Netherlands School of Economics

ECONOMETRIC INSTITUTE

Report 7113

A COMPARISON BETWEEN THE MSE OF TWO PREDICTORS IN THE MULTIPLICATIVE MODEL UNDER TWO ALTERNATIVE STOCHASTIC ASSUMPTIONS

(A Monte-Carlo Study)

by R. Teekens and A.S. Louter

August 27, 1971

Preliminary and Confidential

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1. INTRODUCTION

In a recent paper[1] Teekens and Koerts considered a minimal mean square error (MMSE) estimator of the mean of the dependent variable in a multiplicative model of the type

$$(1.1) \qquad Y = \prod_{k=1}^{K} Z_k^{\delta_k} v$$

in which Y is the dependent variable, Z_k, k = 1, ..., K the non stochastic positive explanatory variable and v a disturbance term. The parameters δ_k, k = 1, ..., K are unknown.

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[1] See Teekens and Koerts (1970, b).

The assumptions under which that study was carried out were rather restrictive i.e. the variance of v was considered as a known parameter - at least in two of the three stochastic models which were investigated.

The aim of this study is to adapt the MMSE-estimator to make it suitable for the unknown-variance case, and to verify whether the MSE of this adapted estimator remains below the MSE of the LS estimator. Furthermore we will compare the MSE of the adapted estimator with the MSE of the original MMSE-estimator, which can be considered as the estimation under ideal conditions. This will be carried out under two assumptions: (i) the disturbance term v is Gamma distributed and (ii) v is Loglaplace distributed.[2]

Equation (1.1) can also be written as:

$$Y = \exp\,(x'\beta + u) \tag{1.2}$$

where

$$x' = (\ln Z_1, \ln Z_2, \ldots, \ln Z_k)$$ [3]

$$\beta' = (\delta_1, \ldots, \delta_K)$$

$$u = \ln v$$

Concerning the disturbance term we assume that

$$E(v) = 1 \tag{1.3}$$

and

$$\text{var } v = \omega^2 \tag{1.4}$$

in which ω^2 is unknown.

If we investigate a sample of size N on Y and x we may write

$$Y_i = \exp\,(x_i'\beta + u_i) \qquad i = 1, \ldots, N \tag{1.5}$$

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[2] The Lognormal case has already been studied in Teekens and Koerts (1970, a) and (1970, b).

[3] We incorporate a multiplicative constant by setting $Z_1 = e$.

Here we maintain the additional assumption that the v_i's (or u_i's) are independently distributed. If we take the logarithm at both sides we get

$$\ln Y_i = x_i'\beta + u_i \qquad i = 1, \ldots, N$$

or in vector notation

$$Y = X\beta + u \tag{1.6}$$

in which $y = [y_i] = [\ln Y_i]$ and x_i' is the i-th row of X.

It can easily be seen that the expectation of Y, given some vector x, equals

$$\eta(x) \equiv E[Y(x)] = \exp\{x'\beta\} \tag{1.7}$$

Summarizing, in the next sections we shall construct an estimator of $\eta(x)$ for unknown variance and compare its MSE with that of the LS-estimator[4] under two alternative stochastic assumptions. This will be done for two different X-matrices, the "textile matrix" and the "automobile matrix".[5]

2. GAMMA DISTRIBUTED DISTURBANCES

2.1. Introduction

If the v_i's are independently Gamma distributed with mean equal to unity and variance equal to ω^2, their density functions can be written as

$$g(v_i) = \frac{1/\omega^2}{\Gamma(1/\omega^2)} \left(\frac{v_i}{\omega^2}\right)^{1/\omega^2-1} . \exp\left(-\frac{v_i}{\omega^2}\right) \qquad i = 1, \ldots, N \tag{2.1}$$

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[4] The LS estimator of $\eta(x)$ is defined as $\hat{Y}(x) = \exp\{x'\hat{\beta}\}$ in which $\hat{\beta}$ is the LS estimator of β in model (1.6).

[5] See the Appendix.

Figure 1 shows three densities of type (2.1) for $\omega^2 = 0.05$, 0.10 and 0.50.

