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SOME STATISTICAL IMPLICATIONS OF THE  
LOG TRANSFORMATION OF MULTIPLICATIVE MODELS

by R. Teekens and J. Koerts

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# SOME STATISTICAL IMPLICATIONS OF THE LOG TRANSFORMATION OF MULTIPLICATIVE MODELS

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## 1. INTRODUCTION

This paper deals with some of the estimation problems inherent in multiplicative models with constant elasticities. Such models (for example, production functions of the Cobb-Douglas type, gravitational

trade flow equations, and multiplicative demand functions) frequently occur in economic theory. In order to estimate the parameters of such relationships, we should introduce a disturbance term into the model. It is logical to introduce such a term into the multiplicative model at the beginning and to make the necessary assumptions about its distribution. After this has been accomplished, the problem is clearly defined and we can pass on to estimation.

The preceding paragraph may seem trivial but it should be realized that there is a lot of confusion about this subject. The estimation problem is normally solved after the multiplicative relation has been transformed so as to put it in the form of the well-known linear model. This procedure gives rise to the following remarks. Firstly, we may ask ourselves whether the transformed problem is identical to the original one. Secondly, in applying this procedure there is a strong inclination towards postponing the introduction of the stochastic model until the derivation of the linear model. This implies a danger that one does not realize what these assumptions mean for the original multiplicative model, with the possibility that the implied stochastic model on the multiplicative relation is not in accordance with one's original ideas. Thirdly, in many cases we are not interested in parameter estimates but in the estimation of the expectation of the dependent variable given a vector of values of the explanatory variables,<sup>1</sup> and the backward transformation may again give rise to inconsistencies.

Indeed, the main issue of this paper is the estimation of  $E(Y | x_p)$ . For the time being some limiting assumptions will be set on the distribution of the disturbance term. Under these assumptions it seems to be possible to construct an estimator which has a number of advantages over the traditional estimators of  $E(Y | x_p)$ .

## 2. THE MULTIPLICATIVE MODEL

We assume that there is an economic process which generates  $Y$ ; let us further assume that this process takes the form of a multiplicative relationship between the variable  $Y$  and  $K - 1$  explanatory variables  $Z_2, \dots, Z_K$  and a disturbance term  $v$ . If we have  $N$  observations on both  $Y$  and the  $Z$ 's, we can write

---

<sup>1</sup> This entity will be denoted by  $E(Y | x_p)$ , where it should be noted that the meaning of this notation differs from the usual one where  $x_p$  is a random variable. In this case  $x_p$  is non-stochastic.

$$(2.1) \quad Y_i = \delta_1 \prod_{k=2}^K Z_{ik}^{\delta_k} v_i \quad i = 1, \dots, N$$

where the parameters  $\delta$  are assumed to be unknown and the  $Z$ 's are nonstochastic. Furthermore we assume

$$(2.2) \quad E(v) = 1$$

and

$$(2.3) \quad \text{var } v = \omega^2 I$$

Taking the logarithm at both sides of (2.1), we obtain

$$(2.4) \quad y = X\beta + u$$

with

$$y = \begin{bmatrix} \ln Y_1 \\ \vdots \\ \ln Y_N \end{bmatrix} \quad X = \begin{bmatrix} 1 & \ln Z_{12} & \dots & \ln Z_{1K} \\ \vdots & \vdots & & \vdots \\ 1 & \ln Z_{N2} & \dots & \ln Z_{NK} \end{bmatrix}$$

$$\beta = \begin{bmatrix} \ln \delta_1 \\ \delta_2 \\ \vdots \\ \delta_K \end{bmatrix} \quad u = \begin{bmatrix} \ln v_1 \\ \vdots \\ \ln v_N \end{bmatrix} \quad \text{and } r(X) = K < N$$

Before we proceed any further, we should justify assumption (2.2). Let us therefore consider the implications for the original model of superimposing a traditional set of assumptions on the linear model (2.4), i.e. let the  $u$ 's be independently normally distributed with

$$(2.5) \quad E(u) = 0$$

$$(2.6) \quad \text{var } u = \sigma^2 I$$

It should be noted that the normality assumption which is rather limiting, is made here for convenience in calculations.<sup>2</sup>

