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MULTI PERIOD MACRO ECONOMIC PLANNING UNDER UNCERTAINTY

by A.R.M. Wennekers and R. Harkema

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1. INTRODUCTION

During the past ten years a great deal of research has been focused on the development of optimal policies for stochastic control systems. Nevertheless, econometricians have as yet paid relatively little attention to using these methods for purposes of planning with macro-economic models. Econometric research in this area has primarily been concerned with planning by means of completely deterministic models or models with known multipliers and stochastic disturbances. Research which is directed towards models with incompletely known multipliers is very scarce and, as far as it exists, almost exclusively concerned with one-period decision problems.[1] To the present authors' knowledge the work by Aoki (1967) constitutes the only exception to this rule.

The purpose of the present paper is to investigate whether optimal control policies can be derived for multi-period macro-economic planning problems when the multipliers of the underlying econometric model are not known with certainty. For reasons of notational convenience we confine ourselves to a two-period decision problem but generalization to any finite number of periods is straightforward. Unfortunately our problem

[1] See, e.g. W.D. Fisher (1962), Zellner and Chetty (1965), and Harkema and Kloek (1969).

will prove to be insoluble unless one is willing to make an approximation. This approximation implies that in calculating the first-period decision of a multi-period decision problem we neglect the fact that in each period additional information will be obtained about the true values of the multipliers of the model. Of course, this does not mean that we completely neglect the fact that information will become available in later periods. We still take into account that decisions in later periods will be influenced by the realized values of the variables in the preceding periods.

The order of discussion is as follows. In Subsection 2.1 we present the problem and give a summary review of the basic elements of a two-period decision problem under uncertainty. In Subsection 2.2 we derive an optimal strategy by application of the well-known principle of backwards induction while Subsection 2.3 exemplifies the difficulties involved in the numerical computation of the optimal first-period decision. In Section 3 we present an approximate solution to our problem and indicate how the approach may be generalized to a larger number of periods. Finally, in Section 4 we summarize our findings and present some suggestions for future research in this area.

2. STATEMENT OF THE PROBLEM AND ITS SOLUTION

2.1. Basic Elements of the Decision Problem

In this paper we shall be concerned with a macro-economic decision maker who wants to optimize the values of a number of target variables y_{ti} $(i = 1, \ldots, m)$ in each of two successive periods t $(t = 1, 2)$. The values of the target variables in each period t are supposed to be generated by the following system of linear equations

$$y_t = \Pi u_t + \xi_t \qquad (t = 1, 2) \tag{2.1}$$

where y_t denotes an m-dimensional vector of endogenous variables, u_t an n-dimensional vector of predetermined variables, ξ_t an m-dimensional vector of disturbances, and Π a real-valued matrix of reduced-form coefficients of order $m \times n$. After partitioning the vector of predetermined variables u_t, the system (2.1) may also be written as

$$(2.2) \qquad y_t = \Pi_1 y_{t-1} + \Pi_2 z_t + \Pi_3 z_{t-1} + \Pi_4 w_t + \Pi_5 w_{t-1} + \Pi_6 x_t + \xi_t$$

where z_t denotes a p-dimensional vector of instrument variables, w_t a q-dimensional vector of non-controlled exogenous variables, x_t an r-dimensional vector of known exogenous variables and

$$(2.3) \qquad u_t' = [y_{t-1}' \quad z_t' \quad z_{t-1}' \quad w_t' \quad w_{t-1}' \quad x_t']$$

$$\Pi = [\Pi_1 \quad \Pi_2 \quad \Pi_3 \quad \Pi_4 \quad \Pi_5 \quad \Pi_6]$$

The vectors of disturbances ξ_t (t = 1, 2) are supposed to satisfy the following assumptions:

(i) the distribution of ξ_t is independent of t and u_t; its distribution is normal with zero mean and positive-definite variance-covariance matrix Ω^{-1};

(ii) the random variables ξ_1 and ξ_2 are independently distributed.

Moreover, we shall assume that the decision maker's preferences with respect to the values of the target variables and the instrument variables in each period t (t = 1, 2) can be represented by a quadratic loss function of the following type[2]

$$(2.4) \qquad \begin{aligned} L_2 = {} & a'y^{(2)} + b'z^{(2)} + \tfrac{1}{2}\{(y^{(2)})'Ay^{(2)} + (z^{(2)})'Bz^{(2)} + \\ & + (y^{(2)})'Cz^{(2)} + (z^{(2)})'C'y^{(2)} + (y^{(2)})'Dw^{(2)} + \\ & + (w^{(2)})'D'y^{(2)} + (z^{(2)})'Ew^{(2)} + (w^{(2)})'E'z^{(2)}\} \end{aligned}$$

with

$$(y^{(2)})' = [y_1' y_2'] = [y_{11} \dots y_{1m} y_{21} \dots y_{2m}]$$

$$(z^{(2)})' = [z_1' z_2'] = [z_{11} \dots z_{1p} z_{21} \dots z_{2p}]$$

$$(w^{(2)})' = [w_1' w_2'] = [w_{11} \dots w_{1q} w_{21} \dots w_{2q}]$$

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2 Note that since the variables $w^{(2)}$ and $x^{(2)}$ can not be influenced by a specific choice of the instrument variables $z^{(2)}$, no linear and quadratic terms in these variables are included in the loss function.

and where a and b denote real-valued vectors of orders 2m and 2p, while A, B, C, D, and E are real-valued matrices of orders 2m × 2m, 2p × 2p, 2m × 2p, 2m × 2q, and 2p × 2q, respectively.[2a]

Our problem now consists of devising an optimal strategy, i.e., a pair of decision functions z_1^0, z_2^0 - where z_t^0 (t = 1, 2) is a function of all information available at the beginning of period t - such that the mathematical expectation of the loss function (2.4) is minimized.

Before proceeding to the solution procedure it may, however, be worthwhile to give a more formal statement of the basic elements of our decision problem. These basic elements are:

(i) a pair of act spaces A_t (t = 1, 2) where A_t contains all admissible values of the instrument variables z_t in period t (t = 1, 2);

(ii) a parameter space P containing all possible values of the parameters of the econometric model (2.1), i.e., all possible values of the matrix of reduced-form coefficients Π and all admissible values of Ω^{-1}, the covariance matrix of the disturbances;

(iii) a pair of future event spaces F_t (t = 1, 2) where F_t contains all conceivable values of the non-controlled exogenous variables w_t and disturbances ξ_t in period t;

(iv) a pair of complete state spaces C_t (t = 1, 2) where each C_t is defined as the Cartesian product space $P \times F_t$. The decision maker believes that the consequence of adopting a particular strategy (i.e. a particular set of values for the instrument variables z_t (t = 1, 2)) depends on the "states of the world" $P \times F_t$ which he can not predict with certainty;

(v) a pair of reduced state spaces R_t (t = 1, 2) where R_t contains the potential values of all observable variables in period t which are unknown at the beginning of that period. In the present case R_t consists of all potential values of the endogenous variables y_t and the non-controlled exogenous variables w_t in period t. The reason for defining a pair of reduced state spaces R_t in addition to the pair of complete state spaces C_t is that most loss functions in planning problems contain observable variables only;

[2a] Without loss of generality A and B will be assumed to be symmetric.