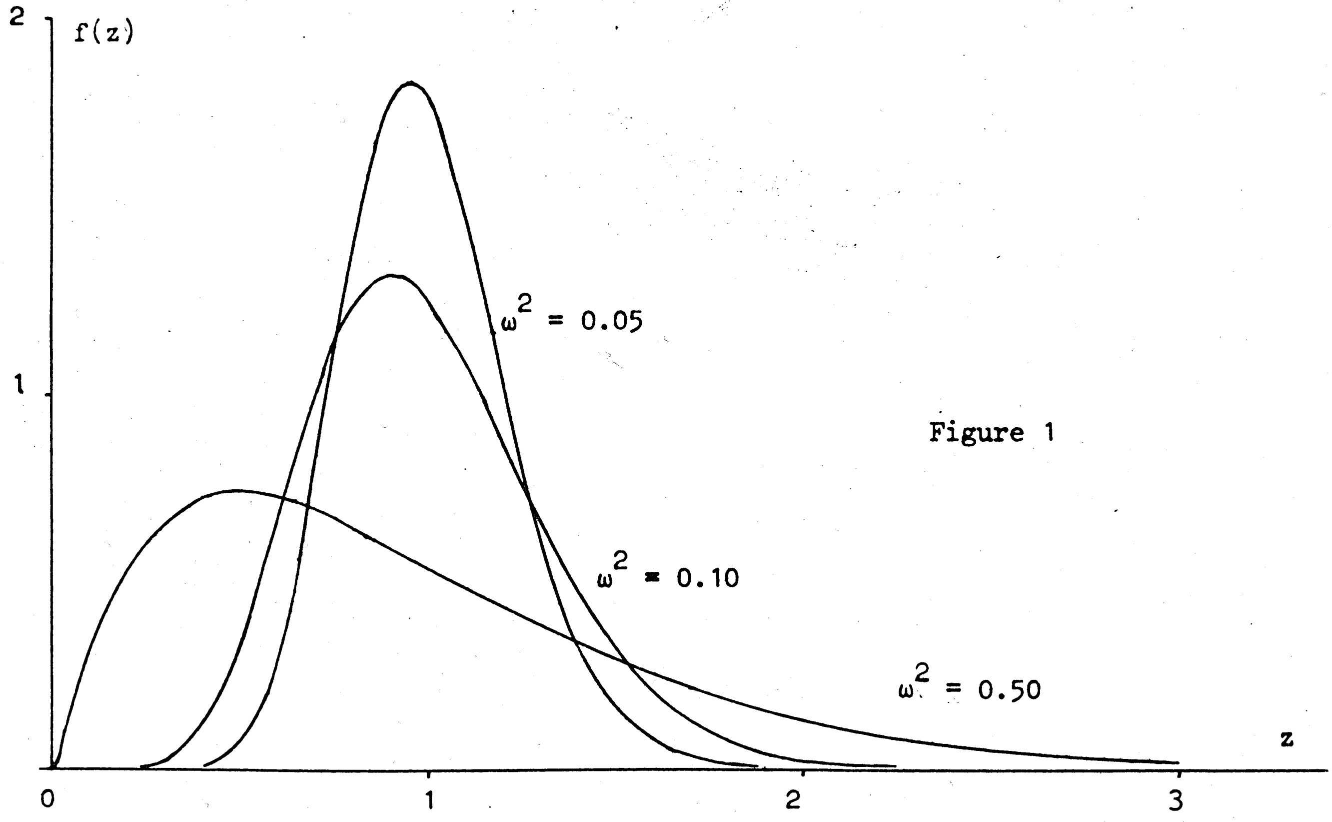

Figure 1

Assuming that ω^2 is a known parameter the MMSE estimator of $\eta(x)$ reads as[6]

$$\text{(2.2)} \qquad \bar{Y}(x) = \frac{1}{\omega^2} \prod_{j=1}^{N} \frac{\Gamma(\lambda_j + 1/\omega^2)}{(2\lambda_j + 1/\omega^2)} \exp\{x'\hat{\beta}\}$$

$$\text{provided } \lambda_j > -\frac{1}{2\omega^2}, \quad j = 1, \ldots, N$$

in which

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) = \ell_x' = x'(X'X)^{-1}X'$$

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[6] See Teekens and Koerts (1970, b), p. 8.

This estimator dominates (in MSE) the LS estimator of $\eta(x)$:[7]

$$(2.3) \qquad \bar{Y}(x) = \exp\{x'\hat{\beta}\}$$

This result tempts us to hope that, in the case of the unknown variance, it may also be possible to construct an estimator which dominates the LS estimator. To that end we shall study the relative MSE of the LS estimator $\hat{Y}(x)$, the adapted MMSE estimator $\bar{Y}'(x)$, and the original MMSE estimator $\bar{Y}(x)$. These quantities are denoted by $\hat{\pi}(x)$, $\bar{\pi}'(x)$ and $\bar{\pi}(x)$ respectively.

We shall see that $\bar{\pi}'(x)$ cannot be determined analytically. Hence this function should be established by simulation. Furthermore, we need $\bar{\pi}(x)$ and $\hat{\pi}(x)$. In Teekens and Koerts (1970, b) we find expressions for both:

$$(2.4) \qquad \bar{\pi}(x) = 1 - M_u^2(\ell_x)/M_u(2\ell_x)$$

and

$$(2.5) \qquad \hat{\pi}(x) = M_u(2\ell_x) - 2M_u(\ell_x) + 1$$

in which $M_u(\tau)$ stands for the moment-generating function of u. In these expressions we have to substitute the moment-generating function of u in the case where v is Gamma distributed. This function equals

$$(2.6) \qquad M_u(\tau) = \frac{\omega^{2\iota'\tau}}{\Gamma^N(\frac{1}{\omega^2})} \prod_{j=1}^{N} \Gamma(t_j + \frac{1}{\omega^2}) \qquad \text{provided } t_j > -\frac{1}{\omega^2},\ j = 1, \ldots, N$$

with $\tau' = (t_1, \ldots, t_N)$.

Substitution of (2.6) into (2.4) and (2.5) yields

$$(2.7) \qquad \bar{\pi}(x) = 1 - \frac{1}{\Gamma^N(\frac{1}{\omega^2})} \prod_{j=1}^{N} \frac{\Gamma^2(\lambda_j + \frac{1}{\omega^2})}{\Gamma(2\lambda_j + \frac{1}{\omega^2})}$$

and

$$(2.8) \qquad \hat{\pi}(x) = \frac{\omega^4}{\Gamma^N(\frac{1}{\omega^2})} \prod_{j=1}^{N} \Gamma(2\lambda_j + \frac{1}{\omega^2}) - \frac{2\omega^2}{\Gamma^N(\frac{1}{\omega^2})} \prod_{j=1}^{N} (\lambda_j + \frac{1}{\omega^2}) + 1$$

provided that $\lambda_j > -\frac{1}{2\omega^2}$
$j = 1, \ldots, N$

[7] This statement has been proved, under more general conditions in Teekens and Koerts (1970, b), p. 3.

The restriction $\lambda_j > -\frac{1}{2\omega^2}$ $j = 1, \ldots, N$ indicates the basic limitation of our approach. In this approach the MSE is the objective function, consequently we can apply our minimal MSE estim-tor only to those cases where the mean square error of Y(x) is defined. Since the MSE-function runs in terms of the m.g.f. of the transformed disturbance vector, a necessary and sufficient condition of the existence of $\pi(x)$ is the existence of $M_u(\ell_x)$ and $M_u(2\ell_x)$. The above-mentioned inequality is precisly the necessary and sufficient condition for the existence of $M_u(\ell_x)$ and $M_u(2\ell_x)$.