Let us now consider the implications for the multiplicative model. From the assumptions above, it follows that the  $v' = (v_1 \dots v_N)$  is lognormally distributed with<sup>3</sup>

$$(2.7) \quad E(v) = e^{\frac{1}{2}\sigma^2} \mathbf{1}$$

and

$$(2.8) \quad \text{var } v = (e^{2\sigma^2} - e^{\sigma^2}) I$$

We may ask ourselves whether these implicit assumptions about the disturbance term in the original relation are in accordance with our ideas about the specification of the multiplicative model. In our opinion the answer must be "no". We therefore use the following analogous reasoning. The fact that, in the linear model, the mathematical expectation of  $u$  is usually assumed to equal zero originates in the belief that, although the relationship between the dependent and the explanatory variables will not hold exactly for a single observation, it should hold in the average. An analogous reasoning with respect to the multiplicative model leads us to the replacement of assumption (2.7) by (2.2).

As for the linear model, we also assume independency and homoskedasticity in the multiplicative model as reflected by (2.3). Then it can easily be verified that

$$u \sim N(-\frac{1}{2} \ln (\omega^2 + 1), \ln (\omega^2 + 1) I)$$

or

$$(2.9) \quad u \sim N(-\frac{1}{2}\sigma^2, \sigma^2 I)$$

where  $u = \ln (\omega^2 + 1)$ .

<sup>2</sup> Further research will be devoted to the question of how robust the estimators to be derived will be with respect to different distributions of  $u$  (or  $v$ ).

<sup>3</sup> See Appendix A for the derivation of (2.7) and (2.8).

Having completely specified our multiplicative model, we are now ready to tackle the estimation problem.

In the next section, the least-squares and the maximum likelihood estimators of  $E(Y | x_p)$  will be derived. As a starting point, it will be assumed that  $\omega^2$  (and consequently  $\sigma^2$ ) are known parameters.

### 3. LEAST SQUARES AND MAXIMUM LIKELIHOOD ESTIMATORS

#### 3.1. The Least Squares Method

##### 3.1.1. Parameter Estimators

Least squares estimation of  $\beta$  in model (2.4) yields

$$(3.1) \quad \hat{\beta} = (X'X)^{-1}X'y$$

Since it is no longer assumed that the expectation of  $u$  equals zero, this estimator is biased with respect to  $\beta$ :

$$(3.2) \quad E(\hat{\beta}) - \beta = -\frac{1}{2}\sigma^2(X'X)^{-1}X'1$$

The first column of  $X$  being a unit vector, (3.2) can be written as

$$(3.3) \quad E(\hat{\beta}) - \beta = -\frac{1}{2}\sigma^2 \begin{bmatrix} 1'1 & 1'X_1 \\ X_1'1 & X_1'X_1 \end{bmatrix}^{-1} \begin{bmatrix} 1'1 \\ X_1'1 \end{bmatrix} = -\frac{1}{2}\sigma^2 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with  $X = (1 \vdots X_1)$

from which it appears that in the linear model only the estimator of the constant term  $\hat{\beta}_1$  is biased and the other  $\beta$ 's are unbiased. However, in this case our interest is not the estimation of the parameters in the linear model but of those in the multiplicative model, and we realize that not  $\beta_1$  but  $\delta_1 = \exp(\beta_1)$  appears as a parameter in the multiplicative model.

Let us therefore consider the bias of  $\hat{\delta}_1 = e^{\hat{\beta}_1}$ . The expectation of  $\hat{\delta}_1$  can easily be derived:

$$(3.4) \quad E(\hat{\delta}_1) = \int_{-\infty}^{\infty} e^{\hat{\beta}_1} f(\hat{\beta}_1) d\hat{\beta}_1 = e^{\beta_1 - \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \frac{\beta_1}{\sigma^2}} = \delta_1 e^{-\frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \frac{\beta_1}{\sigma^2}}$$

where use has been made of the fact that  $\hat{\beta}_1$  is normally distributed with mean  $\beta_1 - \frac{1}{2}\sigma^2$  and variance  $\sigma_{\hat{\beta}_1}^2$ . Hence the bias of  $\hat{\delta}_1$  equals

$$(3.5) \quad E(\hat{\delta}_1) - \delta_1 = \delta_1 [1 - e^{-\frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \frac{\beta_1}{\sigma^2}}]$$

The estimation of the other parameters in the multiplicative model does not raise any problems since they are not affected by the transformation.