(vi) a loss function (see (2.4)) defined on the Cartesian product space $A_1 \times A_2 \times R_1 \times R_2$. The decision maker assigns a loss to choosing a particular strategy and then finding that particular elements of the reduced state spaces R_t (t = 1, 2) obtain;

(vii) an initial probability distribution $f_0(\Pi, \Omega)$ on the parameter space P. For reasons of mathematical tractability we shall take as our initial probability distribution a so-called matrix Normal-Wishart distribution with parameters Π_0, S_0, N_0, and λ_0.[3] The density function corresponding with this distribution is given by

$$f_0(\Pi, \Omega \mid \Pi_0, S_0, N_0, \lambda_0) =$$

$$(2.5) \qquad = \frac{|\Omega|^{\frac{1}{2}n}|N_0|^{\frac{1}{2}m}}{(2\pi)^{\frac{1}{2}mn}} \exp\{-\tfrac{1}{2}\operatorname{tr}\Omega[\Pi - \Pi_0]N_0[\Pi - \Pi_0]'\} \times$$

$$\times \frac{|S_0|^{\frac{1}{2}\lambda_0}\ |\Omega|^{\frac{1}{2}(\lambda_0-m-1)} \exp\{-\tfrac{1}{2}\operatorname{tr}\Omega S_0\}}{2^{\frac{1}{2}\lambda_0 m}\pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma[\tfrac{1}{2}(\lambda_0 + 1 - i)]}$$

Multiplying this initial probability distribution with the likelihood function of the vector y, it can be proved[4] that the distribution $f_1(\Pi, \Omega \mid y_1, w_1, z_1)$ which expresses the information about (Π, Ω) at the beginning of the second period is again a matrix Normal-Wishart distribution with parameters $\Pi^{(1)}$, $S^{(1)}$, $N^{(1)}$ and $\lambda^{(1)}$, where

$$\Pi^{(1)} = [\Pi_0 N_0 + y_1 u_1'][N^{(1)}]^{-1}$$

$$(2.6) \qquad N^{(1)} = N_0 + u_1 u_1'$$

$$\lambda^{(1)} = \lambda_0 + 1$$

$$S^{(1)} = S_0 + \Pi_0 N_0 \Pi_0' + y_1 y_1' - \Pi^{(1)} N^{(1)} [\Pi^{(1)}]'$$

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3 In this paper we shall take it for granted that the information about the parameters of the econometric model (2.1) can be expressed by a matrix Normal-Wishart distribution. A detailed discussion about the question as to how prior and sample information about structures of econometric equations systems may be combined in order to yield a matrix Normal-Wishart distribution on the parameter space P may be found in Harkema (1971).

4 See, e.g., Harkema (1971), Chapter 2.

and

$$y_1' = [y_{11} \ldots y_{1m}]$$

$$u_1' = [u_{11} \ldots u_{1n}]$$

(viii) a probability distribution on the Cartesian product space $F_1 \times F_2$ which expresses the ideas of the decision maker about the growth patterns of the non-controlled exogenous variables and about the disturbances in the periods 1 and 2. It will be assumed that the vector of non-controlled exogenous variables $[w^{(2)}]' = [w_1'w_2']$ and the vector of disturbances $[\xi^{(2)}]' = [\xi_1'\xi_2']$ are statistically independent. From the normality assumption about the disturbances it then follows that the distribution on $F_1 \times F_2$ is most conveniently expressed as the product of the distribution of $[w^{(2)}]'$ times a normal distribution of $[\xi^{(2)}]'$, conditional upon the unknown covariance matrix Ω^{-1}. For the time being we shall not be specific about the shape of the distribution of $[w^{(2)}]'$ but only suppose it to have first and second moments.

2.2. A Solution by Backwards Induction

An optimal strategy can now be calculated by application of the well-known principle of backwards induction.[5] The first step of this solution procedure consists of minimizing the conditional expectation of the loss function (2.4), given each possible set of values of the vectors y_1, w_1 and z_1. In order to compute this conditional expectation we first derive the conditional distribution $f_1(w_2 \mid w_1)$ of w_2, given w_1, from the joint density of w_1 and w_2 specified under (viii). Next, we assign a conditional probability distribution to the complete state space C_2, given the values of y_1, w_1, and z_1, by means of the formula[6]

$$f_1(\xi_2, w_2, \Pi, \Omega \mid y_1, w_1, z_1) = f_1(w_2 \mid w_1) f(\xi_2 \mid \Omega) f_1(\Pi, \Omega \mid y_1, w_1, z_1) \tag{2.7}$$

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[5] See Raiffa and Schlaifer (1961), pp. 7-11.

[6] Note that w_2 is assumed to be independently distributed of Π, Ω, y_1 and z_1, while ξ_2 is assumed to be independently distributed of w_1, Π, y_1, and z_1.

Having calculated this distribution we can derive the conditional distribution $f_1(y_2, w_2 \mid y_1, w_1, z_1)$ on the reduced state space R_2 by means of the relations (2.2). The conditional expectation of L_2 given y_1, w_1, and z_1, is then found by integrating (2.4) with respect to the conditional distribution $f_1(y_2, w_2 \mid y_1, w_1, z_1)$. Minimizing the conditional expectation of L_2 given y_1, w_1, and z_1, with respect to z_2, we obtain the optimal value z_2^0 of the vector of instrument variables in period 2, given each possible set of values of y_1, w_1 and z_1. Substitution of the expression for z_2^0 into the conditional expectation of L_2 given y_1, w_1, and z_1, yields

$$\Lambda_1 = \min_{z_2} \{E(L_2 \mid y_1, w_1, z_1)\} \tag{2.8}$$

where Λ_1 represents the minimum expected loss at the beginning of the second period for each possible set of values of y_1, w_1, and z_1. Evidently z_2^0 as well as Λ_1 are functions of y_1, w_1, and z_1 and since y_1 and w_1 are random variables, z_2^0 and Λ_1 are random variables as well.

In the second step of the solution procedure we have to derive the optimal value z_1^0 of the vector of instrument variables in the first period. Applying the same procedure as in the first step we now have to minimize the mathematical expectation of Λ_1, given the information available at the beginning of the first period. So now we start with computing the marginal distribution of w_1 from the joint density specified under (viii). Next, we assign a probability distribution to the complete state space C_1 by means of the formula

$$\begin{aligned} f_0(\xi_1, w_1, \Pi, \Omega) &= \\ &= f_0(w_1)f(\xi_1 \mid \Omega)f_0(\Pi, \Omega \mid \Pi_0, S_0, N_0, \lambda_0) \end{aligned} \tag{2.9}$$

From this distribution we then derive the distribution $f_0(y_1, w_1)$ on the reduced state space R_1 by means of the relations (2.2). Integrating Λ_1 with respect to the distribution of y_1 and w_1 we obtain the mathematical expectation of Λ_1. After minimizing this mathematical expectation with respect to z_1, we find the optimal value z_1^0 of the vector of instrument variables in the first period. Substitution of the expression for z_1^0 into the mathematical expectation of Λ_1 yields

$$\Lambda_0 = \min_{z_1} \{E(\Lambda_1)\} \tag{2.10}$$

which represents the minimum expected loss for our decision problem, given the information available at the beginning of the first period.

