neighbourhood of the equilibrium position. This is presumably not the kind of behaviour expected of a "fairly shock proof" economy.

Also, as a matter of fact, definitions 1.3 are not the ones usually encountered in dynamical system theory. (Cf. [7], [18]). We shall not adopt the terminology of 1.3. Instead we use (cf. [7], [18]).

2.4. Definitions. (Attractors)

(i) An equilibrium point $e \in M$ is called globally attracting if $\lim_{t\to\infty} x(t;x^o) = e$ for all $x^o \in M$

(ii) An equilibrium point $e \in M$ is called (locally) attracting if $\lim_{t\to\infty} x(t;x^o) = e$ for all x^o in a sufficiently small neighbourhood of e.

(iii) A closed set $F \subset M$ is called globally attracting if $\lim_{t\to\infty} \rho(x(t;x^o),F) = 0$ for all $x^o \in M$

(Here $\rho(y,F) = \inf_{x\in F} ||x - y||$, is the distance of y to the closed set $F \subset M$; $\|\ \|$ denotes the usual norm in $\mathbb{R}^n$).

(iv) A closed set $F \subset M$ is called attracting if $\lim_{t\to\infty} \rho(x(t;x^o),F)$ for all x^o in a sufficiently small neighbourhood of F.

Let E be the set of equilibrium points of (DS). The set E is closed because f is continuous.

(v) (DS) is said to have a pointwise attracting equilibrium set, if for every $x^o \in M$ there is an $e \in E$ such that $\lim_{t\to\infty} x(t;x^o) = e$

This is what was called stability in 2.3. In [17] one also finds a somewhat weaker notion than 2.4(v), called quasi-stability in [17] and [12]. A dynamical system has this property if all its trajectories $\{x(t;x^o)|\ t \geq 0\}$ are bounded and if E, the set of equilibria, is attracting.

If either E or M is bounded, the condition on the boundedness of the trajectories can be omitted. If E is finite or countable, then a dynamical system with bounded trajectories and attracting equilibrium set also has a pointwise attracting equilibrium set [17].

A fairly shock proof equilibrium e one should have the property that a (small) disturbance from e (or from a position in a sufficiently small neighbourhood of e) should not have much effect (also in the future). This leads to

2.5. Definition. (Stability)

An equilibrium e is called stable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $||x^o - e|| < \delta$ implies $||x(t;x^o) - e|| < \varepsilon$ for all $t \geq 0$.

An equilibrium such that both the "facts" cited in 2.2 are represented in our model should be both stable and (globally) attracting.

2.6. Definitions (Asymptotic Stability)

An equilibrium point $e \in M$ is called globally asymptotically stable if it is both globally attracting and stable; it is called (locally) asymptotically stable if it is attracting and stable.

One can of course extend the notion of stability of 2.5 to cover stability of closed sets, etc....

The economic examples of Scarf [14] section 3, cf. 2.8 below, show that even one stable equilibrium point in a tâtonnement process might be too much to hope for. However the situation as a whole is not too bad (from the stability point of view) both the motion m and the trajectory m look stable (intuitively). The precise definition is

2.7. Definition. (Stability of Motion)

A motion $x(t;x^o)$ of (DS) is called stable, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $||x^1 - x^o|| < \delta$ implies $||x(t;x^o) - x(t,x^1)|| < \varepsilon$ for all $t \geq 0$.

Let $x(t;x^o)$ be a motion of (DS) : $\dot{x} = f(x)$. Let $z = x - x(t;x^o)$. Then $z(t;0) = 0$ for all t is a solution of the system

$$\dot{z} = \dot{x} - \dot{x}(t,x^o) = f(x) - f(x(t;x^o)) = f(z + x(t;x^o)) - f(x(t;x^o)) = g(z,t),$$

and the stability of the motion $x(t;x^o)$ is equivalent to the stability of the equilibrium point 0 of the nonautonomous system $\dot{z} = g(z,t)$.

2.8. Reasons for Requiring Structural Stability and Total Stability.

All of the definitions given up to now, relate to one fixed dynamical system

(DS) $\dot{x} = f(x)$

and to one possible disturbance at time t = 0.(One takes different starting points). However, even in physics it is rarely the case that the function f is exactly known. And this is even more so in economics, biology and sociology and the like. Also for a given economic, biological or physical system one will usually have disturbances, not only of the initial position (i.e. at t = 0), but also at many other moments in time. Thus it is intuitively clear (cf. also 3.2) that the systems (2.1) and 2.2 have no chance at all of remaining near equilibrium after a sufficiently long time period has elapsed if there occur small random disturbances not only at time t = 0, but also other moments in time $t_1, t_2, \ldots, \lim_{i\to\infty} t_i = \infty$. Cf. 5.

And, in view of our usually imperfect knowledge of the function f of (DS) it becomes important to examine whether a slight perturbation of (DS):

(DS_{pert}) $\dot{x} = g(x)$

where the function g is close to f in some suitable sense, behaves more or less in the same way as (DS). (For instance with respect to its equilibrium set). This leads to various concepts like structural stability, total stability, Ω-stability, tolerance stability. Cf. [7], [15], [18], [19].

In fact Thom [16] suggests that every (DS) used in applied science to describe a given set of phenomena should be structurally stable. (The actual situation is a (possibly varying) (small) perturbation of the theoretical model). Cf.also [19].

For structural stability one requires that (DS) and (DS_{pert}) are "essentially" the same (Cf. 2.11 and 2.12); for total stability one only requires that solutions to (DS) and (DS_{pert}) are close to each other. The precise definition of the latter follows.

2.9. Definition (Total Stability)

An equilibrium point e of (S) is called totally stable if for every $\varepsilon > 0$ there exist two positive numbers $\delta_1 > 0$, $\delta_2 > 0$ such that $||y(t;x^o) - e|| < \varepsilon$ provided only that $||x^o - e|| < \delta_1$ and that $||g(x) - f(x)|| < \delta_2$ for all $x \in M$ such that $||x - e|| < \delta_1$.
(Here $y(t;x^o)$ denotes the solution to (DS_{pert}) starting in x^o at time t = 0). Note that total stability of e, implies stability of e.

It is easily seen that the requirement that $f(x)$ and $g(x)$ are close to each other for all $x \in M$ offers hardly any guarantee that the systems (DS) and (DS_{pert}) are "the same"(especially in the neighbourhood of equilibrium points). A good notion of nearness in this respect is

2.10. Definition (ε-C^1-Perturbations)

A differentiable function $g : M \to \mathbb{R}^n$ is an ε-C^1-perturbation of the differentiable function $f : M \to \mathbb{R}^n$ if for all $x \in M$ $||f(x) - f(x)|| < \varepsilon$ and $||Df(x) - Dg(x)|| < \varepsilon$.
Here $Df(x)$ denotes the derivative of f at x. Thus the second condition requires that all the first partial derivatives of f and g are close to each other. For a tâtonnement process (T) this is practically the same as requiring that the price elasticities be close to each other.

We still have to define what it means that two dynamical systems are "the same". For this we assume that we are dealing with systems for which condition (B') holds.

2.11. Definition (Equivalent Dynamical Systems)

Two dynamical systems (DS) and (DS_{pert}) on M are equivalent if there exists a homeomorphism[3] $M \to M$ (i.e. a one to one, onto map which is continuous in both directions) which maps the trajectories of (DS) into those of (DS_{pert}) and vice versa.

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3) A homomorphism is a 1-1 onto map which is continuous in both directions. It need not be differentiable.

We can now define

2.12. Definition (Structural Stability)

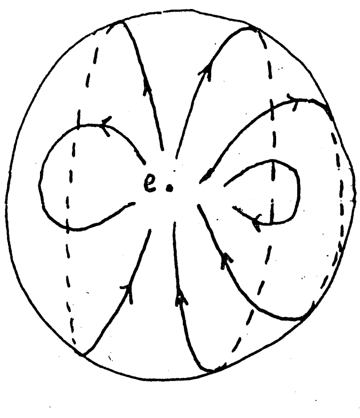
A dynamical system (DS), with differentiable f, is structurally stable if there exists an $\delta > 0$ such that every δ-C_1-perturbation g of f gives an equivalent system.

Remark. One can refine this notion by requiring that for every $\varepsilon > 0$ there be a $\delta > 0$ such that for every δ-C_1-perturbation g of f there exists an ε-homeomorphism M, establishing the equivalence of the perturbed system and the original one. (A homeomorphism $\varphi : M \to M$ is an ε-homeomorphism if $||\varphi(x) - x|| < \varepsilon$ for all $x \in M$).

3. EXAMPLES

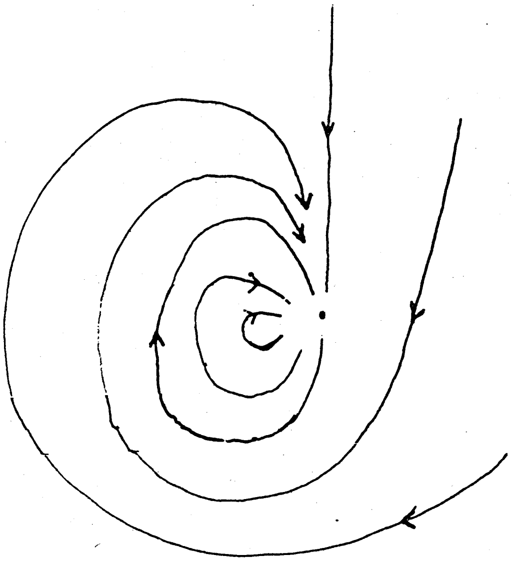
In most of the examples below we have drawn a socalled phase portrait; that is for every $x \in M$, the trajectory of the motion starting in x is depicted. One cannot see from these pictures how fast a given motion is.