### 3.1.2. The Estimator of $E(Y | x_p)$

In the linear model, the dependent variable is a linear function of the regression coefficients; hence if we are able to construct unbiased estimators of these coefficients, it follows that the estimator of the expectation of the dependent variable, given a vector of explanatory variables, is unbiased. It can easily be seen that this implication does not hold for the multiplicative model. The multiplicative model (2.1) can also be written as

$$(3.6) \quad Y_i = e^{x_i' \beta + u_i} \quad i = 1, \dots, N$$

Then the L.S. estimator of the expectation of  $Y$ , given some  $x_p$ -vector, is defined as

$$(3.7) \quad \hat{Y}_p = e^{x_p' \hat{\beta}}$$

The mean of this estimator equals

$$(3.8) \quad \begin{aligned} E(\hat{Y}_p) &= e^{x_p' \beta} \int_{-\infty}^{\infty} e^{x_p' (X'X)^{-1} X u} f(u) du \\ &= e^{x_p' \beta - \frac{1}{2}\sigma^2 (1 - \alpha_p)} \end{aligned}$$

with  $\alpha_p = x_p' (X'X)^{-1} x_p$ .

It can easily be verified that the bias of  $\hat{Y}_p$  equals

$$(3.9) \quad E(\hat{Y}_p) - E(Y_p) = (e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} - 1)e^{x_p'\beta}$$

The variance of  $\hat{Y}_p$  is equal to<sup>4</sup>

$$(3.10) \quad \text{var } \hat{Y}_p = e^{2x_p'\beta - \sigma^2} [e^{2\sigma^2\alpha_p} - e^{\sigma^2\alpha_p}]$$

To ascertain whether we have a good estimator or not, we shall use as our criterion function the mean squared error of the estimator:

$$(3.11) \quad E(T - \theta)^2 = \text{var } T + (\theta - ET)^2$$

where  $T$  is an estimator of  $\theta$ . In our case this function becomes

$$(3.12) \quad \begin{aligned} \hat{\pi}' &= \text{var } \hat{Y}_p + [E\hat{Y}_p - E(Y | x_p)]^2 \\ &= e^{2x_p'\beta} \{ e^{-\sigma^2(1-\alpha_p)} [e^{\sigma^2\alpha_p} - 1] + [e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} - 1]^2 \} \end{aligned}$$

let  $\hat{\pi} = \hat{\pi}' \cdot e^{-2x_p'\beta}$  then

$$(3.13) \quad \hat{\pi} = e^{-\sigma^2(1-\alpha_p)} [e^{\sigma^2\alpha_p} - 1] + [e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} - 1]^2$$

### 3.2. The Maximum Likelihood Method

#### 3.2.1. Parameter Estimators

The maximum likelihood estimators can be derived from the original model (2.1). To that end, we introduce the following notation:  $\Lambda(\mu, \sigma^2)$  stands for a lognormal distribution corresponding to a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . From (2.4) follows

$$(3.14) \quad v_i \sim \Lambda(-\frac{1}{2}\sigma^2, \sigma^2)$$

---

<sup>4</sup> See Appendix B.

From this distribution the distribution of  $Y_i$  can be derived:

$$(3.15) \quad Y_i \sim \Lambda(\ln \delta_1 \prod_{k=2}^K Z_{ik}^{\delta_k} - \frac{1}{2}\sigma^2, \sigma^2)$$

Then the likelihood function of  $(Y_1, \dots, Y_N)$  equals

$$(3.16) \quad L(Y_1, \dots, Y_N) = \frac{1}{\prod_i Y_i} \sigma^{-N} (2\pi)^{-\frac{1}{2}N} \exp \left\{ -\frac{1}{2\sigma^2} \sum_i (\ln Y_i - \ln \delta_1 \prod_{k=2}^K Z_{ik}^{\delta_k} + \frac{1}{2}\sigma^2)^2 \right\}$$

In order to maximize this function with respect to  $\delta$ , we first take the logarithm, differentiate, and derive the first order conditions. The transformed likelihood function takes the following form:

$$(3.17) \quad \ell(y) = -1'y - \frac{1}{2}N \ln \sigma^2 - \frac{1}{2}N \ln 2\pi - \frac{1}{2\sigma^2} (y - X\beta + \frac{1}{2}\sigma^2_1)' (y - X\beta + \frac{1}{2}\sigma^2_1)$$

where

$$\ell(y) = \ln [L(Y_1, \dots, Y_N)]$$

and where the notation of (2.4) has been introduced. The necessary conditions for a maximum of  $\ell(y)$  are

$$(3.18) \quad X'X\beta - X'y - X'1 \cdot \frac{1}{2}\sigma^2 = 0$$

The maximum likelihood estimator of  $\beta$  follows from (3.18):

$$(3.19) \quad \hat{\beta} = (X'X)^{-1} X'(y + \frac{1}{2}\sigma^2_1)$$

The maximum likelihood estimator  $\hat{\beta}$  can also be written as a function of the least squares estimator  $\hat{\beta}$

$$(3.20) \quad \hat{\beta} = \hat{\beta} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \frac{1}{2}\sigma^2$$