2.3. An Illustration of Some Computational Difficulties

It will be clear that the solution procedure outlined above may be generalized so as to include any finite number of periods. Unfortunately, however, practical application of this solution procedure runs up against serious difficulties. These difficulties may be illustrated by means of the following simple example. Suppose a decision maker wants to optimize the values of a (scalar) target variable y_t in each of two successive periods t (t = 1, 2). The values of his target variable in each period t are supposed to be generated by the following single-equation model

$$y_t = \pi_1 y_{t-1} + \pi_2 z_t + \xi_t = \pi' u_t + \xi_t \qquad (t = 1, 2) \tag{2.11}$$

where y_t denotes a scalar target variable, z_t a scalar instrument variable and ξ_t a disturbance term. The disturbances ξ_1 and ξ_2 are supposed to be independently and normally distributed, each with zero mean and variance ω^{-1}. Moreover we shall assume that the decision maker's initial probability distribution of (π, ω) is Normal-gamma[7] with parameters π_0, s_0, N_0, and λ_0

$$f_0(\pi, \omega \mid \pi_0, s_0, N_0, \lambda_0) \propto \omega \exp\{-\tfrac{1}{2}\omega(\pi - \pi_0)' N_0 (\pi - \pi_0)\} \omega^{\frac{1}{2}\lambda_0 - 1} \exp\{-\tfrac{1}{2}\omega\lambda_0 s_0\} \tag{2.12}$$

The distribution $f_1(\pi, \omega \mid y_1, z_1)$ which expresses the decision maker's knowledge about π and ω at the beginning of the second period is then also Normal-gamma with parameters $\pi^{(1)}$, $s^{(1)}$, $N^{(1)}$ and $\lambda^{(1)}$ where

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[7] See Raiffa and Schlaifer (1961) pp. 318-319 and note that the Normal-gamma distribution is just a specific case of the matrix Normal-Wishart distribution defined in (2.5).

$$\pi^{(1)} = [N^{(1)}]^{-1}\{N_0\pi_0 + u_1y_1\}$$

$$N^{(1)} = N_0 + u_1u_1'$$

(2.13)
$$\lambda^{(1)} = \lambda_0 + 1$$

$$s^{(1)} = \{\lambda_0 s_0 + \pi_0'N_0\pi_0 + y_1^2 - [\pi^{(1)}]'N^{(1)}\pi^{(1)}\}/\lambda^{(1)}$$

and

$$u_1' = [y_0 \quad z_1] \qquad\qquad y_1 = [y_1]$$

Finally we shall suppose that the decision maker's preferences can be represented by a quadratic loss function of the following shape

(2.14)
$$L_2 = a'y^{(2)} + b'z^{(2)} + \tfrac{1}{2}\{(y^{(2)})'Ay^{(2)} + (z^{(2)})'Bz^{(2)}\}$$

where $(y^{(2)})' = [y_1 \quad y_2] \qquad (z^{(2)})' = [z_1 \quad z_2]$

In order to calculate an optimal strategy we start with assigning a joint probability distribution to ξ_2, π, and ω, given all information available at the beginning of the second period, i.e., the values of y_1 and z_1. Analogous to (2.7) we now define

(2.15)
$$f_1(\xi_2, \pi, \omega \mid y_1, z_1) =$$

$$f(\xi_2 \mid \omega)f_1(\pi, \omega \mid y_1, z_1) \propto$$
$$\omega^{\frac{3}{2}} \exp\{-\tfrac{1}{2}\omega[(\pi - \pi^{(1)})'\xi_2]\begin{bmatrix} N^{(1)} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \pi - \pi^{(1)} \\ \xi_2 \end{bmatrix}\} \times$$

$$\omega^{\frac{1}{2}\lambda^{(1)}-1} \exp\{-\tfrac{1}{2}\omega\lambda^{(1)}s^{(1)}\}$$

which is again a Normal-gamma distribution. Next, we have to calculate the conditional distribution of y_2, given y_1 and z_1. A convenient procedure to derive this distribution runs as follows. We start with computing the conditional distribution of ξ_2 and π, given ω, y_1, and z_1. From (2.15) it is easily seen that this distribution is normal with

$$E[\xi_2, \pi' \mid \omega, y_1, z_1] = [0 \quad (\pi^{(1)})']$$

(2.16)

$$V[\xi_2, \pi' \mid \omega, y_1, z_1] = \omega^{-1}\begin{bmatrix} 1 & 0 \\ 0 & N^{(1)} \end{bmatrix}^{-1}$$

From the model as specified in (2.11) we then obtain that the conditional distribution of y_2, given ω, y_1, and z_1, is also normal with

$$E[y_2 \mid \omega, y_1, z_1] = u_2'\pi^{(1)}$$

(2.17)

$$V[y_2 \mid \omega, y_1, z_1] = \omega^{-1}[1 + u_2'(N^{(1)})^{-1}u_2]$$

Multiplying the conditional distribution of y_2, given ω, y_1, and z_1, with the marginal distribution of ω, given y_1 and z_1, - which may be found in the last line of (2.15) - we find that the conditional distribution of y_2 and ω, given y_1 and z_1, is Normal-gamma with parameters $u_2'\pi^{(1)}$, $s^{(1)}[1 + u_2'(N^{(1)})^{-1}u_2]^{-1}$, and $\lambda^{(1)}$. On integrating this distribution with respect to ω we finally obtain that the conditional distribution of y_2, given y_1 and z_1 is Student[8] with parameters $u_2'\pi^{(1)}$, $1/\{s^{(1)}[1 + u_2'(N^{(1)})^{-1}u_2]\}$, and $\lambda^{(1)}$. The conditional expectation[9] of L_2, given y_1 and z_1, then follows from integrating (2.14) with respect to the distribution of y_2, given y_1 and z_1. This yields

[8] See Raiffa and Schlaifer (1961), pp. 318-320.