3.1. Example



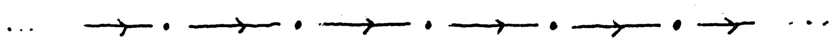
$M = S^2$, the 2-sphere; $S^2 = \{(x_1,x_2,x_3) \in R^3 | x_1^2+x_2^2+x_3^2 = 1\}$. There is one equilibrium point e, which is globally attracting, but not stable.

3.2. Example



$M = \mathbb{R}^2$. There is one equilibrium point e. Because $\mathbb{R}^2$ is diffeomorphic to $\{x \in \mathbb{R}^2 | x_1 > 0, x_2 > 0\}$ one can modify this example to get one on $\{x \in \mathbb{R}^2 | x_1 > 0, x_2 > 0\}$ with the same properties and with the equilibrium point at (1,1), say. The transformation $y_1 = e^{x_1}$ $y_2 = e^{x_2}$ e.g. transforms the given example into the same one (a diffeomorphic one) on $\{x \in \mathbb{R}^2 | x_1 > 0, x_2 > 0\}$. The equilibrium point e is attracting (globally) but not stable.

3.3. Example

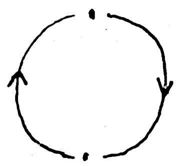


$M = \mathbb{R}$. There are non stable equilibrium points at all integers in $\mathbb{R}$. An equation which has this phase portrait is e.g.

$$\dot{x} = 1 - \cos 2\pi x$$

The system defined by this equation is not structurally stable and none of the equilibrium points is totally stable.

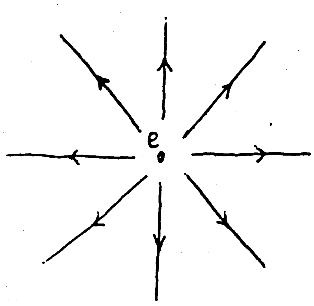
3.4. Example



$M = S^1 = \{ (x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 1 \}$

There are two equilibrium points. Neither is attracting, neither is stable. The system is not structurally stable and not totally stable.

3.5. Example

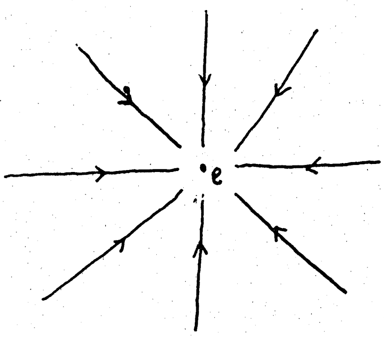


$M = \mathbb{R}^2$, there is one equilibrium points, which is neither stable nor attracting. An equation with this phase portrait is

$$\dot{x} = x$$

$$\dot{y} = y$$

3.6. Example



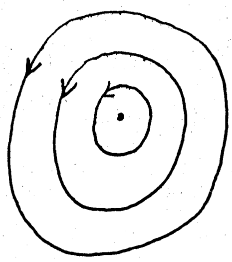
$M = \mathbb{R}^2$; there is one stable and globally attracting equilibrium point, which is therefore globally symptotically stable. An equation with this phase portrait is

$$\dot{x} = -x$$

$$\dot{y} = -y$$

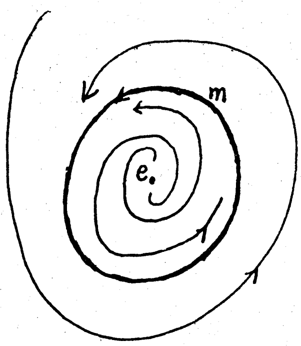
The system defined by these equations is structurally stable.

3.7. Example



$M = \mathbb{R}^2$. There is one equilibrium point which is stable, but not attracting. Scarf in [14], §2, gives an example of a tâtonnement process for prices which has this phasepicture. The system is not structurally stable and not totally stable.

3.8. Example



$M = \mathbb{R}^2$. There is one closed trajectory. There is one equilibrium point which is neither stable nor attracting. The more complicated examples of Scarf [14], §3 are of this type. They are (arbitrarily small) perturbations of 3.7. These systems are structurally stable.

4. DISCUSSION OF THE EXAMPLES.

Remark. Most of the examples given in nr.3 are not derived from an economic dynamic process; it is not clear whether such "pathological" systems occur in economics. In fact an assumption like substitutability in a tâtonnement process rules out examples like (3.1), (3.2) and (3.4)

4.1. Sequences of Disturbances.

Suppose we have a dynamical process (DS), and that there occur disturbances of magnitudes u^1, u^2, ... at various times t_1, t_2,...

If one starts in x^o at time t = 0, then gets a disturbed trajectory

$0 \leq t < t_1 \quad x_{pert}(t;x^o) = x(t,x^o)$

$t = t_1 \quad x_{pert}(t_1;x^o) = x^1$, where $||x^1 - x(t_1;x^o)|| = u_1$

$t_1 < t < t_2 \quad x_{pert}(t;x^o) = x(t-t_1,x^1)$

$t = t_2 \quad x_{pert}(t_2;x^o) = x^2$, where $||x^2 - x(t_2;x^1)|| = u_2$

$t_2 < t < t_3 \quad x_{pert}(t;x^o) = x(t-t_2;x^2)$

. . . . .

(We suppose of course that the disturbances are such that the motion remains in M).

For a natural process it seems reasonable that the u^1, u^2, ... should be bounded, and that during a small time interval there can only be a finite given total amount of disturbance. We shall therefore consider sequences of disturbances (t_1, t_2, ...; u_1, u_2,...) such that there exists a (time interval) T > 0, and a number K such that for all $t \in \mathbb{R}$.

$$\sum_{t \leq t_i \leq t+T} ||u_i|| \leq K$$

$S_{T,K}$ denotes the set of all such sequences of disturbances.

One could also consider sequences of disturbances of finite total disturbance, i.e. such that

$$\sum_{i=1}^{\infty} ||u_i|| < \infty$$

S_K denotes the set of all sequences of disturbances such that

$\sum_{i=1}^{\infty} ||u_i|| \leq K.$

4.2. Stability under Sequences of Disturbances.

It is clear that for systems like 3.1 and 3.2

$$x_{pert}(t;x_o)$$

does not tend to any limit for "almost all" sequences of disturbances $(t_1, t_2, \ldots; u_1, u_2, \ldots) \in S_{T,K}$. This holds for all $T > 0$, $K > 0$. Even for disturbances $(t_1, t_2, \ldots; u_1, u_2, \ldots) \in S_K$ the limit for $t \to \infty$ of $x_{pert}(t; x_o)$ will not exist for many disturbances.

On the other hand for a system like 3.6, the limit

$$\lim_{t\to\infty} x_{pert}(t; x_o)$$

does exist (and is equal to the equilibrium point) for all disturbances $(t_1, t_2, \ldots; u_1, u_2, \ldots) \in S_K$; and for disturbances from $S_{T,K}$ one has a statement of the following form. Cf. also §7.

For every $T > 0$ and $\varepsilon > 0$ there exists a K_o such that

(*) $$||x_{pert}(t; x_o) - e|| < \varepsilon \text{ for all } t \text{ sufficiently large}$$

(depending on x_o) and for all disturbances in $S_{T,K}$, $K \leq K_o$.

For systems like (3.1) and (3.2) such a statement does not hold. The intuitive content of (*) is that $x_{pert}(t; x_o)$ will be close to equilibrium and remain close to equilibrium provided the disturbances affecting the system are not too large.

4.3. Liapunov Functions (cf. [7], and also [17].

A continuous function $\Phi(x)$ on M is called a modified Liapunov function if for every $x^o \in M$, the function

$$y(t) = \Phi(x(t; x^o))$$

is a strictly decreasing function of t for all t except when $x(t; x^o)$ is an equilibrium point.

Systems like 3.3 and 3.6 admit modified Liapunov functions. The systems 3.1, 3.3, 3.4 e.g. do not admit such a function.

If e is an equilibrium point of (DS), and there is a function Φ defined on N, a neighbourhood of e, such that

$$\Phi(x) > 0, \; x \in N \setminus \{e\}, \; \Phi(e) = 0$$

$\Phi(x(t; x^o))$ is strictly decreasing at $t = 0$ for all $x^o \in N$, $x^o \neq e$ then Φ is called a Liapunov function for e.

If Φ is defined on all of M, we say that Φ is a global Liapunov function (e is then the only equilibrium point of (DS)).

4.4. Total Stability and Structural Stability.

The perturbation ($\varepsilon > 0$) : $\dot{x} = 1 + \varepsilon - \cos 2\pi x$, which has no equilibrium points at all, of example 3.3 shows that 3.3 is neither totally stable and nor structurally stable. A very small perturbation of 3.3 yields a system with widely different trajectories. However, example 3.3 has a pointwise attracting equilibrium set (i.e. is globally stable in the sense of [13], [17] and [61]), and does admit a modified Liapunov function. It follows that these notions are not particularly relevant whenever the dynamical systems involved are not exactly known (as is usually the case in physics, economics, biology, etc..).