In the next sections we shall investigate how $\bar{Y}(x)$ can be modified into $\bar{Y}'(x)$ so as to be able to face the problem of unknown variance.

2.2. The Estimation of the Variance and the Definition of $\bar{Y}'(x)$

The construction of $\bar{Y}'(x)$ will be accomplished by replacing the unknown ω^2, which appears in $\bar{Y}(x)$, by its estimator.

The estimation of the variance ω^2 of the multiplicative disturbances can be accomplished either by estimation of the variance σ^2 of the transformed disturbances (the indirect method) or by direct estimation using the following estimator

$$(2.9) \qquad \Omega^2 = \frac{1}{N-K}\left[\sum_{i=1}^{N} \hat{v}_i^2 - \frac{1}{N}\left\{\sum_{i=1}^{N} \hat{v}_i\right\}^2\right]$$

with $\hat{v}_i = \exp(\hat{u}_i)$, $\hat{u}_i$ being the computed i-th residual, obtained by application of least squares to the transformed model.

The indirect method provides us with an estimator of ω^2 by applying the functional relationship between ω^2 and σ^2 to the estimator

$$(2.10) \qquad \hat{\sigma}^2 = \frac{1}{N-K}\, y'[I - X(X'X)^{-1}X']y = \frac{1}{N-K}\, u'[I - X(X'X)^{-1}X']u$$

This relationship is[8]

$$(2.11) \qquad \sigma^2 = \Psi'(1/\omega^2)$$

with

$$\Psi'(x) \equiv \frac{d^2[\ln \Gamma(x)]}{dx^2}$$

Function (2.11) is one-to-one on the positive axes, hence $\hat{\omega}^2$ follows uniquely from

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[8] See Teekens and Koerts (1970, b), Appendix A.

$$(2.12) \qquad \hat{\sigma}^2 = \psi'(1/\hat{\omega}^2)$$

After this brief description of the two estimation methods, we now have to decide which of the methods should be used. There are certain indications which suggest the application of the indirect method: Finney (1941) showed that for the estimation of the variance of a lognormal distribution an indirect method is more efficient than the direct one. Therefore, we shall construct $\bar{Y}'(x)$ by replacing ω^2 by its estimator $\hat{\omega}^2$ as defined in (2.10) and (2.12):

$$(2.13) \qquad \bar{Y}'(x) = \frac{1}{\hat{\omega}^2} \prod_{j=1}^{N} \frac{\Gamma(\lambda_j + 1/\hat{\omega}^2)}{(2\lambda_j + 1/\hat{\omega}^2)} \exp\{x'\hat{\beta}\} \quad \text{provided that } \lambda_j > -\frac{1}{2\hat{\omega}^2}, \; j = 1, \ldots, N$$

There is, however, a difficulty arising from the restriction $\lambda_j > -1/2\hat{\omega}^2$, $j = 1, \ldots, N$. It is clear that $\bar{Y}'(x)$ is not defined if $\hat{\omega}^2$ does not obey this restriction. In that case we define $\bar{Y}'(x) = \hat{Y}(x)$, hence

$$(2.14) \qquad \begin{cases} \bar{Y}'(x) = \frac{1}{\hat{\omega}^2} \prod_{j=1}^{N} \frac{\Gamma(\lambda_j + 1/\hat{\omega}^2)}{(2\lambda_j + 1/\hat{\omega}^2)} \exp\{x'\hat{\beta}\} & \text{if all } \lambda_j > -1/2\hat{\omega}^2 \\ \bar{Y}(x) = \exp\{x'\hat{\beta}\} & \text{if at least one } \lambda_j \leq -1/2\hat{\omega}^2 \end{cases}$$

It will be clear from (2.12) and (2.14) that it is practically impossible to determine analytically the mean square error of $\bar{Y}'(x)$. Therefore, $\bar{\pi}'(x)$ will be approximated by simulation. Section 4 gives a brief description of the simulation procedure.

3. LOGLAPLACE DISTRIBUTED DISTURBANCES

3.1. Introduction

First, we have to define the LogLaplace distribution. If u_i is Laplace distributed then $v_i = \exp(u_i)$ is LogLaplace distributed; this nomenclature is an analogon of the one used for the normal and the lognormal distributions. If we assume that v_i has a mean equal to unity, then its density function reads as:[9]

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[9] See Teekens and Koerts (1970, b) pp. 9-10.

$$
\begin{aligned}
g(v_i) &= \frac{1}{2\gamma}(1 - \gamma^2)^{-1/\gamma} \, . \, v_i^{1/\gamma - 1} && \text{for} \quad 0 < v_i \leq 1 - \gamma^2 \\
(3.1) \qquad g(v_i) &= \frac{1}{2\gamma}(1 - \gamma^2)^{1/\gamma} \, . \, v_i^{-1/\gamma - 1} && \text{for} \quad 1 - \gamma^2 < v_i < \infty \\
&= 0 && \text{elsewhere}
\end{aligned}
$$

It is important to note here that the first and second moments around zero exist only under the condition:

$$(3.2) \qquad 0 < \gamma < \tfrac{1}{2}$$

Furthermore, we give the functional relations between γ^2 and ω^2:

$$(3.3) \qquad \omega^2 = \gamma^2(2 + \gamma^2)/(1 - 4\gamma^2)$$

and its inverse:

$$(3.4) \qquad \gamma^2 = -1 - 2\omega^2 + (4\omega^4 + 5\omega^2 + 1)^{\frac{1}{2}}$$

and between γ and σ^2:

$$(3.5) \qquad \sigma^2 = 2\gamma^2$$

Three densities of type (3.1) are drawn in figure 2.