It can easily be verified that  $\hat{\beta}$  is an unbiased estimator of  $\beta$ , and that - for known  $\sigma^2$  -  $\text{var } \hat{\beta}$  equals  $\text{var } \hat{\beta}$ . That the second-order condition is fulfilled can easily be seen, since the Hessian of (3.17) being equal to  $\{-2X'X\}$  is negative definite.

### 3.2.2. The Estimator of $E(Y | x_p)$

The M.L.-estimator of  $E(Y | x_p)$  equals

$$(3.21) \quad \hat{\bar{Y}}_p = e^{x_p' \hat{\beta}}$$

Substitution of (3.20) yields

$$(3.22) \quad \hat{\bar{Y}}_p = e^{x_p' \hat{\beta} + \frac{1}{2} \sigma^2}$$

Thus

$$(3.23) \quad E\hat{\bar{Y}}_p = e^{\frac{1}{2} \sigma^2} E\hat{Y}_p = e^{x_p' \hat{\beta} + \frac{1}{2} \sigma^2 \alpha_p}$$

and

$$(3.24) \quad \text{var } \hat{\bar{Y}}_p = e^{\sigma^2} \text{var } \hat{Y}_p = e^{2x_p' \hat{\beta}} [e^{2\sigma^2 \alpha_p} - e^{\sigma^2 \alpha_p}]$$

The mean squared error function of  $\hat{\bar{Y}}$  follows from (3.23) and (3.24):

$$(3.25) \quad \hat{\pi}' = e^{2x_p' \hat{\beta}} \{ (e^{2\sigma^2 \alpha_p} - e^{\sigma^2 \alpha_p}) + (e^{\frac{1}{2} \sigma^2 \alpha_p} - 1)^2 \}$$

or

$$(3.26) \quad \hat{\pi} = e^{\sigma^2 \alpha_p} (e^{\sigma^2 \alpha_p} - 1) + (e^{\frac{1}{2} \sigma^2 \alpha_p} - 1)^2$$

with  $\hat{\pi} = \hat{\pi}' \cdot e^{-2x_p' \hat{\beta}}$ .

### 3.3. A Comparison of the Least Squares and the Maximum Likelihood of $E(Y | x_p)$

As has already been stated before, the comparison of the two estimators will take place on the basis of their mean squared errors. Usually, however, such a comparison is impossible because the mean squared errors are functions of the unknown  $\beta$ . In our case, however, both mean squared error functions can be written as  $\pi' = e^{2x_p' \beta} \cdot \pi$  i.e. they have the exponential expression  $e^{2x_p' \beta}$  in common and since  $\beta$  does not appear elsewhere in the functions the comparison can take place on the basis of the  $\pi$ -functions without the necessity of taking into account the  $\beta$ -vector.

The functions  $\hat{\pi}$  and  $\pi^*$  both depend on  $\sigma^2$  and  $\alpha_p$ . Let us consider  $\alpha_p$  in more detail; it has been defined in (3.9) as

$$\alpha_p = x_p'(X'X)^{-1}x_p$$

In Appendix D it has been shown that

$$(3.27) \quad \alpha_p \geq \frac{1}{N}$$

for all  $x_p$  and that

$$(3.28) \quad \frac{1}{N} \leq \alpha_p \leq 1$$

for  $x_p$  coinciding with a row vector of the  $X$  matrix, which is the case if the  $X$  matrix is not only used for estimation of the parameters, but also for the explanation of  $Y_i$ ,  $i = 1, \dots, N$ . If, however,  $x_p$  stands for a projection of the explanatory variables (3.28) is not always satisfied, i.e.  $\alpha_p$  may then be larger than unity.

In the literature on the linear model it is frequently assumed that

$$(3.29) \quad \lim_{N \rightarrow \infty} \frac{1}{N} X'X = A$$

where  $A$  is a bounded non-singular matrix. Let us trace the impact of this assumption on  $\alpha_p$ . From (3.29) it follows that the matrix  $(X'X)^{-1}$  tends to the zero matrix for  $N$  to infinity. Thus,

$$(3.30) \quad \lim_{N \rightarrow \infty} \alpha_p = \lim_{N \rightarrow \infty} x_p'(X'X)^{-1}x_p = 0$$

for all  $x_p$ .