One can prove that the systems described by the equations of 3.5 and 3.6 are structurally stable and the equilibrium point of 3.6 is also totally stable. Quite generally one can prove that an asymptotically stable equilibrium point e of a system (DS) is totally stable, and that a system (DS) is structurally stable in a neighbourhood of such a point provided that $x(t,x_o)$ for x_o close to e moves fast enough towards e. Thus

$$\dot{x} = -x$$

$$\dot{y} = -y$$

is structurally stable. But (cf. 5.6)

$$\dot{x} = -x^3$$

$$\dot{y} = -y^3$$

(which has the same phase portrait)is not structurally stable. At least in our sense (cf. 1.11).

5. MOST PRICE ADJUSTMENT PROCESSES HAVE A FINITE SET OF EQUILIBRIA.

In this section we show as an application of transversality theory that most price adjustment processes have a finite set of equilibria. We first deal with tâtonnement processes (5.7 and 5.6) and then go on to non tâtonnement processes. We need some standard results of differential topology recalled in 5.3 and 5.4 (transversality, partitions of unity). First we recall some conditions commonly found in discussions on price adjustment processes.

5.1. Walras-Law, Homogeneity, etc.

We are dealing with processes (T) or (NT). Cf. 1.1.

First the Walras-law:

(W) Prices and excess demand are related by $\sum_{i=1}^{n} p_i h_i = 0$

Homogeneity of the demand functions reflects that if all prices go up by the same factor, excess demand should be the same. This is the condition:

(H) $$h_i(\lambda p) = h_i(p) \quad \text{for all } \lambda > 0$$

We do not go deeply into the question of existence of equilibria (cf. however, § 8) and shall therefore have occasion to assume

(E) There exists a positive equilibrium price vector $p^* = (p_1^*, \ldots, p_n^*),\ p_i^* > 0$

The following two conditions for the s_{ij} in (NT) follow from the asumption that the total amount available of each commodity should remain constant.

(C) $$\sum_i s_{ij}(t) = c_j$$ $$\sum_i g_{ij}(p,s) = 0$$

For background material on all these conditions, cf [12]. In this section we shall for simplicity assume that we are dealing with a process given by

(T') $$\dot{p}_i = h_i(p) = f_i(p)$$

instead of

(T) $$\dot{p}_i = f_i(p)$$

Because we assume $\text{sign}(f_i(p)) = \text{sign}(h_i(p))$, this makes no difference as far as the equilibria are concerned. However, in order to apply arguments of this section to (T) instead of (T'), small (i.e. $\varepsilon - C^1$)

changes in the h_i should correspond to small changes in the f_i.
If e.g. $f_i = F_i(h_1, \ldots, h_n)$ and $|\det (\frac{\partial F_j}{\partial h_i})| \geq \delta$ for some fixed $\delta > 0$ this is assured. One could for instance take

$$(T_r) \qquad \dot{p}_i = r_i h_i(p)$$

where the r_i are positive rates of adjustment. For these processes (T_r) the arguments of this section go through unchanged.
Let $g : U \to \mathbb{R}^n$ be a C^1-map (= continuously differentiable map), where U is an open subset of $\mathbb{R}^n$, let $A \subset U$ be compact subset of U.

An ε-C^1-A-perturbation of $g : U \to \mathbb{R}^n$ (resp. the system $\dot{p} = g(p)$ on U) is a C^1-function $g' : U \to R^n$ (resp. a system $\dot{p} = g'(p)$ on U such that $||g(x) - g'(x)|| < \varepsilon$ for all $x \in A$ and $|\frac{\partial g_i}{\partial x_k}(x) - \frac{\partial g_i'}{\partial x_k}(x)| < \varepsilon$ for all $i, k = 1, \ldots, n$, $x \in A$.

5.2. <u>Definition</u> (<u>Transversality in a Point</u>)

Let $x \in A$, $y \in \mathbb{R}^n$ a fixed point. The map g is said to be transversal to y in x if either

(i) $y \neq g(x)$, or

(ii) $y = g(x)$ and $Dg(x)$ has rank n.

The map g is tranversal in A to y, if it is transversal to y for all $x \in A$.
Cf. e.g. [1], where a far more general notion of transversality is discussed.
Of transversality theory (cf. [1], [10]) we need the (fairly weak) results:

5.3. <u>Proposition</u>. ([10] Weak Transversality Theorem p. 27, Lemma 1 p.45)

(i) Let $g : U \to \mathbb{R}^n$ be a C^1-map, $y \in R^n$, $A \subset U$, A compact and let g be transversal to y in A. Then there exists an $\varepsilon > 0$ such that every ε-C^1-A perturbation g' of g is also transversal to y in A.

(ii) Let $g : U \to R^n$ be any C^1-map; y and A as before. Then for every $\varepsilon > 0$ there exists an ε-C^1-A perturbation of g which is transversal to y in A.

(iii) Let $g : U \to R^n$ be a C^1-map which is transversal to $y = f(x)$ in x. Then for every neighbourhood V of x in U, there exists an $\varepsilon > 0$ such that for every ε- C^1-perturbation $g' : U \to R^n$ there is an $x' \in V$, with $g'(x') = y$ and g' transversal to y in x'. Moreover, if we take V and ε small enough, there is precisely one $x' \in V$ such that $g'(x') = y$.

Proof. For (i), (ii) and the first part of (iii) we refer to [10].

The second part of (iii) is then proved by a standard argument. Because of transversality of g to y in x, and because $\{h \in R^n \mid ||h|| = 1\}$ is compact there is a positive number m such that

$$(*) \qquad ||\frac{\partial g}{\partial h}(x)|| \geq m$$

for all h, $||h|| = 1$. Take V small enough so that

$$|\frac{\partial g_i}{\partial h}(x) - \frac{\partial g_i}{\partial h}(x')| < \varepsilon$$

for all $x' \in \bar{V}$, $h \in R^n$, $||h|| = 1$, $i = 1, \ldots, n$. This can be done, again, because the set of these h is compact. Now let g' be an ε-C^1-perturbation of g. And suppose that there are two different solutions z, $z' \in V$ of $g'(x) = y$. Then for each $i = 1, \ldots, n$, there is a ϑ_i, $0 < \vartheta_i < 1$, such that

$$\frac{\partial g_i'}{\partial h}(z + \vartheta_i h) = 0$$

where $h = z' - z$. We then have for this particular h

$$|\frac{\partial g_i}{\partial h}(x)| \leq |\frac{\partial g_i}{\partial h}(x) - \frac{\partial g_i}{\partial h}(z + \vartheta_i h)| + |\frac{\partial g_i}{\partial h}(z + \vartheta_i h) - \frac{\partial g_i'}{\partial h}(z + \vartheta_i h)| + |\frac{\partial g_i'}{\partial h}(z + \vartheta_i h)|$$

$$\leq \varepsilon + \varepsilon + 0 = 2\varepsilon$$

Thus $||\frac{\partial g}{\partial h}(x)|| \leq 2.\sqrt{n}.\varepsilon$ for this particular h, which contradicts (*) if ε is small enough. q.e.d.

The C^1 compact open topology on $C^1(U,\mathbb{R}^n)$, the set of C^1-functions $U \to \mathbb{R}^n$, is defined by taking as open sets, the sets $V_{A,\varepsilon}(f) = \{g \in C^1(U,R^n) \mid g$ is an ε-C^1-A perturbation of $f\}$ for all $\varepsilon > 0$, $A \subset U$, A compact. With this topology on $C^1(U,\mathbb{R}^n)$, (i) and (ii) of the proposition above say that the maps transversal to y in A are open and dense in $C^1(U,\mathbb{R}^n)$.
Suppose that $x \in U$, $g(x) = y$, and g transversal to y in x. Then because dim U = n = dim R^n, we have by the inverse function theorem that there exist an open neighbourhood V of x and W of y such that g induces an homeomorphism $g : V \to W$.

It follows that there are finitely many solutions (or none) of $g(x) = y$ if g is transversal to y in A. (Because A is compact).
In the following we shall often deal with functions $f : U \to \mathbb{R}^n$ defined on a subset $U \subset S^{n-1}$, such that $p_1f_1(p) + \ldots + p_nf_n(p) = 0$ for all $p \in U$. By "abus de language" we shall call such a function "transversal" in A to $0 \in \mathbb{R}^n$ if for every $a \in A$ there is an i for which $a_i \neq 0$ and for which the function

$$(f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n) : U \to \mathbb{R}^{n-1}$$

is transversal to $0 \in R^{n-1}$ in $a \in U$.
A second tool we need is the existence of certain functions (Partitions of unity). The proposition below is rather special and covers precisely the case we need. Let $S^{n-1} = \{x \in R^n \mid x_1^2 + \ldots + x_n^2 = 1\}$. If F_1 and F_2 are two disjoint closed subsets of S^{n-1}, then their distance

$\rho(F_1,F_2) = \inf_{x\in F_1, y\in F_2} ||x - y||$ is positive (because F_1 and F_2 are compact).

5.4. <u>Proposition</u>.

Let $\delta > 0$, then there exists a constant K depending only on δ, such that for every two closed subsets $F_1, F_2 \subset S^{n-1}$ with $\rho(F_1, F_2) \geq \delta$, there exists a C^1-function $\phi : S^{n-1} \to \mathbb{R}$ with the properties.
a. $\phi(x) \in [0,1]$ for all $x \in S^{n-1}$
b. $\phi(x) = 1$ if $x \in F_1$
c. $\phi(x) = 0$ if $x \in F_2$

d. $\frac{\partial\phi}{\partial y}(x) = 0$ if $x \in F_1 \cup F_2$ for all directions $y \in R^n$ tangent to S^{n-1} in x

e. $|\frac{\partial\phi}{\partial y}(x)| < K$ for all $x \in S^{n-1}$ and all directions $y \in R^n$, $||y|| = 1$ tangent to S^{n-1} in x.