[9] We assume that the distribution of y_2, given y_1 and z_1, possesses sufficient degrees of freedom to ensure the existence of first and second moments.

$$E[L_2 | y_1, z_1] = a_2 u_2' \pi^{(1)} + b_2 z_2 +$$

(2.18)
$$+ \tfrac{1}{2} A_{22} [u_2' \pi^{(1)} (\pi^{(1)})' u_2 + \frac{\lambda^{(1)} s^{(1)}}{\lambda^{(1)} - 2} \{1 + u_2' (N^{(1)})^{-1} u_2\}] +$$
$$+ A_{12} y_1 u_2' \pi^{(1)} + \tfrac{1}{2} B_{22} z_2^2 + B_{12} z_1 z_2 + L_1$$

where

$$L_1 = a_1 y_1 + b_1 z_1 + \tfrac{1}{2} A_{11} y_1^2 + \tfrac{1}{2} B_{11} z_1^2$$

and

$$u_2' = [y_1 \quad z_2]$$

On differentiating this expression with respect to z_2 the optimal value z_2^0 of z_2 is found to be equal to

$$z_2^0 = -\{A_{22} [\pi_2^{(1)}]^2 + \frac{A_{22} \lambda^{(1)} s^{(1)} [N^{(1)}]^{22}}{\lambda^{(1)} - 2} + B_{22}\}^{-1} \times$$

(2.19)
$$\times \{a_2 \pi_2^{(1)} + b_2 + [A_{22} \pi_1^{(1)} \pi_2^{(1)} + \frac{A_{22} [N^{(1)}]^{12} \lambda^{(1)} s^{(1)}}{\lambda^{(1)} - 2} + A_{12} \pi_2^{(1)}] y_1$$
$$+ B_{12} z_1\}$$

After inserting the expression for z_2^0 into (2.18) and applying some algebraic rearrangements, the minimum of the conditional expectation of L_2, given y_1 and z_1, may be written as

$$\Lambda_1 = \min_{z_2} \{E(L_2 | y_1, w_1, z_1)\} =$$
$$= (a_1 + a_2 \pi_1^{(1)} + \tfrac{1}{2} A_{22} \pi_1^{(1)} \pi_2^{(1)} z_2^0 + \frac{\tfrac{1}{2} A_{22} \lambda^{(1)} s^{(1)} [N^{(1)}]^{12} z_2^0}{\lambda^{(1)} - 2} +$$

(2.20)
$$+ \tfrac{1}{2} A_{12} \pi_2^{(1)} z_2^0) y_1 + (b_1 + \tfrac{1}{2} B_{12} z_2^0) z_1 +$$
$$+ \tfrac{1}{2} [A_{11} + A_{22} (\pi_1^{(1)})^2 + \frac{A_{22} \lambda^{(1)} s^{(1)}}{\lambda^{(1)} - 2} [N^{(1)}]^{11} + 2 A_{12} \pi_1^{(1)}] y_1^2 +$$
$$+ \tfrac{1}{2} B_{11} z_1^2 + \frac{\tfrac{1}{2} A_{22} \lambda^{(1)} s^{(1)}}{\lambda^{(1)} - 2} + \tfrac{1}{2} a_2 \pi_2^{(1)} z_2^0 + \tfrac{1}{2} b_2 z_2^0$$

In the second step of the solution procedure we have to calculate the mathematical expectation of Λ_1, given all information available at the beginning of the first period. Proceeding along the same lines as before we now assign a joint probability distribution to ξ_1, π, and ω by means of the formula

$$(2.21) \qquad f_0(\xi_1, \pi, \omega) = f(\xi_1 \mid \omega) f_0(\pi, \omega \mid \pi_0, s_0, N_0, \lambda_0)$$

where $f_0(\pi, \omega \mid \pi_0, s_0, N_0, \lambda_0)$ has been defined in (2.12). Applying the same procedure as in the first step of the solution procedure it then follows that the marginal distribution of y_1 is again a Student distribution with parameters $u_1'\pi_0$, $1/\{s_0[1 + u_1'N_0^{-1}u_1]\}$, and λ_0. In order to calculate the mathematical expectation of Λ_1, we now have to integrate (2.20) with respect to the distribution of y_1. At this point, however, we are faced with serious difficulties. Since $\pi^{(1)}$, $s^{(1)}$ and z_2^0 (compare (2.13) and (2.19)) depend upon y_1, (2.20) represents a rather complicated function of y_1. To make this clear it will be worthwhile to have a closer look at its last term $\frac{1}{2}b_2z_2^0$. In terms of y_1 this last term may be written as

$$(2.22) \qquad \tfrac{1}{2}b_2z_2^0 = \frac{f_0 + f_1y_1 + f_2y_1^2 + f_3y_1^3}{f_4 + f_5y_1 + f_6y_1^2}$$

where the coefficients f_i $(i = 0, \ldots, 6)$ do not depend on y_1 but represent rather intricate functions of the instrument variable z_1. For the time being integration of an expression like (2.22) with respect to a Student distribution seems to go beyond our technical possibilities. Thus at this point the solution procedure breaks down.

It may be interesting to note that a less negative result is obtained when the loss function is assumed to be linear in the target variables. In that case we have $A = 0$ so that the expressions for z_2^0 and Λ_1 reduce to

$$(2.19^*) \qquad z_2^0 = -\frac{1}{B_{22}}\{a_2\pi_2^{(1)} + b_2 + B_{12}z_1\}$$

and

$$\Lambda_1 = (a_1 + a_2\pi_1^{(1)})y_1 + (b_1 + \tfrac{1}{2}B_{12}z_2^0)z_1 + \tag{2.20*}$$
$$+ \tfrac{1}{2}B_{11}z_1^2 + \tfrac{1}{2}a_2\pi_2^{(1)}z_2^0 + \tfrac{1}{2}b_2z_2^0$$

As the elements of $\pi^{(1)}$ (compare (2.13)) are linear functions of y_1, (2.20*) represents a quadratic function of y_1 and z_1. So we may now compute an optimal first-period decision along the same lines as in the first step of the solution procedure for the more general problem treated before. It must be stressed that even when the loss function is linear in the target variables an optimal first-period decision can be computed for a two-period decision problem only. Since (2.20*) is quadratic in y_1, attempts to extend the analysis to a larger number of periods are bound to run into the same difficulties as met before.

From the preceding discussions it will be clear that the difficulties arise because of our inability to handle the fact that in each period additional information will be obtained about the true values of the reduced-form coefficients of the model. In the next section we shall present therefore an approach which neglects this type of information. More specifically, we shall assume that the uncertainty about the reduced-form parameters is the same in each period and is expressed by means of the initial probability distribution of the reduced-form parameters at the beginning of the first period. Especially when the planning period is not too long it may be hoped that the various probability distributions which express our knowledge about the reduced-form parameters at the beginning of each period will not be very different. In such cases the initial probability distribution of the reduced-form parameters may provide a reasonable substitute for the probability distributions of the reduced-form parameters in later periods. Of course, this does not mean that we completely neglect the fact that information will become available in later periods. We still take into account the fact that decisions in later periods may be influenced by the realized values of the variables in the preceding periods.

3. AN APPROXIMATE SOLUTION

In the present section the solution procedure, as outlined in the last paragraph of Section 2.3 will be carried out analytically. Formally, the decision problem can be stated as[10]

$$\text{(3.1)} \qquad \min_{z_1} E_{(y_1,w_1)} \{\min_{z_2} E_{(y_2,w_2|y_1,w_1,z_1)}[L_2]\}$$

subject to

$$y_t = \Pi u_t + \xi_t \qquad (t = 1, 2)$$

or equivalently

$$y_t = \Pi_1 y_{t-1} + \Pi_2 z_t + \Pi_3 z_{t-1} + \Pi_4 w_t + \Pi_5 w_{t-1} + \Pi_6 x_t + \xi_t$$

where L_2 is defined by (2.4).