The minimal MSE estimator of $\eta(x)$, derived for known $\gamma^2(\omega^2$ or $\sigma^2)$, equals:

$$(3.6) \qquad \bar{Y}(x) = \frac{1}{1 - \gamma^2} \prod_{j=1}^{N} \frac{1 - 4\gamma^2\lambda_j^2}{1 - \gamma^2\lambda_j^2} \exp\,(x'\hat{\beta}) \qquad \text{provided } |\lambda_j| < \frac{1}{2\gamma}, \; j = 1, \ldots, N$$ [10]

Again, the question arise, if we replace γ^2 by its estimator, whether $\bar{Y}'(x)$ is still better in MSE than the LS estimator.

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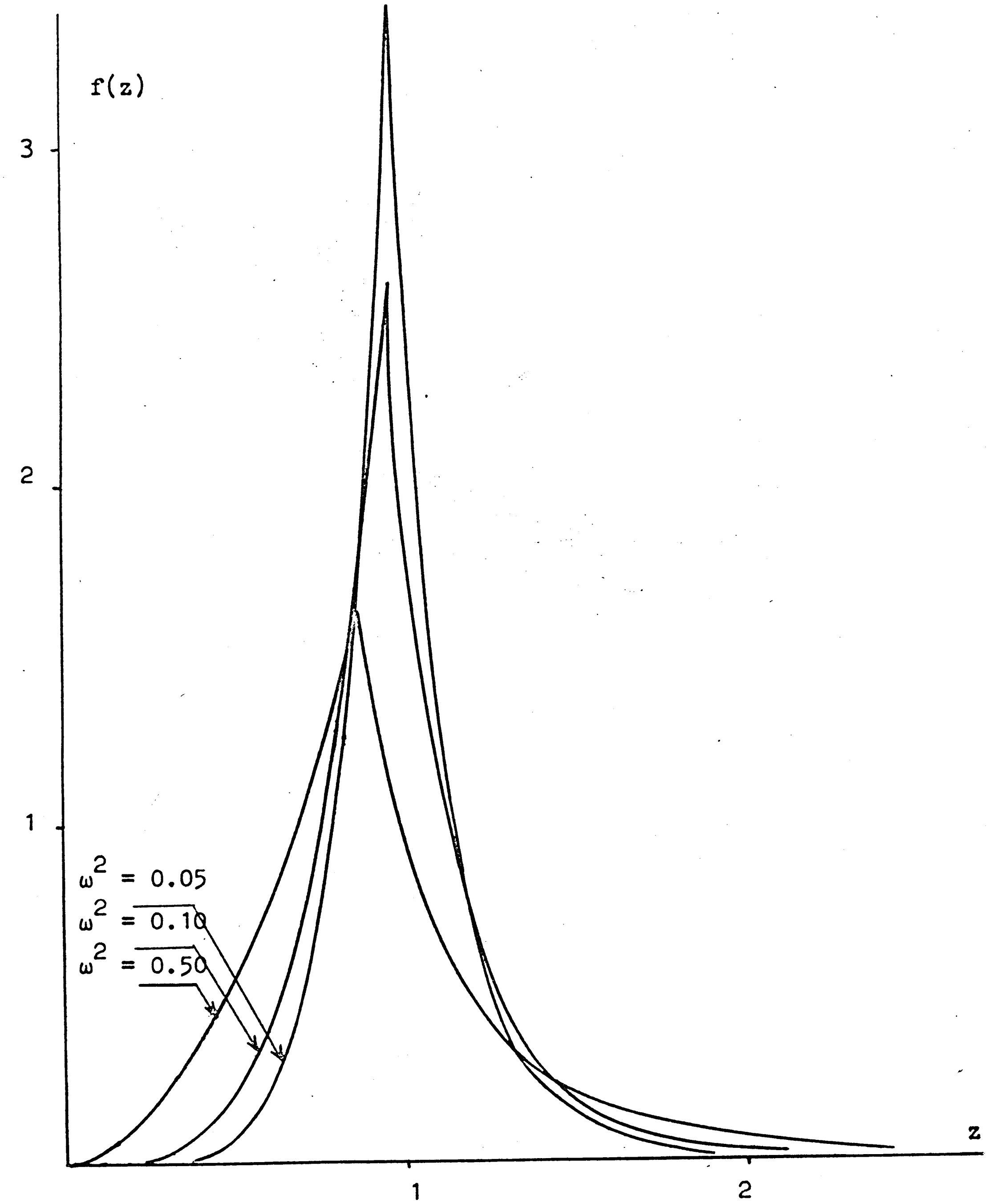
[10] Here again we are confronted with the limitations of the applicability of the MSE principle; see Section 2.1.



3.2. <u>The Estimation of the Parameter γ^2 and the Definition of $\bar{Y}'(x)$</u>

Looking for an estimator of γ^2, it seems that, on the basis of (3.5), $\frac{1}{2}\hat{\sigma}^2$ will be appropriate. There are, however, difficulties. We shall consider those distributions only for which $E(v_i)$ and $E(v_i^2)$ exist. Hence we exclude all estimators of γ^2 other than those which produce estimates greater than zero and smaller than $\frac{1}{4}$ (see condition 3.2). Hence $\frac{1}{2}\hat{\sigma}^2$ is, in general not admissable as an estimator of γ^2, and we adhere to the following estimator - based on (3.4):

$$\hat{\gamma}^2 = 1 - 2\hat{\omega}^2 + 4(\hat{\omega}^2)^2 + 5\hat{\omega}^2 + 1)^{\frac{1}{2}} \qquad (3.7)$$

with

$$(3.8) \qquad \hat{\omega}^2 = \frac{1}{N-K}\Big[\sum_{i=1}^{N} \hat{v}_i'^2 - \frac{1}{N}\{\sum_{I=1}^{N} \hat{v}_i\}^2\Big]$$