5.5. Boundary Conditions.

We now return to the tâtonnement process (T). Prices will in any case be assumed to be non negative. Assume that the Walras law (W) holds. Then the prices move along the spheres $\sum_{i=1}^{n} p_i^2$ = constant. Indeed,

$\frac{d}{dt}(\sum_{i=1}^{n} p_1^2) = 2\sum_{i=1}^{n} p_i\dot{p}_i = 2\sum_{i=1}^{n} p_i h_i(p) = 0$ by (W). We can therefore assume that we have $\sum_{i=1}^{n} p_i^2 = 1$, and from now on in this section, we shall do so.

We examine two types of boundary behaviour of the function f when one or more of the prices tend to zero.

(P) There exist (small) constants $c > 0$, $d > 0$ such that $f_i(p) \geq d$ if $0 < p_i < c$, $i = 1, \ldots, n$

This is a condition rather similar to the one used by Debreu in [5], and it reflects that for each i there is someone who desires the i-th commodity. Cf. also [8] for further details. This condition implies that a solution to (T) starting in $p^o > 0$, has $p(t;p^o) > 0$ for all $t \geq 0$.

Another possibility is that nothing special happens to the f_i if one or more of the prices tend to zero; especially: f_i does not become infinite as p_i goes to zero. For such a system it seems not unreasonable to assume that f is continuously differentiable on the compact subset $A = \{p \in S^{n-1} | p_i \geq 0, i = 1, \ldots, n\}$ of the sphere $S^{n-1} = \{p \in R^n | p_1^2 + \ldots + p_n^2 = 1\}$. Here, as in [11], we interpret differentiability to mean that there exist an open subset $U \subset S^{n-1}$, containing A, such that there exists a C^1-function f' on U which agrees with f on A. Thus we get the condition:

(D) f is differentiable on $A = \{p \in S^{n-1} | p_i \geq 0, i = 1, \ldots, n\}$

Let (T'): $\dot{p} = f(p)$ be a tâtonnement process on $U = \{p \in S^{n-1} | p_i > 0, i = 1, \ldots, n\}$. If (T') satisfies (p), there are c, d > 0 such that $f_i(p) \geq d$ if $0 < p_i < c$. We then denote by $A_f \subset U$ the subset of U, $A_f = \{p \in U | p_i \geq c, i = 1, \ldots, n\}$. There are of course many subsets A_f which can be obtained in this way, but it generally does not matter which one we pick.

5.6. Theorem.

(i) Let (T) : $\dot{p} = f(p)$ be a tâtonnement process on U satisfying (P) and (W). Then for every $\varepsilon > 0$ there exists a ε-C^1-perturbation $\dot{p} = g(p)$ of (T'), satisfying (P) and (W), such that g is transversal to $0 \in R^n$ in A_f. In particular, the perturbed process has only finitely many equilibria.

(ii) Let (T'): $\dot{p} = f(p)$ be a tâtonnement process on U satisfying (P) and (W), such that f is transversal to $0 \in R^n$ in A_f. Then there exists an $\varepsilon > 0$ such that any ε-C^1-perturbation, $\dot{p} = g(p)$, satisfying (W) also satisfies (P), and such that g is transversal to $0 \in R^n$ in A_f. In particular all ε-C^1-perturbations of (T') which satisfy (W), also have only finitely many equilibria.

(iii) Let (T) be as above in (ii). Then there exists an $\varepsilon > 0$, such that every ε-C^1-perturbation of (T') which satisfies (T') has the same number of equilibrium points as (T').

We topologize $C^1_{PW}(U,R^n) = \{f \in C^1(U,R^n) | (P)$ and (W) are satisfied} by means of the open sets $V_\varepsilon(f) = \{g \in C^1_{PW}(U,R^n)\} |$ g is an ε-C^1-perturbation of f}. Then 4.6 (i), (ii) say that there is an open and dense set in $C^1_{PW}(U,R^n)$ of processes with only finitely many equilibrium points.

Proof. (i). Let $U_f = \{p \in U | p_i > \frac{1}{2}c, i = 1, \ldots, n\}$ and let d > 0 be such that $f_i(p) \geq d$ if $0 < p_i < c$. Consider

$$V = \{(p_1, \ldots, p_{n-1}) \in R^{n-1} | p_1^2 + \ldots + p_{n-1}^2 < 1; p_i > 0\}$$

There is a 1-1 correspondence between V and U (which is a homeomorphism) given by

$$(p_1, \ldots, p_{n-1}) \leftrightarrow (p_1, \ldots, p_n)$$

where p_n is the unique positive number such that $p_1^2 + \ldots + p_n^2 = 1$.
Now let

$$f' : V \to \mathbb{R}^n$$

be defined by

$$f'(p_1, \ldots, p_{n-1}) = (f_1(p_1, \ldots, p_n), \ldots, f_n(p_1, \ldots, p_n))$$

where p_n is determined as above. Take in any case $\varepsilon < d$. Now apply 5.3(ii) to find an ε'-C^1-B-perturbation h' of f' which is transversal to $B = \{(p_1, \ldots, p_{n-1}) | (p_1, \ldots, p_n) \in \bar{U}_f\}$, where p_n is determined as before.
Define,

$$h : U \to \mathbb{R}^n$$

by, $h_1(p) = h_1'(p_1, \ldots, p_{n-1}), \ldots, h_{n-1}(p) = h_{n-1}'(p_1, \ldots, p_{n-1})$,

$$h_n(p) = -\frac{1}{p_n} \sum_{i=1}^{n-1} p_i h_i(p).$$

Then for sufficiently small ε', the function h is an ε-C^1-$\bar{U}_f$-perturbation of f (because $p_n \geq \frac{1}{2}c$ and $p_1, \ldots, p_{n-1} \leq 1$ on $\bar{U}_f$), and h is transversal to $0 \in \mathbb{R}^n$ in $\bar{U}_f$ and hence certainly in A_f.

Now let $F_1 = A_f$ and $F_2 = \{p \in S^{n-1} | \exists_i$ such that $p_i \leq \frac{1}{2}c\}$
Then $F_1 \cap F_2 = \phi$ and both are closed. We can therefore apply 5.4 to find a function $\phi : S^{n-1} \to R$ with the properties listed in 5.4.
Now define

$$g : U \to R^n$$

by the formula

$$g(p) = \phi(p)h(p) + (1 - \phi(p))f(p)$$

We then have

$$||g(p) - f(p)|| = ||\phi(p)h(p) - \phi(p)f(p)|| = \phi(p)||h(p) - f(p)|| < \varepsilon$$

for all $p \in U$. Indeed if $p \in \bar{U}_f$ we have $||h(p) - f(p)|| < \varepsilon$ and

$0 \leq \phi(p) \leq 1$ and if $p \in U \setminus \bar{U}_f \subset F_2$, $\phi(p) = 0$.

And for a tangent direction y to $p \in U$, $||y|| = 1$, we have

$$\frac{\partial g_i}{\partial y}(p) = \frac{\partial \phi}{\partial y}(p)\, h_i(p) + \phi(p)\frac{\partial h_i(p)}{\partial y} - \frac{\partial \phi}{\partial y}(p) f_i(p) - \phi(p)\frac{\partial f_i}{\partial y}(p) + \frac{\partial f_i}{\partial y}(p)$$

and hence

$$\left|\frac{\partial g_i}{\partial y}(p) - \frac{\partial f_i}{\partial y}(p)\right| \leq \phi(p)\left|\frac{\partial h_i}{\partial y}(p) - \frac{\partial f_i}{\partial y}(p)\right| + \left|\frac{\partial \phi}{\partial y}(p)\right|\, |h_i(p) - f_i(p)| \leq$$

$$\leq \varepsilon K + \varepsilon.$$

Indeed if $p \in \bar{U}_f$, $|h_i(p) - f_i(p)| < \varepsilon$, and $\left|\frac{\partial \phi}{\partial y}(p)\right| \leq K$ and $\left|\frac{\partial h_i}{\partial y}(p) - \frac{\partial f_i}{\partial y}(p)\right| < \varepsilon$, and if $p \in U \setminus \bar{U}_f$, $\frac{\partial \phi}{\partial y}(p) = 0$ and $\phi(p) = 0$. Moreover K does not depend on ε. (cf. 5.4). Thus if we had started with $\frac{\varepsilon}{K+1}$ instead of ε, we would have found the desired ε-perturbation. Note that g satisfies (W) and (P), and that g is transversal to 0 in A_f, because $g(p) = h(p)$ and $\frac{\partial g}{\partial y}(p) = \frac{\partial h}{\partial y}(p)$ for $p \in A_f$.

(ii) This follows immediately from 5.3 (i)

(iii) To prove (iii), let $e^1, \ldots, e^m$ be the equilibrium points of (T). For each $i = 1, \ldots, m$ take ε_i and V_i small enough so that 5.3 (iii) applies (with respect to the function $f' : U \to R^{n-1}$). Let $a = \min ||f(p)||$, $p \in A_f \setminus \bigcup_i V_i$. Take $\varepsilon = \min\{\frac{1}{2}d, \varepsilon_1, \ldots, \varepsilon_m, a\}$. (Note that $a > 0$ because $A_f \setminus \bigcup_i V_i$ is compact). q.e.d.

Remarks. 1. If we take (T_r) instead of (T') prices move along ellipsoids $\sum_{i=1}^{n} r_i^{-1} p_i^2$ = constant, if (W) is satisfied. The same proof works in this case.