Our first task will be to calculate the conditional density $f_1(y_2, w_2 \mid y_1, w_1, z_1)$ on the reduced state space R_2. As stated before we are unable to handle the element of learning about the true values of the reduced-form parameters (Π, Ω). Therefore, we assume that in each period the uncertainty about the reduced-form parameters (Π, Ω) may be approximated by means of the initial probability distribution defined in (2.5), i.e.,

$$\text{(3.2)} \qquad f_1(\Pi, \Omega \mid y_1, w_1, z_1) = f_0(\Pi, \Omega)$$

Adhering to the procedure described in Section 2.2, we then assign a probability distribution to the complete state space C_2 by defining[11]

---

[10] Note that we use the symbol $E_{(a/b)}[c]$ to indicate the mathematical expectation of c with respect to the conditional distribution of a, given b. So $E_{(a/b)}[c] = \int cf(a/b)da$.

[11] Later on it will turn out that we only need the first and second moments of the conditional distribution $f(w_2|w_1)$ of w_2, given w_1. Therefore we shall not explicitly specify this distribution.

$$f_1(\Pi\ ,\ \Omega,\ w_2,\ \xi_2\ |\ y_1,\ w_1,\ z_1) = f_1(\Pi,\ \Omega,\ w_2,\ \xi_2\ |\ w_1) \propto f(w_2\ |\ w_1)\ \times$$

$$(3.3)\qquad |\Omega|^{\frac{1}{2}(n+1)} \exp\{-\tfrac{1}{2}\ \mathrm{tr}\ \Omega[\Pi - \Pi_0,\ \xi_2]\begin{bmatrix} N_0 & 0 \\ 0 & 1 \end{bmatrix}[\Pi - \Pi_0,\ \xi_2]'\ \times$$

$$|\Omega|^{\frac{1}{2}(\lambda_0-m-1)} \exp\{-\tfrac{1}{2}\ \mathrm{tr}\ \Omega S_0\}$$

In order to derive the conditional distribution of y_2 and w_2, given y_1, w_1 and z_1, from (3.3) it appears to be convenient to switch to the conditional distribution of Π and ξ_2, given Ω, w_2 and w_1. It is easily seen that the conditional distribution of the vector $h' = [\pi_{11} \ldots \pi_{1n}\xi_{21}, \ldots, \pi_{m1} \ldots \pi_{mn}\xi_{2m}]$, given Ω, w_2 and w_1 is normal with

$$(3.4)\qquad \begin{aligned} E[h'\ |\ \Omega,\ w_2,\ w_1] &= [\pi_{011} \ldots \pi_{01n}0,\ \ldots,\ \pi_{0m1} \ldots \pi_{0mn}0] \equiv h_0' \\ V[h\ |\ \Omega,\ w_2,\ w_1] &= \Omega^{-1} \otimes \begin{bmatrix} N_0 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \equiv H_0 \end{aligned}$$

Rewriting the reduced-form equations (3.1) in the second period as[12]

$$(3.5)\qquad y_2 = [I_m \otimes (u_2^*)']\ h$$

where

$$(u_2^*)' = [u_2'\quad 1] = [y_1'\ \ z_2'\ \ z_1'\ \ w_2'\ \ w_1'\ \ x_2'\ \ 1]$$

it follows that the conditional distribution of y_2, given Ω, w_2, y_1, w_1 and z_1, is again normal with

$$E[y_2\ |\ \Omega,\ w_2,\ y_1,\ w_1,\ z_1] = [I_m \otimes (u_2^*)']h_0 = \Pi_0 u_2$$

$$\begin{aligned} V[y_2\ |\ \Omega,\ w_2,\ y_1,\ w_1,\ z_1] &= [I_m \otimes (u_2^*)']H_0[I_m \otimes (u_2^*)']' \\ &= [1 + u_2'N_0^{-1}u_2]\Omega^{-1} \equiv \beta_2\Omega^{-1} \end{aligned}$$

---

[12] Arithmetic rules for Kronecker matrix products may be found in Dhrymes (1970).

Multiplication of the above conditional distribution of y_2 with the marginal distribution of Ω, the kernel of which may be found in the last line of (3.3), yields

$$f(y_2, \Omega \mid w_2, y_1, w_1, z_1) \propto$$

$$(3.7) \qquad |\Omega|^{\frac{1}{2}} \exp\{-\tfrac{1}{2} \operatorname{tr} \Omega[y_2 - \Pi_0 u_2][y_2 - \Pi_0 u_2]'/\beta_2\} \times$$

$$|\Omega|^{\frac{1}{2}(\lambda_0 - m - 1)} \exp\{-\tfrac{1}{2} \operatorname{tr} \Omega S_0\}$$

By integrating this expression over all admissible values of Ω, the conditional distribution of y_2, given w_2, y_1, w_1 and z_1, can be proved to be multivariate Student with parameters $\Pi_0 u_2$, $(\lambda_0 - m + 1)S_0^{-1}/\beta_2$ and $\lambda_0 - m + 1$. The conditional distribution of y_2 and w_2, given y_1, w_1, and z_1, then follows from multiplying the conditional distribution of y_2, given w_2, y_1, w_1, and z_1, with the conditional distribution $f(w_2 | w_1)$ of w_2, given[13] w_1.

A convenient way of calculating the conditional expectation of the loss function L_2, given y_1, w_1, and z_1, is provided by the well-known formula

$$(3.8) \qquad E_{(y_2, w_2 | y_1, w_1, z_1)}[L_2] = E_{(w_2|w_1)}[E_{(y_2|w_2, y_1, w_1, z_1)}[L_2]]$$

As the first and second moments of the distribution of y_2, given w_2, y_1, w_1 and z_1, are given by[14]

$$E[y_2 \mid w_2, y_1, w_1, z_1] = \Pi_0 u_2$$

$$(3.9)$$

$$E[y_2 y_2' \mid w_2, y_1, w_1, z_1] = \frac{\beta_2 S_0}{\lambda_0 - m - 1} + \Pi_0 u_2 u_2' \Pi_0'$$

---

13 It is assumed that the conditional distribution of w_2, given w_1, is independent of y_1 and z_1.

14 See Raiffa and Schlaifer (1961), pp. 256-257.

it is easily seen that

$$E_{(y_2|w_2,y_1,w_1,z_1)}[L_2] = a_2'\Pi_0 u_2 + b_2' z_2 + \tfrac{1}{2}\{\alpha u_2' N_0^{-1} u_2 + u_2'\Pi_0' A_{22}\Pi_0 u_2$$

$$(3.10) \quad + 2y_1' A_{12}\Pi_0 u_2 + z_2' B_{22} z_2 + 2z_1' B_{12} z_2 + 2y_1' C_{12} z_2 + 2u_2'\Pi_0' C_{21} z_1 + 2u_2'\Pi_0' C_{22} z_2$$

$$+ 2y_1' D_{12} w_2 + 2u_2'\Pi_0' D_{21} w_1 + 2u_2'\Pi_0' D_{22} w_2 + 2z_1' E_{12} w_2 + 2z_2' E_{21} w_1$$

$$+ 2z_2' E_{22} w_2\} + L_1 + k$$

with

$$L_1 = a_1' y_1 + b_1' z_1 + \tfrac{1}{2}\{y_1' A_{11} y_1 + z_1' B_{11} z_1 + y_1' C_{11} z_1$$

$$+ z_1' C_{11}' y_1 + y_1' D_{11} w_1 + w_1' D_{11}' y_1 + z_1' E_{11} w_1 + w_1' E_{11}' z_1\}$$

and where

$$\alpha = \frac{tr(A_{22} S_0)}{\lambda_0 - m - 1}$$

and k denotes a known but irrelevant constant.