with $\hat{v}_i = \exp(\hat{u}_i)$.
This estimator of γ^2 assumes values m between zero and $\frac{1}{4}$.
There is a second restriction on the use of $\hat{\gamma}^2$: it should obey the inequality

$$(3.9) \qquad \lambda_j^2 < \frac{1}{4\hat{\gamma}^2} \qquad j = 1, \ldots, N$$

(see equation (3.6)). From the discussion in Section 2.2 it will be clear that we selected the following solution

$$(3.10) \quad \begin{cases} \bar{Y}'(x) = \dfrac{1}{1-\hat{\gamma}^2} \displaystyle\prod_{j=1}^{N} \dfrac{1 - 4\hat{\gamma}^2\lambda_j^2}{1-\hat{\gamma}^2\lambda_j^2} \exp\{x'\hat{\beta}\} & \text{if all } \lambda_j^2 < \dfrac{1}{4\hat{\gamma}^2} \\ \text{and} & \\ \bar{Y}'(x) = \exp\{x'\hat{\beta}\} & \text{if at least one } \lambda_j^2 \geq \dfrac{1}{4\hat{\gamma}^2} \end{cases}$$

The next section is devoted to the simulation procedure used to approximate $\pi'(x)$.

4. SIMULATION

The simulation under both assumptions - gamma distributed v_i's and LogLaplace distributed v_i's - was carried out on the same material. We selected two different X-matrices with dimensions 15 × 3 i.e. fifteen observations on three explanatory variables. Both of them are from emperical studies: the TEXTILE and the AUTOMOBILE matrix.

From these matrices, which are time series, we selected the first 10 rows as the sample from which β had to be estimated by least squares, and from the remaining 5 rows we made a selection for the "prediction" of the vector of explanatory variables x. These x-vectors were used to obtain the "conditional predictions" of the dependent variable Y.

The computation of the relative mean square error of $\bar{Y}'(x)$ is based on the following derivation[11]

$$(4.1)\quad \bar{\pi}'(x) = E[\bar{Y}'(x) - \eta(x)]^2/\eta^2(x) = E(c \exp\{x'\beta + \ell_x'u\} - \eta(x))^2/\eta^2(x) =$$

$$= E[c.\eta(x)\exp(\ell_x'u) - \eta(x)]^2/\eta^2(x)$$

$$= E[c.e^{\ell_x'u} - 1]^2$$

and since c is a function of ℓ_x and the distribution parameter of v_i only, the relative MSE does not depend on β.

For each prediction vector x_i - related to a specific X-matrix (Textile or Automobile) the relative MSE of $\bar{Y}'(x_i)$ was estimated by taking the mean of 10.000 simulated values of

$$(4.2)\qquad p_{ij} = [c_{ij} \exp\{\ell_{x_i}' u_j\} - 1]^2$$

i.e.

$$(4.3)\qquad \bar{P}'(x_i) = \sum_{j=1}^{10.000} p_{ij}/10.000$$

The values of p_{ij} were obtained by generating a vector u given the stochastic assumption for different values of ω^2 $(0 < \omega^2 \leq .5)$.[12] From (2.10), (2.12) and (2.14) it can be seen how the values of c_{ij} depends on the vector u_j and x_i in case of Gamma distributed multiplicative disturbances. In the case of Loglaplace distributed multiplicative disturbances the relation between c_{ij} and u_j and x_i follows from (3.7), (3.8) and (3.10).

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[11] Here we write $\bar{Y}'(x) = c.\exp\{x'\hat{\beta}\}$, in which the definition of c depends on the stochastic assumptions (see (2.14) and (3.10).

[12] We note that for the Gamma distributed v_i's, we considered only those values of ω^2 which correspond to integer numbers of freedom.

5. RESULTS

In Tables 1 and 2 the results of the simulation and the computation of the relative MSE of $\bar{Y}(x)$ and $\hat{Y}(x)$ are summarized. For the sake of convenience we present the natural logarithms of the relative MSE's. For each vector x_i we also computed a number indicated by α. This quantity is defined as

$$(5.1) \qquad \alpha = \ell_x'\ell_x = x_i'(X'X)^{-1}x_i$$

From Teekens and Koerts (1970, a) and (1970, b) it can be seen that for lognormally distributed multiplicative disturbances the quantity was the only function of x_i and X which determined the relative MSE of $\bar{Y}(x)$ and $\bar{Y}'(x)$. This is no longer true if we impose other stochastic assumptions on the model. From the tables, however, it can be seen that α is yet a good indicator of the level of the three MSE functions. The figures, shown in Table 1 and 2, are presented in figures 3 and 4.

The first conclusion which can be drawn from the results is that the adapted MMSE estimator is better (in MSE) than the LS estimator - at least for all but one of the cases under investigation.

Furthermore, the results confirm the conclusions based on the study of the multiplicative model with lognormally distributed disturbances, viz. that for low values of α ($\alpha < 1$) and low values of ω^2 ($\omega^2 \leq 0.05$) the differences in relative MSE between $\bar{Y}'(x)$ and $\hat{Y}(x)$ are negligible. Consequently, in these cases the LS-approach can be considered as sufficiently accurate.

There are other phenomena, which are worth mentioning. Figures 3 and 4, in particular, show that the relative MSE of $\bar{Y}'(x)$ deviates much less from $\bar{\pi}(x)$ in the LogLaplace case than in the case of Gamma distributed disturbances.

TABLE 1 The Logarithms of the Relative MSE of $\overline{Y}(x)$, $\overline{Y}'(x)$ and $\hat{Y}(x)$ in Case of Gamma distributed Multiplicative Disturbances.