Remarks.2. If we take instead of (W) the weaker condition (W'): $\sum_{i=1}^{n} p_i h_i \leq 0$ then $p(t;p^o)$ remains in $\sum_{i=1}^{n} r_i^{-1} p_i^2 \leq r$, a solid ellipsoid, if p^o is in this solid ellipsoid.
(We are dealing with (T_r), for (T') take $r_i = 1$, $i = 1, \ldots, n$).

Let $U_r = \{p \in R^n | p_i > 0, i = 1, \ldots, n; \sum_{i=1}^{n} r_i^{-1} p_i^2 \leq r\}$.

Assume that (P) is satisfied on U_r for some $r > 0$. Then the analogue of theorem 5.6 holds with U replaced by U_r. The proof is similar.

Now suppose that the second type of boundary behaviour occurs; i.e. that condition (D) is satisfied. Let $A = \{p \in S^{n-1} | p_i \geq 0 \ i = 1, \ldots, n\}$ and U some open neighbourhood of A in S^{n-1}, on which f is defined (and differentiable).

5.7. Theorem.

(i) For every process (T') : $\dot{p} = f(p)$ such that (W) and (D) are satisfied and every $\varepsilon > 0$ there exists an ε-C_1-perturbation $\dot{p} = g(p)$ satisfying (W) and (D), such that g is transversal to 0 in A. In particular the perturbed system has only finitely many equilibria.

(ii) Let (T') $\dot{p} = f(p)$ satisfy (W) and (D) and suppose that f is transversal to 0 in A. Then there exists an $\varepsilon > 0$ such that every ε-C^1-perturbation $\dot{p} = g(p)$ of (T'), satisfying (W) and (D), has g transversal to 0 in A. Hence all ε-C^1-perturbations of f also have only finitely many equilibrium points.

Note that there does not exist a precise analogue of 5.6 (iii), because a boundary equilibrium point (i.e. an equilibrium point with at least one price zero) can disappear into the region where at least one price is negative under a small perturbation. If all the equilibrium points in A of a given process $\dot{p} = f(p)$ are in the interior of $A \subset S^{n-1}$ and f is transversal to 0 in A then one proves , as in 5.6 (iii), that a small perturbation of $\dot{p} = f(p)$ has the same number of equilibrium points.

<u>Proof</u>. Let $U_i = \{p \in U | p_i > 0\}$. Note that $\bigcup_i U_i$ is an open set in S^{n-1} which contains A. Thus by restricting U a bit if necessary we can assume that $\bigcup_i U_i = U$. Let $f(i) : U_i \to R^{n-1}$ be the map $f(i)(p) = (f_1(p), \ldots, f_{i-1}(p), f_{i+1}(p), \ldots, f_n(p))$.

B looking at each of the f(i) in turn one now easily proves (ii) as a consequence of 5.3 (i).

As to (i) : Let $G_i = \{p \in A | p_i = 0\}$.

Because the G_i are compact, there is a $\delta > 0$ such that

$$V = \{p \in S^{n-1} | p_i > -\delta \text{for all } i\}$$

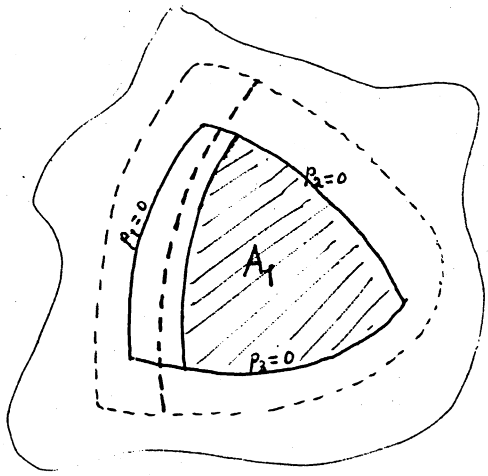
is contained in U. By taking δ a bit smaller if necessary we can also see to it that $\bar{V} \subset U$. Further let $\delta' < \frac{1}{\sqrt{n}}$, and let

$$A_i = \{p \in A | p_i \geq \delta'\}$$

then $\bigcup_{i=1}^{n} A_i = A$ and the A_i are compact.

Finally let $V_i = \{p \in V | p_i > \frac{1}{2}\delta'\}$.

Choose $\varepsilon > 0$. We now use the same arguments as in the proof of Theorem 5.5 to construct a $\frac{\varepsilon}{2}$ - C^1-perturbation of f which is transversal to $0 \in R$ in A_1. In the construction, U is replaced by U, U_f is replaced by V_1, A_f is replaced by A_1. Let $\delta_2 > 0$ be such that any δ_2-C^1-perturbation of g' is still transversal 0 in A_1. Such a δ_2 exists by 5.6 (ii), which has already been proved.



The thin outer line is the boundary of U; the dotted line is the boundary of V. The lines $p_1 = 0$, $p_2 = 0$, $p_3 = 0$ form the boundary of A, and the ///// part of A is A_1. The boldly dotted line marks the boundary of V_1 in V.

Let $\varepsilon_2 = \min(\frac{\varepsilon}{4}, \delta_2)$ and using (g', U, V_2, A_2) instead of (f, U, V_1, A_1) construct a ε_2-C^1-perturbation g" of g' which is transversal to 0 in A_2.

Because $\varepsilon_2 \leq \delta_2$, we then have that g" is transversal to 0 in $A_1 \cup A_2$. Let $\delta_3 > 0$ be such that any δ_3-C^1-perturbation of g" is still transversal to $0 \in R^n$ in $A_1 \cup A_2$, etc. etc. The construction of 5.5 yields functions which satisfy the Walras law. Therefore we finally (after n steps) wind up with an ε-perturbation g of f which is transversal to 0 in $\bigcup_{i=1}^{n} A_i = A$ and which satisfies (W) and (D) (by construction) q.e.d.

5.8. Non Tâtonnement Processes.

For the process (NT) with conditions (W), (C) and either (P) or (D) one can derive theorems similar to (5.6) and (5.7). (The conditions (C) are easy to handle; far easier than (W)).

5.9. Remark.

It may happen that for some prices boundary condition (D) applies, while for other we have (P). This case can be dealt with in more or less the same way.

6. LIAPUNOV FUNCTIONS

First we recall the definitions. As in nr.2 we consider a system (DS) : $\dot{x} = f(x)$ defined on a set $M \subset R^n$ such that (B) is satisfied. Let e be an equilibrium of (DS).

6.1. Definitions.

(i) A function Φ defined in a neighbourhood N of e such that $\Phi(e) = 0$, $\Phi(x) > 0$ for all $x \in N \setminus \{e\}$ and such that $\Phi(x(t;x^o)) < \Phi(x^o)$ for all $x^o \in N$, $x^o \neq e$ and $t > 0$ such that $x(t;x^o) \in N$, is called a local Liapunov function for (DS) near e.

(ii) If Φ is defined on all of M, it is called a global Liapunov function.

(iii) A function Φ on M such that $\Phi(x(t;x^o)) \leq \Phi(x^o)$ for all $t > 0$ unless x^o is an equilibrium point is called a modified Liapunov function (cf. [17]) for (DS).

The main theorem concerning Liapunov function is:

6.2. Theorem.

Let the dynamical system (DS) satisfy (B); let e be an equilibrium point of (DS). Then e is (locally) asymptotically stable if and only if (DS) admits a Liapunov function near e. The point e is globally asymptotically stable if and only if (DS) admits a global Liapunov function.

(cf. [4] Ch. V §2).

Concerning modified Liapunov functions Uzawa proved (cf. [17]):

6.3. Proposition.

Let (DS) satisfy (B). And suppose that

(i) Every motion $x(t;x^o)$ is contained in some compact subset of Ω $(= \{x \in R^n \mid x_i > 0, i=1,\dots,n\})$

(ii) (DS) admits a modified Liapunov function

(iii) The set E of equilibria of (DS) is countable.

Then (DS) has a pointwise attracting equilibrium set E.

Now suppose that the set E of equilibria of (DS) is finite, and let conditions (i) and (ii) of 6.3 also be satisfied. Let $E = \{e^1, \dots, e^m\}$. Consider $\Phi(e^1), \dots, \Phi(e^m)$, where Φ is a modified Liapunov function. Renumbering the e^i if necessary we can assume that in $e = e^1$, $\Phi(e) \leq \Phi(e^i)$ $i = 1, \dots, n$. Now let $x^o \in M$ be a nonequilibrium point. There is an index i such that $\lim_{t\to\infty} x(t;x^o) = e^i$ and we have $\Phi(x(t;x^o)) < \Phi(x^o)$.

Therefore $\Phi(x^o) > \Phi(e^i) \geq \Phi(e)$. It follows that the function

$$\Psi(x) = \Phi(x) - \Phi(e)$$

is a Liapunov function for (DS) near e. We have proved (using 6.3 and 6.2)

6.4. Proposition.

Let (DS) satisfy (B). And suppose that:

(i) Every motion $x(t;x^o)$ is contained in some compact subset of M.

(ii) (DS) admits a modified Liapunov function

(iii) The set E of equilibria is finite.

Then there is an asymptotically stable equilibrium point of (DS).

The property "(DS) has an asymptotically stable equilibrium point" is a good notion with respect to perturbations of (DS). In fact

6.5. Proposition (cf.[7] Ch. VII section 56)

An asymptotically stable equilibrium point of (DS) is totally stable.