In order to determine the conditional expectation of the loss function L_2, given y_1, w_1 and z_1, we only need to know the first and second moments of the conditional distribution of w_2, given w_1. We will assume that the decision maker's ideas about the growth patterns of the non-controlled exogenous variables can be represented by

$$(3.11) \quad \begin{aligned} E[w_2 \mid w_1] &= \Gamma w_1 \equiv \bar{w}_2 \\ E[w_2 w_2' \mid w_1] &= W_1 \Delta W_1 \end{aligned}$$

where W_1, Γ, and Δ denote real-valued diagonal matrices of orders $q \times q$. The diagonal elements of W_1 are the values w_{1i} $(i = 1, \ldots, q)$ of the non-controlled exogenous variables in the first period; the diagonal elements of Γ and Δ represent the means and second moments of the probability distributions which represent the decision maker's ideas about the growth rates of the non-controlled exogenous variables.

The conditional expectation of the loss function L_2 given y_1, w_1 and z_1, then equals

$$
\begin{aligned}
&E_{(y_2, w_2 | y_1, w_1, z_1)}[L_2] = a_2'\Pi_0\bar{u}_2 + b_2'z_2 + \tfrac{1}{2}\{\alpha \text{ tr } (N_0^{-1}\hat{U}^2) \\
(3.12) \quad &+ \text{tr } (\Pi_0'A_{22}\Pi_0\hat{U}^2) + 2y_1'A_{12}\Pi_0\bar{u}_2 + z_2'B_{22}z_2 + 2z_1'B_{12}z_2 + 2y_1'C_{12}z_2 \\
&+ 2\bar{u}_2'\Pi_0'C_{21}z_1 + 2\bar{u}_2'\Pi_0'C_{22}z_2 + 2y_1'D_{12}\bar{w}_2 + 2\bar{u}_2'\Pi_0'D_{21}w_1 + 2 \text{ tr } (\Pi_0'D_{22}\hat{U}_{4.}^2) \\
&+ 2z_1'E_{12}\bar{w}_2 + 2z_2'E_{21}w_1 + 2z_2'E_{22}\bar{w}_2\} + L_1 + k
\end{aligned}
$$

where

$$\bar{u}_2 = E_{(w_2|w_1)}[u_2] = [y_1' \quad z_2' \quad z_1' \quad \bar{w}_2' \quad w_1' \quad x_2']$$

$$(3.13) \qquad \hat{U}^2 = E_{(w_2|w_1)}[u_2u_2'] =$$

$$
= \begin{bmatrix} \hat{U}_{11}^2 & \cdots & \hat{U}_{16}^2 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \hat{U}_{61}^2 & \cdots & \hat{U}_{66}^2 \end{bmatrix}
= \begin{bmatrix}
y_1y_1' & y_1z_2' & y_1z_1' & y_1\bar{w}_2' & y_1w_1' & y_1x_2' \\
z_2y_1' & z_2z_2' & z_2z_1' & z_2\bar{w}_2' & z_2w_1' & z_2x_2' \\
z_1y_1' & z_1z_2' & z_1z_1' & z_1\bar{w}_2' & z_1w_1' & z_1x_2' \\
\bar{w}_2y_1' & \bar{w}_2z_2' & \bar{w}_2z_1' & E[w_2w_2'] & \bar{w}_2w_1' & \bar{w}_2x_2' \\
w_1y_1' & w_1z_2' & w_1z_1' & w_1\bar{w}_2' & w_1w_1' & w_1x_2' \\
x_2y_1' & x_2z_2' & x_2z_1' & x_2\bar{w}_2' & x_2w_1' & x_2x_2'
\end{bmatrix}
$$

and

$$\hat{U}_{4.}^2 = [\hat{U}_{41}^2 \; \ldots \; \hat{U}_{46}^2]$$

The next step in our solution procedure consists of minimizing (3.12) with respect to the vector z_2. In order to find this minimum we will differentiate (3.12) with respect to z_2, where it must be noted that z_2 is also a subvector of $\bar{u}_2$. This yields the following first-order conditions

$$
\begin{aligned}
(3.14)\quad & \Pi'_{02}a_2 + b_2 + (\alpha N_0^{21} + \Pi'_{02}A_{22}\Pi_{01} + C'_{12} + \Pi'_{02}A_{21} + C'_{22}\Pi_{01})y_1 \\
& + (\alpha N_0^{22} + \Pi'_{02}A_{22}\Pi_{02} + B_{22} + \Pi'_{02}C_{22} + C'_{22}\Pi_{02})z_2 + (\alpha N_0^{23} + \Pi'_{02}A_{22}\Pi_{03} \\
& + B'_{12} + \Pi'_{02}C_{21} + C'_{22}\Pi_{03})z_1 + (\alpha N_0^{24} + \Pi'_{02}A_{22}\Pi_{04} + E_{22} + \Pi'_{02}D_{22} + C'_{22}\Pi_{04})\bar{w}_2 \\
& + (\alpha N_0^{25} + \Pi'_{02}A_{22}\Pi_{05} + E_{21} + \Pi'_{02}D_{21} + C'_{22}\Pi_{05})w_1 + \\
& + (\alpha N_0^{26} + \Pi'_{02}A_{22}\Pi_{06} + C'_{22}\Pi_{06})x_2 = 0
\end{aligned}
$$

where the N_0^{2j} ($j = 1, \ldots, 6$) denote the appropriate submatrices of

$$
N_0^{-1} = \begin{bmatrix} N_0^{11} & \cdots\cdots & N^{16} \\ \vdots & & \vdots \\ N_0^{61} & & N_0^{66} \end{bmatrix}
$$