For time series assumption (3.29) looks rather realistic, provided that the explanatory variables do not show a decreasing tendency over time. As to cross section data its relevance is not clear. In Section 5.3, therefore, we shall use two alternatives with respect to the  $\lim$  of  $\alpha_p$ : first, assumption (3.30) and second, the assumption that

$$(3.31) \quad 0 < \lim_{N \rightarrow \infty} \alpha_p < \alpha^*$$

with  $\alpha^*$  some finite number.

However, for any finite sample size  $N$ ,  $\alpha_p$  may assume any positive value. And given  $X$ ,  $\alpha_p$  can be regarded as a characteristic of  $x_p$ . In the next paragraph we investigate for which values of  $\alpha_p$  the L.S.-estimator has a lower mean squared error than the M.L.-estimator and for which values of  $\alpha_p$  the reverse holds.

Consider

$$(3.32) \quad \hat{\pi} - \hat{\pi} = (e^{-\sigma^2} - 1)(e^{2\sigma^2\alpha_p} - e^{\sigma^2\alpha_p}) + (e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} - 1)^2 - (e^{\frac{1}{2}\sigma^2\alpha_{p-1}})^2 \\ = e^{\frac{1}{2}\sigma^2\alpha_p}(e^{-\frac{1}{2}\sigma^2} - 1)\{e^{\frac{3}{2}\sigma^2\alpha_p}(e^{-\frac{1}{2}\sigma^2} + 1) - 2\}$$

This expression equals zero for  $\sigma^2 \neq 0$  if

$$e^{\frac{3}{2}\sigma^2\alpha_p} = 2(1 + e^{-\frac{1}{2}\sigma^2})^{-1}$$

Consequently for

$$(3.33) \quad \alpha_p = \alpha_0 \equiv \frac{2}{3\sigma^2} \{ \ln 2 - \ln (1 + e^{-\frac{1}{2}\sigma^2}) \}$$

the mean squared errors are equal. Moreover, if

$$\alpha_p > \alpha_0 \quad \text{then} \quad \hat{\pi} < \hat{\pi}$$

and

$$0 < \alpha_p < \alpha_0 \quad \text{then} \quad \hat{\pi} < \hat{\pi}$$

Hence we can conclude that the L.S.-estimator of  $E(Y | x_p)$  is superior to the M.L.-estimator (in the sense of having a lower mean squared error) for  $\alpha_p > \alpha_0$  and that the reverse is true for  $\frac{1}{N} < \alpha_p < \alpha_0$ . This conclusion only holds for given values of  $\sigma^2$ .

Now the question arises as to whether it is possible to construct an estimator of  $E(Y | x_p)$  which has a lower mean squared error than the least squares and the maximum likelihood estimators for known  $\sigma^2$ . Authors like Goldberger [5], Bradu & Mundlak [2], and Heien [6] confine themselves to minimum variance unbiased estimators which are derived with the help of an estimation function introduced by Finney [4]. The condition of unbiasedness may be very limiting in the sense that it rules out biased estimators with a possibly smaller mean squared error.<sup>5</sup>

<sup>5</sup> The use of the Finney-estimator will be one of the subjects of further research.

#### 4. A NEW ESTIMATOR OF $E(Y | x_p)$

Notice that, if  $\frac{1}{N} \leq \alpha_p < 1$  the least-squares estimator always underestimates whereas the maximum likelihood estimator always overestimates  $E(Y | x_p)$  (see (3.8) and (3.23)). This fact gives us an intuitive argument for introducing a class of estimators which allows for a convex combination of least squares and maximum likelihood estimators. Thus we introduce the following class:

$$(4.1) \quad C_p = \{e^{x_p' \hat{\beta} + \xi_p \sigma^2} \mid -\infty < \xi_p < \infty\}$$

where  $\hat{\beta}$  is the least squares estimator of  $\beta$  and  $\xi_p$  is a constant. Our goal is to determine the estimator  $Y_p \in C_p$  with the lowest mean squared error. To that end we minimize the mean squared error function of

$$(4.2) \quad Y_{\xi_p} = e^{x_p' \hat{\beta} + \xi_p \sigma^2}$$

with respect to  $\xi_p$ .