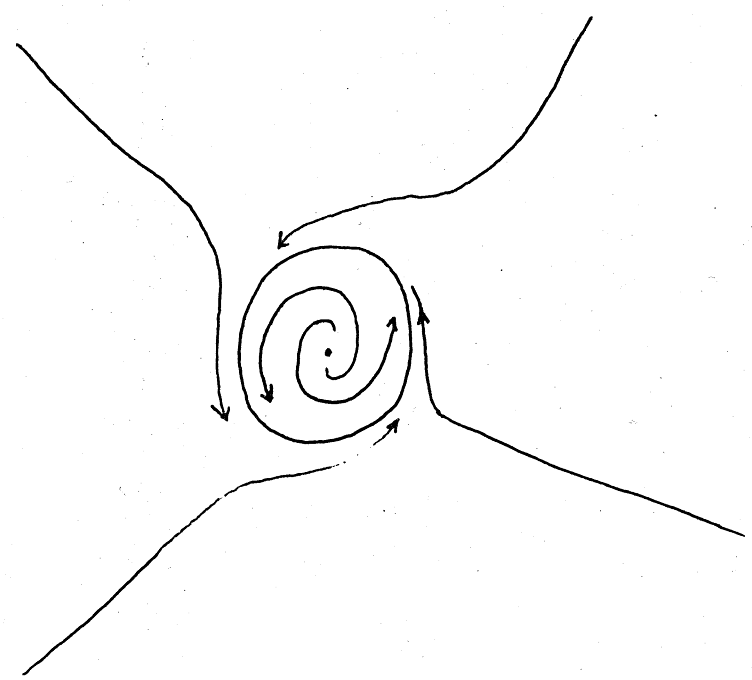
Systems (DS) satisfying conditions (i), (ii), (iii) of 6.4 also behave nicely under repeated disturbances. One can show that for every $\varepsilon > 0$ there are a $K > 0$, $T > 0$ such that for almost all sequences of disturbances in $S_{T,K}$ $||x_{pert}(t,x^o) - e^i|| < \varepsilon$ for some asymptotically stable equilibrium point e^i provided t is large enough; and $\lim_{t\to\infty} x_{pert}(t;x^o) = e^i$ for almost all sequences of disturbances in S_K. Cf. [9].

6.6. Asymptotic stability and structural stability.

Asymptotically stable equilibrium points do not behave as nicely with respect to structural. Consider for example the system

$$(*) \qquad \begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y^3 \end{aligned}$$

on R^2. For every $\varepsilon > 0$, here is an ε-C^1-perturbation of (*) which has a phase portrait like



In fact, let $\psi(r)$ be a differentiable function $\psi : [0,\infty) \to [0,2]$ such that $\psi(x) = 0$, $x \geq 2\varepsilon$,

$\psi(0) = 2$, $\psi(t) \in (1,2)$ if $0 < t < \varepsilon$, $\psi(\varepsilon) = 0$, $\psi(t) \in (0,1)$ if $\varepsilon < t < 2\varepsilon$, and $|\frac{d\psi}{dt}| \leq \frac{2}{\varepsilon}$. (Such a function exists). Define $\phi(x,y) = \psi(\sqrt{x^2+y^2})$. The system

$$\dot{x} = -x^3(1 - \phi(x,y)) - \phi(x,y)y^3$$
$$\dot{y} = -y^3(1 - \phi(x,y)) - \phi(x,y)x^3$$

is then a $(6\varepsilon + 2\sqrt{\varepsilon})$ - C^1-perturbation of $\dot{x} = -x^3$, $\dot{y} = -y^3$; and this perturbed system has the phase portrait sketched above.

However, let

$$\dot{x} = f(x)$$

be a (DS) defined on an open subset $U \subset R^n$, which contains the origin 0. Let 0 be an equilibrium point of (S). Consider the matrix

$$A = (\frac{\partial f_i}{\partial x_j}(0))$$

If all the eigenvalues of A have a nonzero real part, then 0 is a structurally stable equilibrium point. Cf [15] 11.2

One can further prove that if M is a smooth differentiable manifold and (DS) a smooth dynamical system on M, with a globally asymptotically stable equilibrium point e such that the matrix $\frac{df}{dx}(e)$ has no eigenvalues with zero real part, where $x = (x_1, \ldots, x_n)$ is a smooth coordinate system for a neighbourhood of e in M, then (DS) is structurally stable Cf. [15].

7. GROSS SUBSTITUTABILITY AND REVEALED PREFERENCES.

We again consider a tâtonnement process (T) defined on $\Omega = \{(p_1, \ldots, p_n) | p_i > 0\}$. Almost the same arguments as in 5.4 give

7.1. Proposition.

Let (T) satisfy (B), and suppose that

(i) (T) admits a modified Liapunov function

(ii) (T) has a globally attracting equilibrium point.

Then this equilibrium point is globally asymptotically stable.

Now suppose that (T) satisfies (H). The price vector p is an equilibrium iff λp is an equilibrium, and $f(\lambda p) = f(p)$. Therefore we can and shall view the process as taking place on U,
$U = \{(p_1, \ldots, p_n) | p_i > 0, p_1^2 + \ldots + p_n^2 = 1\}$.

(If we are only interested in equilibrium points and their stability this does not matter; if one is also interested in the time it takes to get into the neighbourhood of an equilibrium point, this does matter, however).

The commodities involved in the process (T) are said to be <u>strongly gross substitutes if</u>:

the excess demand function $h(p) = (h_1(p), \ldots, h_n(p))$ is differentiable

(S) at all points $p \in \Omega$ and

$$\frac{\partial h_i}{\partial p_j} > 0 \text{ for all } i \neq j$$

7.2. <u>Theorem</u>.

Let the process (T) satisfy (B), (H), (E) and (S). Then (T) has a globally asymptotically stable equilibrium point.

Proof. Arrow, Block, Hurwicz [3] show that under this condition there is precisely one (up to scalar multiples) equilibrium price vector $(p_1^*, \ldots, p_n^*)$, $p_i^* > 0$.

Uzawa [17], then proves that the function $\Lambda(p) = \max_{j=1,\ldots,n} \frac{p_j}{p_j^*}$

is a modified Liapunov function for (T). It follows that

$\Lambda(p) = \max_j \frac{p_j}{p_j^*} - 1$ is a global Liapunov function, which proves 7.2. q.e.d.

We now examine, as in [17] a process

$$(T^*) \qquad \dot{p}_i = \begin{cases} 0 & \text{if } p_i = 0,\ f_i(p) < 0 \\ f_i(p) & \text{otherwise} \end{cases}$$

$$f_i(p) = r_i h_i(p)$$

where the r_i are positive numbers (speeds of adjustment). The weak axiom of revealed preference says

(RP) $$p^* h(p) = \sum_{j=o}^{n} p_j^* h_i(p) > 0$$

for all equilibria p^* and nonequilibria p

7.3. Theorem.

(i) If the process (T^*) satisfies (H), (E), (W), (B), (PR) then (T^*) has a pointwise attracting equilibrium set E.

(ii) Every $e \in E$ is stable.

(iii) If T^* has only finitely many equilibrium points I, then it has precisely one equilibrium point which is globally asymptotically stable.

Proof. (i) is proved by Uzawa. He shows that the function $\Phi_*(p) = \sum_{i=1}^{n} \frac{1}{r_i}(p_i - p_i^*)^2$ (p^* a fixed equilibrium point) is a modified Liapunov function for all p^*. This implies (ii). The condition of (iii) makes of course no sense if we consider (T^*) as a process on Ω. Because of (W) and (H), however, we can just as well examine the behaviour of (T) on $A = \{p \in S^{n-1} | p_i \geq 0$ (or (on) $U = \{p \in S^{n-1} | p_i > 0\}$)

The space A is connected, i.e. it can not be written as the union of two relatively open disjoint subsets of A. Let E be finite. For each $e \in E$, we define $U_e = \{p^o \in A | \lim_{t\to\infty} p(t;p^o) = e\}$. Then U_e is open. We proceed to prove this. The function $d(p,p') = \sqrt{\sum_{i=1}^{n} \frac{1}{r_i}(p_i - p_i')^2}$ is a metric on U. It follows that if $p^o \in A$, is such that $d(p^o,e) < d(p^o,e'), e' \in E \setminus \{e\}$ then $\lim_{t\to\infty} p(t;p^o) = e$. Thus every $e \in E$ has a small open neighbourhood V_e such that $\lim_{t\to\infty} p(t;p^o) = e$ if $p^o \in V_e$. (We have therefore shown that the points $e \in E$ are all asymptotically stable).

Now let $p^o \in U_e$. There exists a t_o such that $p(t_o;p^o) \in V_e$ Because $p(t_o;p^o)$ is a continuous function of p^o and V_e is open, there exists a neighbourhood V of p^o such that $p(t_o,p^1) \in V_e$ for all $p^1 \in V$, but then $\lim_{t\to\infty} p(t;p^1) = e$. This proves that all the U_e are open. However, because of (i) we have $\bigcup_{e\in E} U_e = A$, and of course $U_e \cap U_{e'} = \phi$

if $e \neq e'$. This contradicts the fact that A is connected, unless $E = \{e\}$. q.e.d.

Remarks. 1. Part (iii) of this theorem can also be formulated as: if there are finitely many equilibrium rays of (T*) then there is precisely one equilibrium ray, which is globally asymptotically stable.

2. Without the requirement that E be finite one can show that E must be connected.

8. STABILITY UNDER REPEATED DISTURBANCES.

Let (DS) be a dynamical system on $M \subset R^n$. As in 4.2 we consider disturbed motions $x_{pert}(t;x^o)$ under a sequence of disturbances of magnitudes $u_1, u_2, \ldots$ occurring at times $0 < t_1 < t_2 < \ldots$.