Thus the second-period optimal values of the decision variables z_2 are given by

$$
(3.15)\qquad z_2^0 = \ell_2 + P_2y_1 + Q_2z_1 + R_2w_1
$$

where

$$
\ell_2 = -F_2^{-1}\,[\Pi'_{02}a_2 + b_2 + (K_{26}^2 + C'_{22}\Pi_{06})x_2]
$$

$$
P_2 = -F_2^{-1}\,[K_{21}^2 + C'_{12} + \Pi'_{02}A_{21} + C'_{22}\Pi_{01}]
$$

$$
Q_2 = -F_2^{-1}\,[K_{23}^2 + B'_{12} + \Pi'_{02}C_{21} + C'_{22}\Pi_{03}]
$$

$$
\begin{aligned}
R_2 = -F_2^{-1}\,[& K_{25}^2 + K_{24}^2\Gamma + E_{21} + E_{22}\Gamma + \Pi'_{02}D_{21} + \Pi'_{02}D_{22}\Gamma + \\
& + C'_{22}\Pi_{05} + C'_{22}\Pi_{04}\Gamma]
\end{aligned}
$$

$$F_2 = K^2_{22} + B_{22} + \Pi'_{02}C_{22} + C'_{22}\Pi_{02}$$

and

$$K^2_{ij} = \alpha N_0^{ij} + \Pi'_{0i}A_{22}\Pi_{0j} \qquad (i, j = 1, \ldots, 6)$$

provided that the matrix of second-order derivatives is positive definite. This matrix can be written as

$$[I \quad \Pi'_{02}] \begin{bmatrix} \alpha N_0^{22} + B_{22} & C'_{22} \\ C_{22} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ \Pi_{02} \end{bmatrix}$$

Evidently, a sufficient but not necessary[15] condition for z_2^0 to be optimal is that the matrix of the quadratic part of the loss function is positive definite.

Substitution of the expression (3.15) for z_2^0 into the conditional expectation of L_2, given y_1, w_1 and z_1 yields

$$\Lambda_1 = \min_{z_2} E_{(y_2, w_2 | y_1, w_1, z_1)}[L_2] \tag{3.16}$$

where Λ_1 represents the minimum expected loss at the beginning of the second period for each possible set of values of y_1, w_1 and z_1. Evidently Λ_1 is a function of y_1, w_1 and z_1, and can be written in the following form

$$\begin{aligned} \Lambda_1 = {} & (a_1^*)'y_1 + (b_1^*)'z_1 + \tfrac{1}{2}\{y_1'A_{11}^*y_1 + z_1'B_{11}^*z_1 + y_1'C_{11}^*z_1 \\ & + z_1'(C_{11}^*)'y_1 + y_1'D_{11}^*w_1 + w_1'(D_{11}^*)'y_1 + z_1'E_{11}^*w_1 + w_1'(E_{11}^*)'z_1\} + k \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} (a_1^*)' = {} & a_1' + a_2'(\Pi_{01} + \Pi_{02}P_2) + b_2'P_2 + \ell_2'(K^2_{21} + F_2P_2 + \Pi'_{02}A_{21} + C'_{12} + C'_{22}\Pi_{01}) \\ & + x_2'(K^2_{61} + K^2_{62}P_2 + \Pi'_{06}A_{21} + \Pi'_{06}C_{22}P_2) \end{aligned}$$

---

[15] In fact, the present approach only requires the loss function to be quadratic in at least one of the second-period target variables.

$$(b_1^*)' = b_1' + a_2'(\Pi_{03} + \Pi_{02}Q_2) + b_2'Q_2 + \ell_2'(K_{23}^2 + F_2Q_2 + \Pi_{02}'C_{21} + B_{12}' + C_{22}'\Pi_{03})$$
$$+ x_2'(K_{63}^2 + K_{62}^2Q_2 + \Pi_{06}'C_{21} + \Pi_{06}'C_{22}Q_2)$$

$$A_{11}^* = A_{11} + K_{11}^2 + P_2'F_2P_2 + P_2'K_{21}^2 + K_{12}^2P_2 + P_2'C_{12}' + C_{12}P_2$$
$$+ (\Pi_{01}' + P_2'\Pi_{02}')A_{21} + A_{21}'(\Pi_{01} + \Pi_{02}P_2) + P_2'C_{22}'\Pi_{01} + \Pi_{01}'C_{22}P_2$$

$$B_{11}^* = B_{11} + K_{33}^2 + Q_2'F_2Q_2 + Q_2'K_{23}^2 + K_{32}^2Q_2 + Q_2'B_{12}' + B_{12}Q_2$$
$$+ (\Pi_{03}' + Q_2'\Pi_{02}')C_{21} + C_{21}'(\Pi_{03} + \Pi_{02}Q_2) + Q_2'C_{22}'\Pi_{03} + \Pi_{03}'C_{22}Q_2$$

$$C_{11}^* = C_{11} + K_{13}^2 + P_2'F_2Q_2 + P_2'K_{23}^2 + K_{12}^2Q_2 + P_2'B_{12}' + C_{12}Q_2$$
$$+ (\Pi_{01}' + P_2'\Pi_{02}')C_{21} + A_{21}'(\Pi_{03} + \Pi_{02}Q_2) + P_2'C_{22}'\Pi_{03} + \Pi_{01}'C_{22}Q_2$$

$$D_{11}^* = D_{11} + D_{12}\Gamma + K_{15}^2 + K_{14}^2\Gamma + P_2'F_2R_2 + P_2'(K_{25}^2 + K_{24}^2\Gamma) + K_{12}^2R_2$$
$$+ P_2'(E_{21} + E_{22}\Gamma) + C_{12}R_2 + (\Pi_{01}' + P_2'\Pi_{02}')(D_{21} + D_{22}\Gamma)$$
$$+ A_{21}'(\Pi_{05} + \Pi_{04}\Gamma + \Pi_{02}R_2) + P_2'C_{22}'(\Pi_{05} + \Pi_{04}\Gamma) + \Pi_{01}'C_{22}R_2$$

$$E_{11}^* = E_{11} + E_{12}\Gamma + K_{35}^2 + K_{34}^2\Gamma + Q_2'F_2R_2 + Q_2'(K_{25}^2 + K_{24}^2\Gamma) + K_{32}^2R_2$$
$$+ Q_2'(E_{21} + E_{22}\Gamma) + B_{12}R_2 + (\Pi_{03}' + Q_2'\Pi_{02}')(D_{21} + D_{22}\Gamma)$$
$$+ C_{21}'(\Pi_{05} + \Pi_{04}\Gamma + \Pi_{02}R_2) + Q_2'C_{22}'(\Pi_{05} + \Pi_{04}\Gamma) + \Pi_{03}'C_{22}R_2$$