Automobile Case

	x_1 (α=0.64)			x_2 (α=2.51)			x_3 (α=1.32)		
ω^2	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$
.05	-3.46	-3.44	-3.44	-2.14	-2.08	-1.93	-2.75	-2.71	-2.67
.10	-2.77	-2.72	-2.74	-1.51	-1.40	-1.08	-2.09	-2.00	-1.92
.153	-2.35	-2.32	-2.30	-1.14	-1.00	-0.47	-1.68	-1.61	-1.42
.20	-2.10	-2.05	-2.02	-0.92	-0.77	-0.05	-1.45	-1.37	-1.10
.25	-1.88	-1.84	-1.79	-0.75	-0.59	0.35	-1.25	-1.17	-0.81
.333	-1.61	-1.54	-1.48	-0.54	0.14	0.96	-1.00	-0.87	-0.39
.40	-1.44	-1.38	-1.28	-0.42	-0.18	1.43	-0.85	-0.71	-0.10
.50	-1.23	-1.17	-1.02	-0.28	0.88	2.13	-0.67	-0.50	0.31

Textile Case

	x_1 (α=1.30)			x_2 (α=2.14)			x_3 (α=1.84)		
ω^2	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$
.05	-2.77	-2.76	-2.69	-2.29	-2.25	-2.11	-2.43	-2.39	-2.29
.10	-2.12	-2.09	-1.95	-1.64	-1.59	-1.29	-1.78	-1.73	-1.50
.153	-1.72	-1.68	-1.47	-1.26	-1.15	-0.72	-1.39	-1.31	-0.96
.20	-1.49	-1.42	-1.16	-1.04	-0.93	-0.33	-1.17	-1.05	-0.59
.25	-1.30	-1.22	-0.89	-0.86	-0.74	0.04	-0.98	-0.87	-0.25
.333	-1.06	-0.96	-0.51	-0.64	-0.27	0.60	-0.75	-0.62	0.25
.40	-0.92	-0.82	-0.24	-0.51	-0.13	1.01	-0.61	-0.43	0.62
.50	-0.75	-0.61	0.11	-0.35	0.13	1.64	-0.45	-0.02	1.16

the relative MSE of $\overline{Y}'(x)$ has been approximated.

TABLE 2 The Logarithms of the Relative MSE of $\overline{Y}(x)$, $\overline{Y}'(x)$ and $\hat{Y}(x)$ in Case of Log Laplace distributed Multiplicative Disturbances.

Automobile Case

	x_1 (α=0.64)			x_2 (α=2.51)			x_3 (α=1.32)		
ω^2	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$
.05	-3.55	-3.55	-3.54	-2.18	-2.13	-1.99	-2.82	-2.81	-2.75
.10	-2.95	-2.96	-2.92	-1.59	-1.61	-1.22	-2.22	-2.20	-2.08
.15	-2.63	-2.67	-2.59	-1.27	-1.24	-0.75	-1.90	-1.93	-1.71
.20	-2.42	-2.42	-2.37	-1.06	-1.00	-0.41	-1.69	-1.72	-1.45
.25	-2.26	-2.29	-2.21	-0.92	-0.99	-0.13	-1.54	-1.57	-1.25
.30	-2.15	-2.18	-2.09	-0.80	-0.80	0.09	-1.43	-1.49	-1.10
.35	-2.05	-2.06	-1.99	-0.72	-0.83	0.29	-1.33	-1.36	-0.97
.40	-1.98	-2.01	-1.91	-0.64	-0.62	0.46	-1.26	-1.32	-0.86
.45	-1.91	-1.93	-1.84	-0.58	-0.61	0.61	-1.20	-1.15	-0.77

Textile Case

	x_1 (α=1.30)			x_2 (α=2.14)			x_3 (α=1.84)		
ω^2	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$	$\overline{Y}(x)$	$\overline{Y}'(x)$	$\hat{Y}(x)$
.05	-2.82	-2.85	-2.75	-2.33	-2.32	-2.17	-2.48	-2.43	-2.36
.10	-2.21	-2.25	-2.07	-1.73	-1.74	-1.43	-1.88	-1.84	-1.64
.15	-1.88	-1.85	-1.68	-1.40	-1.42	-0.98	-1.55	-1.52	-1.21
.20	-1.66	-1.70	-1.40	-1.19	-1.16	-0.65	-1.34	-1.27	-0.91
.25	-1.49	-1.54	-1.19	-1.03	-1.05	-0.39	-1.18	-1.21	-0.67
.30	-1.37	-1.43	-1.02	-0.91	-0.93	-0.18	-1.06	-1.09	-0.48
.35	-1.27	-1.19	-0.88	-0.82	-0.91	0.01	-0.97	-0.87	-0.31
.40	-1.19	-1.18	-0.76	-0.74	-0.74	0.17	-0.89	-0.92	-0.17
.45	-1.12	-1.19	-0.66	-0.68	-0.65	0.32	-0.82	-0.88	-0.04

the relative MSE of $\overline{Y}'(x)$ has been approximated.