Let $\bar{x}$ be an asymptotically stable equilibrium point of (S). Let $U = \{x^o \in M | \lim_{t\to\infty} x(t;x^o) = \bar{x}\}$. Then U is open in M. (If $\bar{x}$ is globally asymptotically stable $U = M$). There is a Liapunov function Φ defined on U.

A Liapunov function Φ is a kind of generalized energy function. It is therefore not unreasonable (especially if Φ arises in a natural way) to measure the boundedness of the disturbances in terms of Φ. Given $c > 0$, we define $S^{\Phi}_{T,c}$ as the family of those disturbances $(t_1, t_2,\ldots; u_1, u_2,\ldots)$, such that

$$\sum_{t \le t_i \le T} u_i \le c$$

The u_i give the magnitude of the disturbance at time t_i in terms of Φ. Thus if $x_{pert}(t;x^o)$ is the disturbed motion one has

$0 < t < t_1$	$x_{pert}(t;x^o) = x(t;x^o)$
$t = t_1$	$x_{pert}(t_1;x^o) = x^1_m$, where $\Phi(x^1) - \Phi(x(t_1;x^o)) = u_1$
$t_1 < t < t_2$	$x_{pert}(t;x^o) = x(t-t_1;x^1)$
$t = t_2$	$x_{pert}(t;x^o) = x^2$, where $\Phi(x^2) - \Phi(x(t_2-t_1;x^1)) = u_2$
$t_2 < t < t_3$	$x_{pert}(t;x^o) = x(t-t_2;x^2)$

. . . . .

Let $\bar{V} \subset U$. We define $e_{\bar{V}}$ as

$$e_{\bar{V}} = \sup \{d \mid x \in M,\ \Phi(x) - \max_{y\in\bar{V}} \Phi(y) < d \Rightarrow x \in U\}$$

Note that there always exist $\bar{V}$ in U, such that $e_{\bar{V}} > 0$.

To prove theorems we need a slightly better situation then just a Liapunov function on U. We need a differentiable Liapunov function. Fortunately these always exist under very mild conditions. For instance when f satisfies a global Lipschitz condition on M, there is a differentiable Liapunov function on U. Cf. [7].

If f is differentiable in a neighbourhood W of $\bar{x}$, there is a $U \subset W$ such that $\bar{U} \subset W$ and f satisfies a Lipschitz condition on U.

8.1. Theorem.

Let (DS) a dynamical system on M, $\bar{x}$ an asymptotically stable equilibrium point of M and Φ a differentiable Liapunov function defined on an open neighbourhood U of $\bar{x}$ such that $\bar{U}$ is compact. Then for every compact $\bar{V}$ such that $e_{\bar{V}} > 0$, every $\varepsilon > 0$, and every $T > 0$ there exist c',c" and t' > 0, t" > 0 such that

(i) $\quad \Phi(x_{pert}(t;x^{o})) < \varepsilon \qquad$ for $t \geq t'$

(ii) $\quad \|x_{pert}(t;x^{o}) - \bar{x}\| < \varepsilon \qquad$ for $t \geq t''$

for all $x^{o} \in \bar{V}$ and all disturbances of $S^{\Phi}_{T,c'}$ in case (i) and all disturbances of $S^{\Phi}_{T,c''}$ in case (ii).

Proof. Part(ii) follows from part (i) by choosing a $\delta > 0$ such that $\Phi(x_{pert}(t;x^{o})) < \delta \Rightarrow \|x_{pert}(t;x^{o}) - \bar{x}\| < \varepsilon$ and then applying part (i) with δ instead of ε. It remains to prove (i). Let $A_{\varepsilon} = \{x \in M \mid \Phi(x) = \varepsilon\}$. If ε is small enough A_{ε} is contained in $\bar{W} \subset U$ where $\bar{W} = \{x \in M \mid \exists y \in \bar{V},\ \Phi(x) \leq \Phi(y)\}$. For each $0 < \varepsilon' < \varepsilon$, let $B_{\varepsilon'} = \{x \in M \mid \Phi(x) \leq \varepsilon'\}$ and $U_{\varepsilon'} = \{x \in M \mid \Phi(x) < \varepsilon'\}$. Choose $e < e_{\bar{V}}$. Let $\bar{W}_e = \{x \in M \mid \exists y \in \bar{W},\ \Phi(x) - \Phi(y) \leq e\}$.
For each $x^{o} \in \bar{W}_e - U_{\varepsilon'}$, $\frac{d}{dt}\Phi(x(t;x^{o}))(0) < 0$. Let

$$\lambda = \max_{x^{o}} \frac{d}{dt}\Phi(x(t;x^{o}))(0),\ x^{o} \in \bar{W}_e \setminus U_{\varepsilon'}.$$

Then $\lambda < 0$, because $\bar{W}_e \setminus U_{\varepsilon'}$ is compact.

Because $e < e_{\bar{V}}$, $\bar{W}_e \subset U$. During each interval $[\bar{t}, \bar{t} + T]$ the loss in Φ due to the undisturbed motions occurring is at least $-\lambda T$ if $x_{pert}(t;x^o)$ remains in $\bar{W}_e \setminus U_{\varepsilon'}$.

We take

$$c' < \min\{\varepsilon - \varepsilon', -\lambda T, e\} = c$$

Then $x_{pert}(t;x^o)$ remains in any case in $\bar{W}_e$, if $x^o \in \bar{W}$. And

$$\Phi(x_{pert}(\bar{t};x^o)) - \Phi(x_{pert}(\bar{t} + T;x^o)) \geq c - c'$$

unless $\Phi(x_{pert}(t;x^o))$ passes through $U_{\varepsilon'}$ for some $t^* \in [\bar{t}, \bar{t} + T]$, but then $x_{pert}(t;x^o) \in U_\varepsilon$ for all $t \in [t^*, t^* + T]$. This proves the theorem.

8.2. Corollary (of the proof).

If $x_{pert}(t;x^o) \in U_{\varepsilon'}$ for some t, then $x_{pert}(t';x^o) \in U_\varepsilon$ for all $t' \geq t$.

Remarks.

1. If ε' in the proof goes to zero, $\lambda \to 0$ (monotonically). An optimum c' is obtained by taking ε' such that $\varepsilon - \varepsilon' = -\lambda T$.
2. If $U = M$, then the restriction $e_V > 0$ in the theorem can be removed.

If Φ is not a naturally arising function on M, it seems more reasonable to put the boundedness conditions on the disturbances in terms of the distances a point is moved by a disturbance, as in 4.1.

As before let $U \subset M$ be a neighbourhood of $\bar{x}$ such that there is a differentiable Liapunov function defined on U, and such that $\bar{U}$ is compact. If $\bar{V} \subset \bar{U}$ we define d_V by

$$d_V = \sup\{d \mid x \in M \text{ and } \exists y \in \bar{W},\ ||x - y|| \leq \delta \Rightarrow x \in U\}$$

where as before $\bar{W} = \{x \in M \mid \exists \in \bar{V},\ \Phi(y) \geq \Phi(x)\}$. Note that there are always $\bar{V}$ such that $\bar{W} \subset \bar{U}$ and $d_V > 0$.

8.3. Theorem.

Let (DS) be a dynamical system on M, $\bar{x}$ an asymptotically stable equilibrium point of M and Φ a differentiable Liapunov function defined on a neighbourhood U of $\bar{x}$ such that $\bar{U}$ is compact. Then for every compact $\bar{V}$ such that $\bar{W} \subset \bar{U}$ and $d_V > 0$, every $\varepsilon > 0$

and every $T > 0$ there exist a $K > 0$ and a $t_o > 0$ such that

$$||x_{pert}(t;x^o) - \bar{x}|| < \varepsilon$$

for all $t \geq t_o$, all $x^o \in \bar{V}$ and all disturbances in $S_{T,K}$

<u>Proof</u>. Let $A_\varepsilon = \{x \in M| \; ||x - \bar{x}|| = \varepsilon\}$. The set A_ε is compact, and $A_\varepsilon \subset \bar{W}$ if ε is small enough. Let $c_1 = \min \Phi(x)$, $x \in A_\varepsilon$.
Then $c_1 > 0$. Now for each δ let $c_2(\delta) = \max \Phi(x)$, $x \in B_\delta$, where $B_\delta = \{x \in M| \; ||x - \bar{x}|| \leq \delta\}$. Let $U_\delta = \{x \in M| \; ||x - \bar{x}|| < \delta\}$. Choose $d < d_\gamma$, and let $\bar{W}_d = \{x \in M| \exists y \in \bar{W}, \; ||x - y|| \leq d\}$.
For all $x \in \bar{W}_d \setminus U_\delta$, and all tangent directions y to M in x, $||y|| = 1$,

$$\frac{\partial}{\partial y}\Phi(x) < 0$$

Let $\mu = \max |\frac{\partial}{\partial y}\Phi(x)|$, $x \in \bar{W}_d \setminus U_\delta$, $||y|| = 1$, y tangent to M in x.
Then $\mu > 0$. As $\delta \to 0$, $c_2(\delta) \to 0$; choose some δ such that $c_2(\delta) < c_1$.
As in the proof of 8.1 let $\lambda = \max \frac{d}{dt}(\Phi(x(t;x^o))$, $x^o \in \bar{W}_d \setminus U_\delta$. Then also $\lambda < 0$. We have $|\Phi(x) - \Phi(x')| \leq |\mu| \; ||x - x'||$. Choose

$$K < \min \{d, \mu^{-1}(c_1 - c_2(\delta)), -\mu^{-1}\lambda T\} = K'$$

Then for every disturbance in $S_{T,K}$, $x_{pert}(t;x^o)$ will remain in $\bar{W}_d$, if $x^o \in \bar{V}$. The loss in Φ during the undisturbed parts of motion during interval $[\bar{t}, \bar{t} + T]$ is at least $-\lambda T$ if $x_{pert}(t;x^o)$ remains in $\bar{W}_d \setminus U_\delta$. The gain due to disturbances is at most μK. Thus during every interval $[\bar{t}, \bar{t} + T]$, Φ will diminish along $x_{pert}(t;x^o)$ by at least

$$-\lambda T - \mu K \geq \mu(K' - K)$$

unless $x_{pert}(t;x^o)$ passes through U_δ during $[\bar{t}, \bar{t} + T]$, but if $x_{pert}(t^*,x^o) \in U_\delta$, then $x_{pert}(t;x^o) \in U_\varepsilon$ for all $t \in [t^*, t^* + T]$. q.e.d.