In the second step of the solution procedure we have to derive the optimal value z_1^0 of the vector of instrument variables in the first period. Applying the same procedure as in the first step we now have to minimize the mathematical expectation of Λ_1, given the information at the beginning of the first period. Analogous to (3.9) and (3.11) we now have

$$E[y_1 \mid w_1, y_0, w_0, z_0] = \Pi_0 u_1 \tag{3.18}$$

$$E[y_1 y_1' \mid w_1, y_0, w_0, z_0] = \frac{\beta_1 S_0}{\lambda_0 - m - 1} + \Pi_0 u_1 u_1' \Pi_0'$$

where

$$\beta_1 = 1 + u_1' N_0^{-1} u_1$$

and

$$E[w_1 \mid w_0] = \Gamma w_0 \equiv \bar{w}_1 \tag{3.19}$$

$$E[w_1 w_1' \mid w_0] = W_0 \Delta W_0$$

with

$$W_0 = \begin{bmatrix} w_{01} & & 0 \\ & & \\ 0 & & w_{0q} \end{bmatrix}$$

and where Γ and Δ are specified in (3.11). So the first-period optimal decision is directly given by

$$z_1^0 = \ell_1 + P_1 y_0 + Q_1 z_0 + R_1 w_0 \tag{3.20}$$

where

$$\ell_1 = -F_1^{-1}[\Pi_{02}' a_1^* + b_1^* + (K_{26}^1 + (C_{11}^*)'\Pi_{06})x_1]$$

$$P_1 = -F_1^{-1}[K_{21}^1 + (C_{11}^*)'\Pi_{01}]$$

$$Q_1 = -F_1^{-1}[K^1_{23} + (C^*_{11})'\Pi_{03}]$$

$$R_1 = -F_1^{-1}[(K^1_{24} + (C^*_{11})'\Pi_{04} + \Pi'_{02}D^*_{11} + E^*_{11})\Gamma + K^1_{25} + (C^*_{11})'\Pi_{05}]$$

$$F_1 = K^1_{22} + B^*_{11} + \Pi'_{02}C^*_{11} + (C^*_{11})'\Pi_{02}$$

and

$$K^1_{ij} = \alpha N_0^{ij} + \Pi'_{0i}A^*_{11}\Pi_{0j} \qquad (i, j = 1, \ldots, 6)$$

Generalization of this procedure to decision problems which involve more than two periods is immediate, provided that the matrices in the quadratic part of the loss function are band matrices. Suppose that our loss function for T periods is given by

$$\text{(3.21)} \quad \begin{aligned} L_T = {} & a'y^{(T)} + b'z^{(T)} + \tfrac{1}{2}\{(y^{(T)})'Ay^{(T)} + (z^{(T)})'Bz^T + \\ & + (y^{(T)})'Cz^{(T)} + (z^{(T)})'C'y^{(T)} + (y^{(T)})'Dw^{(T)} + \\ & + (w^{(T)})'D'y^{(T)} + (z^{(T)})'Ew^{(T)} + (w^{(T)})'E'z^{(T)}\} \end{aligned}$$

where the matrices A, B, C, D, and E are band matrices of the following form

$$\text{(3.22)} \qquad A = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & & & 0 \\ A_{21} & A_{22} & A_{23} & 0 & \cdots & & \vdots \\ 0 & A_{32} & A_{33} & & & & \vdots \\ \vdots & & & & & & \vdots \\ \vdots & & & & A_{T-2,T-2} & A_{T-2,T-1} & 0 \\ \cdots & & & 0 & A_{T-1,T-2} & A_{T-1,T-1} & A_{T-1,T} \\ 0 & \cdots & & & 0 & A_{T,T-1} & A_{TT} \end{bmatrix}$$

while analogous specifications hold for B, C, D, and E.

Then our loss function can also be written as

$$L_T = a_T'y_T + b_T'z_T + \frac{1}{2}\{y_T'A_{TT}y_T + 2y_{T-1}'A_{T-1,T}y_T$$

$$(3.23) \qquad + z_T'B_{TT}z_T + 2z_{T-1}'B_{T-1,T}z_T + \ldots + 2z_{T-1}'E_{T-1,T}w_T +$$

$$+ 2z_T'E_{T,T-1}w_{T-1} + 2z_T'E_{TT}w_T\} + L_{T-1}$$

On calculating the mathematical expectation of (3.23) with respect to the conditional distribution of y_T, given w_T, y_{T-1}, w_{T-1}, and z_{T-1}, we obtain an expression which is quite analogous to (3.10). So the optimal value z_T^0 of the instrument variables at the beginning of the T-th period is found by replacing all appropriate superscripts and subscripts 2 and 1 in formula (3.15) by T and T-1, respectively. In the same way the minimum expected loss Λ_{T-1} at the beginning of the T-th period may be derived by appropriate adjustment of formula (3.17). Continuing in this way we can next derive the optimal value z_{T-1}^0 of the instrument variables in period T - 1 and the corresponding minimum expected loss Λ_{T-2}. This procedure, which can be carried out on any computer of sufficient size, finally yields the vectors and matrices which determine the minimum expected loss Λ_1 at the beginning of the second period. The optimal first-period decision is then given by (3.20).

When the matrices of the quadratic part of the loss function are not of the form (3.22) we may of course apply the same procedure as outlined before. In addition to y_{T-1}, w_{T-1}, and z_{T-1}, the optimal value z_T^0 of the instrument variables in the T-th period then turns out to depend also on the values y_{T-2} through y_0, w_{T-2} through w_0, and z_{T-2} through z_0. This implies that our formulas (3.15), (3.17), and (3.20) cannot simply be generalized so as to include this case as well. Development of formulas which cover this case is a straightforward, although rather tedious affair which is outside the scope of the present paper.

4. CONCLUDING REMARKS

In this paper we have investigated whether optimal control policies can be derived for multi-period macro-economic planning problems when the multipliers of the underlying model are not known with certainty. In this section we draw some conclusions and indicate possible directions for future research.

To start with, we found that one runs into insoluble computational problems if one tries to handle the fact that in each period additional information will be obtained about the true values of the reduced-form parameters. It turned out that these problems can only be solved for a two-period decision problem with a loss function which is linear in the target variables.

We therefore followed an approach that neglects this type of information. More specifically, we assumed that the uncertainty about the reduced-form parameters is the same in each period and is expressed by their initial probability distribution at the beginning of the first period. Using this approximation we found that application of the principle of backwards induction leads to a simple linear decision rule for the first period decision.

Unfortunately we cannot be sure whether the initial probability distribution of the reduced-form parameters provides a reasonable substitute for the probability distributions in later periods. At any rate, it may be hoped that, if the horizon of our planning problem is short, the various probability distributions which express our knowledge about the reduced-form parameters will not be very different.

It might be worthwhile to do some research in this area with a numerically specified model. In a two-period decision problem with a loss function which is linear in the target variables, our approximate solution can be compared numerically with the optimal solution as described in Section 2.3.

For a general multi-period decision problem a simulation study can be performed, in which decisions are derived by means of our approximate solution procedure as well as Theil's certainty equivalence approach.[16] First-period decisions can be calculated numerically and then substituted into the model underlying our decision problem. Subsequently first-period values for the endogenous and non-controlled exogenous variables can be generated on a computer.

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[16] See Theil (1964).

Second-period decisions can then be calculated by simple substitution of the first-period values of the relevant variables into the second-period decision rules. Continuing in this fashion one can calculate decisions and generate values for the endogenous and non-controlled exogenous variables in all periods. Substitution of these values into the loss function yields a final loss for both our approximate solution procedure and the certainty equivalence approach, which enables one to compare these two methods.

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