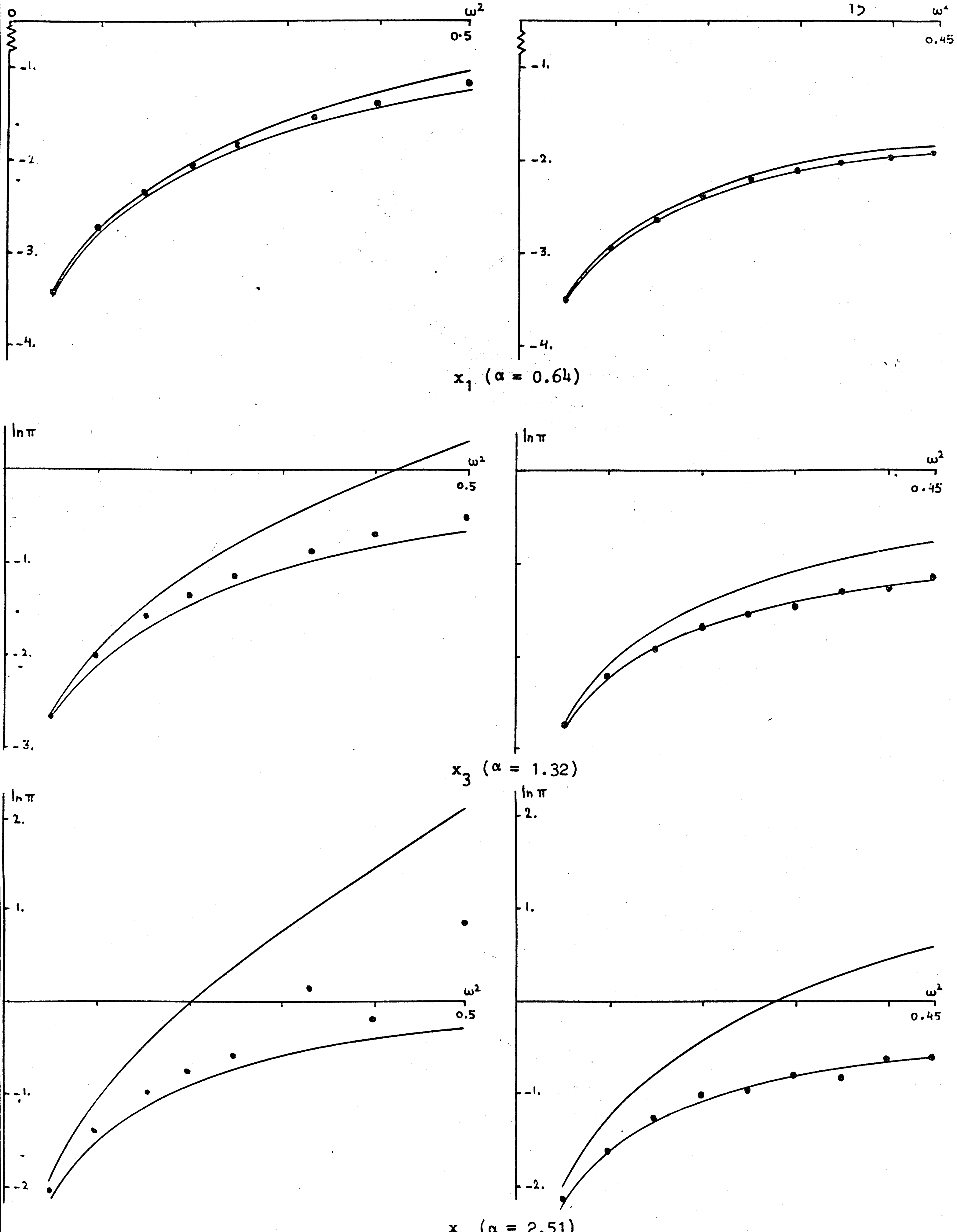

Figure 3. The logarithmic relative MSE's of the predictors $\bar{Y}(x)$, $\bar{Y}'(x)$ and $\hat{Y}(x)$ in the AUTOMOBILE case for Gamma (left side) and Loglaplace (right side) distributed disturbances (the dots stand for the approximated $\bar{\pi}(x)$; in each graph the upper line stands for $\hat{\pi}(x)$ and the lower are for $\bar{\pi}(x)$.