8.4. <u>Corollary (of the proof).</u>

If $x_{pert}(t^*,x^o) \in U_\delta$ then $x_{pert}(t,x^o) \in U_\varepsilon$ for all $t \geq t^*$

Remarks.

1. As $\delta \to 0$, $c_2(\delta) \to 0$ (monotonically) and $\lambda \to 0$ (monotonically). An optimal K' is found by taking δ such that $c_1 - c_2(\delta) = \lambda T$.

2. If U = M. the condition $d_Y > 0$ can be removed.

9. EXISTENCE OF POSITIVE EQUILIBRIUM VECTORS.

In theorems on stability, those of nr. 7 e.g., the condition (E), that there be a positive equilibrium point repeatedly turns up. In this section we prove the existence of such an equilibrium point provided (P) and (W) are satisfied. (In fact one only needs to have a disk-like compact invariant region in Ω). We need a slightly stronger continuity condition on the solutions $p(t;p^o)$ of the dynamical system (DS):

(B") Condition (B) is satisfied and the function $p(t;p^o) : M \times \mathbb{R}(\geq 0) \to M$ is continuous as a function of (t,p^o)

This condition is e.g. satisfied if the function f of $\dot{p} = f(p)$ satisfies a global Lipschitz condition on M.

We define the n-dimensional ball D^n (n dimensional disk) as $D^n = \{ x \in R^n | x_1^2 + \ldots + x_n^2 \leq 1\}$

9.1. Proposition.

Let (DS) be a dynamical system on M such that (B") is satisfied. Suppose that M is homeomorphic to a disk. Then there is an equilibrium point of (DS) in M.

Proof. For each $n \in N$, let $f_n : M \to M$ be the function $f_n(x) = p(\frac{1}{n};x)$

Because M is homeomorphic to a disk, the Brouwer fixed point theorem can be applied to the maps f_n. Let x_n be a fixed point of f_n. The topological space M is compact (being homeomorphic to a disk), therefore there exists a subsequence $\{x_{k_n}\}$ of

x_n which converges to a point $\bar{x} \in M$. We show that $\bar{x}$ is a fixed point. Suppose not, then there exist a $t_o \geq 0$ and open neighbourhoods V of $\bar{x}$ and V' of $p(t_o;\bar{x})$ such that $V \cap V' = \emptyset$. The function $p(-;-)$: $M \times \mathbb{R}(\geq 0) \to M$ is continuous, thus there exist a neighbourhood $W \subset V$ of $\bar{x}$ and a $\delta > 0$ such that $p(t;x) \in V'$ if $|t - t_o| < \delta$ and $x \in W$.

Because $\lim_{n\to\infty} x_{k_n} = \bar{x}$, and $\lim_{n\to\infty} \frac{1}{k_n} = 0$, there exists a $k_{n_o} = j$ such that

1° $x_j \in W$

2° there is a multiple $t_1 = \frac{n_1}{j}$ of $\frac{1}{j}$ such that $|t_1 - t_o| < \delta$.

We then have on the one hand that $p(t_1;x_j) \in V'$. On the other hand $p(\frac{n_1}{j};x_j) = x_j \in W$ because x_j is a fixed point of f_j. A contradiction.

q.ed.

Now let $\Omega = \{p \in \mathbb{R}^n | p_i > 0\}$ and consider processes

(T') $$\dot{p}_i = h_i(p) = f_i(p)$$

or more generally

(T_r) $$\dot{p}_i = r_i h_i(p) = f_i(p), \; r_i > 0$$

Suppose that (W) is satisfied. Then prices move along spheres $\sum_{i=1}^{n} p_i^2 = r^2$ (resp. along ellipsoids $\sum_{i=1}^{n} \frac{p_i^2}{r_i} = r^2$).

Now suppose that ( ) is satisfied on $U_r = \{p \in U | \frac{p_1^2}{r_1} + \ldots + \frac{p_n^2}{r_n} = r^2\}$.

Let $c, d > 0$ be numbers such that $p \in U_r$, $p_i < c \Rightarrow f_i(p) \geq d$.

Let $A = \{p \in U_r | p_i \geq c, i = 1, \ldots, n\}$. Then A is homeomorphic to a(n-1)-disk provided c is small enough, and every solution starting in A remains in A. We can therefore apply 9.1.We have proved

9.2. Theorem.

Let (T_r), defined on $\Omega = \{p \in R^n | p_i > 0; i = 1, \ldots, n\}$, satisfy (W). Then for every r such that (P) is satisfied on $U_r = \{p \in \Omega | \Sigma r_i^{-1} p_i^2 = r^2\}$ there exists an equilibrium point on U_r (which therefore has all prices positive).

If (T_r) satisfies (H) and (P) there is an equilibrium ray in Ω.

9.3. Corollary.

Let (T_r) defined on $\Omega = \{p \in R^n | p_i > 0, i = 1, \ldots, n\}$ satisfy (W'). Then for every r such that (P) is satisfied on $U_r = \{p \in \Omega | \Sigma r_i^{-1} p_i^2 \leq r^2\}$ there exists an equilibrium point in U_r with all prices positive.

Same Proof.

LIST OF (CONDITIONS ON) DYNAMICAL SYSTEMS.

p_i is the price of commodity i (at a given moment in time); $h_i(p_1, \ldots, p_n) = h_i(p)$ is the excess demand for commodity i at prices $p_1, \ldots, p_n$; $f_i(p)$ is a function such that $\text{sign}(f_i(p)) = \text{sign}(h_i(p)$; s_{ij} is the amount of commodity j held by the i-th individual. The systems considered are

(DS) $\dot{x} = f(x)$, $x \in M \subset R^n$, f any continuous n-vectorvalued function

(NT) $\dot{p}_i = f_i(p;s)$

$\dot{s}_{ij} = g_{ij}(p;s)$

(T) $\dot{p}_i = f_i(p)$, $i = 1, \ldots, n$

(T') $\dot{p}_i = h_i(p) = f_i(p)$, $i = 1, \ldots, n$

(T_r) $\dot{p}_i = f_i(p) = r_i h_i(p)$, $r_i > 0$, $i = 1, \ldots, n$

(T^*) $\dot{p}_i = \begin{cases} 0 \text{ if } p_i = 0 \quad f_i(p) = r_i h_i(p) < 0 \\ r_i h_i(p) = f_i(p) \quad \text{otherwise} \end{cases}$, $r_i > 0$, $i = 1, \ldots, n$

The conditions on systems considered are

(B) For every $x^o \in M$, there exists a unique solution $x(t;x^o)$ of (DS) $x(t;x^o) \in M$ for all $t \geq 0$ such that $x(0;x^o) = x^o$. For a fixed $t > 0$ $x(t;x^o)$ is a continuous function of x^o.

(B') For every $x^o \in M$, there exists a unique solution $x(t;x')$ of (DS) defined for all $t \in R$ passing through x^o at time $t = 0$. For a fixed $t \in R$, $x(t;x^o)$ is a continuous function of x^o.

(B") Condition (B) is satisfied and the function p(-;-); $M \times \mathbb{R}(\geq 0) \to M$, $(t;p^o) \to p(t;p^o)$ is continuous as a function of (t,p^o).

(C) $\sum_i s_{ij}(t) = c_j$ (this is a condition on (NT))

$\sum_i g_{ij}(p;s) = 0$

(D) The function f is differentiable on $A = \{p \in S^{n-1} | p_i \geq 0\}$, $i = 1, \ldots, n\}$ (a condition on (T) or (T'); $S^{n-1} = \{p \in R^n | \Sigma p_i^2 = 1\}$

(E) There exists a positive equilibrium price vector $p^* = (p_1^*, \ldots, p_n^*)$, $p_i^* > 0$, $i = 1, \ldots, n$

(H) $h_i(\lambda p) = h_i(p)$, for all $\lambda > 0$,

(P) There exist (small) constants $c > 0$, $d > 0$ such that $f_i(p) \geq d$ if $0 < p_i < c$.

(RP) $p^*.h(p) = \sum_j p_j^* h_j(p) > 0$ for all equilibria p^* and nonequilibria p.

(S) The excess demand functions $h_i(p)$, $i = 1, \ldots, n$ are differentiable at all points $p \in \Omega$ and $\frac{\partial h_i}{\partial p_j}(p) > 0$ for all $i \neq j$, $p \in \Omega$.

(W) $\sum_i p_i h_i(p) = 0$

(W') $\sum_i p_i h_i(p) \leq 0$

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