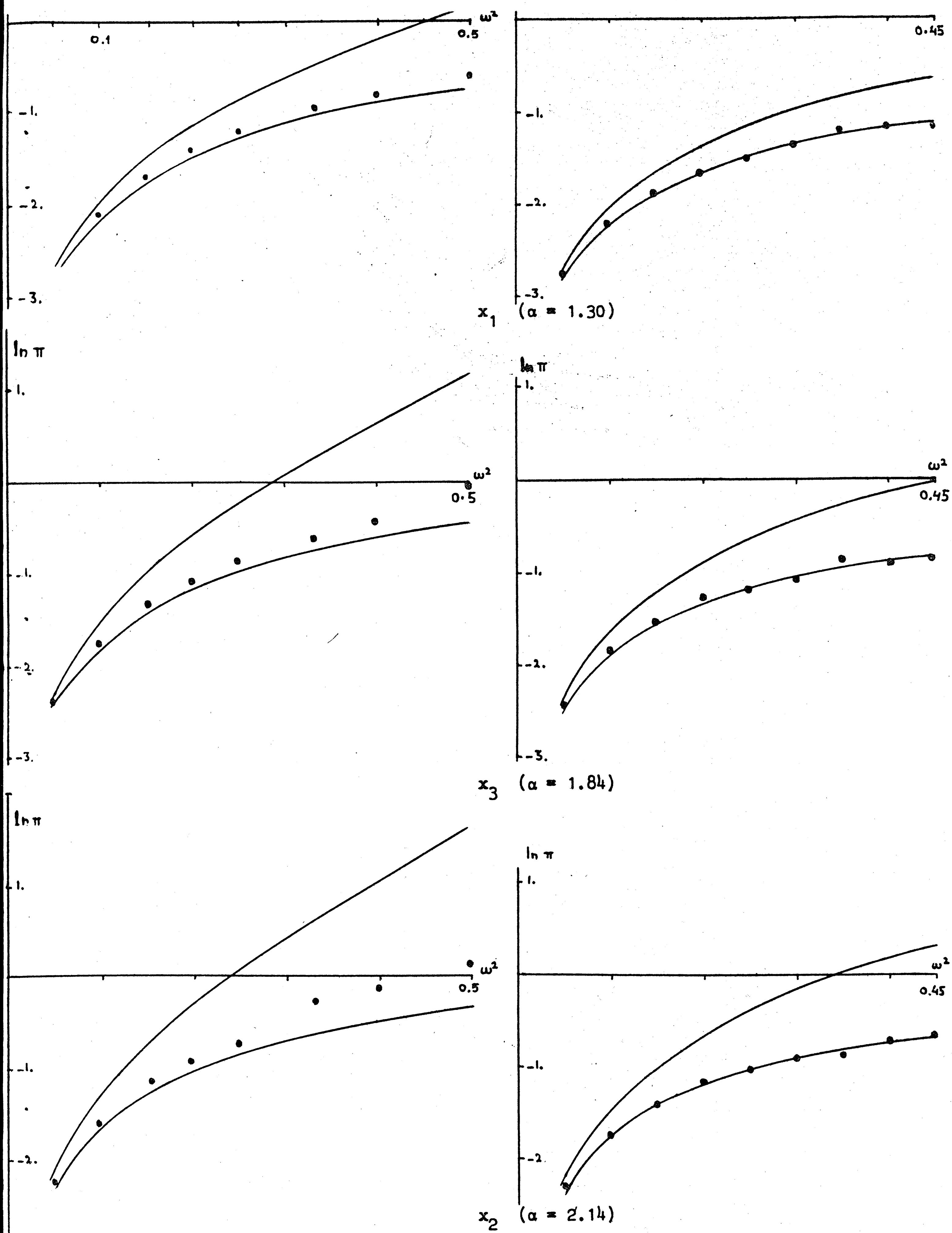

Figure 4. The logarithmic relative MSE's of the predictors $\bar{Y}(x)$, $\bar{Y}'(x)$ and $\hat{Y}(x)$ in the TEXTILE case for Gamma (left side) and Loglaplace (right side) distributed disturbances (the dots stand for the approximated $\hat{\pi}'(x)$; in each graph the upper line stands for $\hat{\pi}(x)$ and the lower are for $\bar{\pi}(x)$.

This is, in our opinion, closely connected with the two different estimation procedures (applied to obtain estimates of ω^2 and γ^2).
It was seen in the previous section that, under certain conditions, i.e., if $\hat{\omega}^2$ or $\hat{\gamma}^2$ become greater than some upperbound fixed by the vector ℓ_x, we are forced to set $\bar{Y}'(x) = \hat{Y}(x)$ (see (2.14) and (3.10)).
Let us now consider the case of Gamma distributed disturbances.
If the population parameter ω^2 becomes larger, the probability that $\hat{\omega}^2$ exceeds the above-mentioned upper limit will increase and the estimator $\bar{Y}'(x)$ will more probably coincide with $\hat{Y}(x)$. Hence $\bar{\pi}'(x)$ will tend to $\hat{\pi}(x)$ for increasing ω^2.

The situation in the case of Loglaplace distributed disturbances is different, however. In Section 3.2 we adopted an estimation procedure which guarantees that $0 < \hat{\gamma}^2 < \frac{1}{4}$, for all $\hat{\omega}^2$. It is clear that by imposing this a priori restriction on $\hat{\gamma}^2$, we reduce the probability that this estimator will exceed the upper limit which is fixed by the vector ℓ_x, and thereby the probability that $\bar{Y}'(x)$ coincides with $\hat{Y}(x)$. Hence, for Loglaplace distributed v_i's, $\bar{\pi}'(x)$ tends to remain closer to $\bar{\pi}(x)$ than for Gamma distributed disturbances.

6. CONCLUSIONS

Finally, we summarize the main conclusions.

(1) In the case of unknown variance, it is possible to construct an estimator $\bar{Y}'(x)$ of $\eta(x)$, derived from the minimal mean square error estimator $\bar{Y}(x)$ by substituting the unknown parameters in $\bar{Y}(x)$ by their estimators, which, in MSE, dominates the LS estimator $\hat{Y}(x)$ of $\eta(x)$ - at least for the cases which were investigated. This conclusion also holds good for Gamma distributed as for Loglaplace distributed multiplicative disturbances.

(2) For low values of α ($\alpha < 1$) and ω^2 ($\omega^2 \leq .05$) the LS estimator can be considered as sufficiently accurate. For higher values of these parameters, however, the adapted TK-estimator may have a considerable lower relative mean square error.

(3) In the case of Gamma distributed disturbances the relative MSE of $\bar{Y}'(x)$ approaches the relative MSE of $\hat{Y}(x)$ if ω^2 tends to its maximum value i.e. the maximum value for which the mean square error function is defined. Such a phenomenon has not been observed for Loglaplace distributed disturbances.

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APPENDIX

(i) Automobile data ; Given by Chow (1957) p.32. The X matrix stands -apart from a constant term- for the time series (1921-35) of log Automobile Stock per Capita and log Personal Money Stock per Capita.

	1.0	1.569657	5.090062
	1.0	1.691202	5.232712
	1.0	1.929781	5.249652
	1.0	2.060386	5.314191
	1.0	2.130372	5.341856
	1.0	2.189192	5.309752
	1.0	2.171793	5.424069
	1.0	2.211566	5.365509
	1.0	2.271920	5.331268
	1.0	2.207395	5.357529
x_1	1.0	2.092605	5.410306
x_2	1.0	1.933693	5.457456
x_3	1.0	1.846879	5.367843
x_4	1.0	1.820023	5.376666
x_5	1.0	1.876407	5.479805

(ii) Textile data ; Given by Theil and Nagar (1961). The X matrix stands -apart from a constant term- for the time series (1923-37) of log Real Income per Capita and log Relative Price.

	1.0	1.985430	2.004320
	1.0	1.991670	2.000430
	1.0	2.000000	2.000000
	1.0	2.020780	1.957130
	1.0	2.020780	1.937020
	1.0	2.039410	1.952790
	1.0	2.044540	1.957130
	1.0	2.050380	1.918030
	1.0	2.038620	1.845720
	1.0	2.022430	1.815580
x_1	1.0	2.007320	1.787460
x_2	1.0	1.979550	1.795880
x_3	1.0	1.984080	1.803460
x_4	1.0	1.989450	1.720990
x_5	1.0	2.010300	1.775970