In order to establish the mean squared error function we determine the mean and the variance of  $Y_{\xi_p}$ , which are respectively equal to

$$(4.3) \quad E(Y_{\xi_p}) = e^{x_p' \hat{\beta} - \frac{1}{2} \sigma^2 (1 - \alpha_p - 2\xi_p)}$$

$$(4.4) \quad \text{var } Y_{\xi_p} = e^{2x_p' \hat{\beta} - \sigma^2 (1 - 2\xi_p)} [e^{2\sigma^2 \alpha_p} - e^{\sigma^2 \alpha_p}]$$

Hence the mean squared error function is equal to

$$(4.5) \quad \pi'_{\xi_p} = e^{2x_p' \hat{\beta} - \sigma^2 (1 - 2\xi_p)} [e^{2\sigma^2 \alpha_p} - e^{\sigma^2 \alpha_p}] + [e^{-\frac{1}{2} \sigma^2 (1 - \alpha_p - 2\xi_p)} - 1]^2 e^{2x_p' \hat{\beta}}$$

or

$$(4.6) \quad \pi_{\xi_p} = e^{\sigma^2 (2\xi_p + 2\alpha_p - 1)} - 2e^{\frac{1}{2} \sigma^2 (2\xi_p + \alpha_p - 1)} + 1$$

$$\text{with } \pi_{\xi_p} = \pi'_{\xi_p} e^{-2x_p' \hat{\beta}}.$$

To find the minimum M.S.E. estimator within  $C_p$  we have to minimize (4.6). We therefore differentiate this expression with respect to  $\xi_p$  and get

$$(4.7) \quad \frac{\partial \pi_{\xi_p}}{\partial \xi_p} = 2\sigma^2 e^{\sigma^2(2\xi_p + 2\alpha_p - 1)} - 2\sigma^2 e^{\frac{1}{2}\sigma^2(2\xi_p + \alpha_p - 1)}$$

The first order condition for a minimum of (4.6) is fulfilled if

$$(4.8) \quad \xi_p^0 = \frac{1}{2}(1 - 3\alpha_p)$$

To verify whether  $\xi_p^0$  constitutes a minimum, we consider the second derivative in the point  $\xi_p^0$ :

$$(4.9) \quad \frac{\partial^2 \pi_{\xi_p}}{\partial \xi_p^2} [\xi_p = \xi_p^0] = 2\sigma^4 e^{-\sigma^2 \alpha_p} > 0$$

From (4.9) we conclude that  $\pi_{\xi_p}$  reaches a minimum for  $\xi_p = \xi_p^0$ . Hence

$$(4.10) \quad \bar{y}_p = e^{x_p' \hat{\beta} + \frac{1}{2}\sigma^2(1-3\alpha_p)}$$

has minimal M.S.E. in  $C_p$ . Since the least squares and the maximum likelihood estimators of  $E(Y | x_p)$  both belong to  $C_p$ , the new estimator is better than these traditional estimators, provided that  $\sigma^2$  is known.

Normally, however, we have to estimate  $\sigma^2$  (or  $\omega^2$ ). And now the question arises as to whether in this case our new estimator of  $E(Y | x_p)$  is still uniformly better in  $C_p$ .

## 5. A COMPARISON OF THE THREE ESTIMATORS OF $E(Y | x_p)$ IN THE CASE OF UNKNOWN VARIANCE

### 5.1. An Estimator of the Variance in the Multiplicative Model

A well-known estimator of  $\sigma^2$  is

$$(5.1) \quad S^2 = \frac{1}{N - K} y'My$$

It can easily be verified that  $S^2$  is an unbiased estimator of  $\sigma^2$ , even in our case where  $E\epsilon \neq 0$ . Moreover it can be proved that, under our assumptions,  $S^2$  has minimum variance in the class of unbiased estimators.<sup>6</sup> Let us therefore consider

<sup>6</sup> See Corsten [1].

$$(5.2) \quad \hat{\omega}^2 = e^{S^2}$$

as an estimator of  $\omega^2$ . The mathematical expectation of  $\hat{\omega}^2$  can be derived by making use of the fact that  $y'My/\sigma^2$  is  $\chi^2$ -distributed.<sup>7</sup>

It is equal to

$$(5.3) \quad E(\hat{\omega}^2) = \int_0^\infty e^{\frac{\sigma^2}{N-K} \cdot q} \chi^2(N-K) dq = \left(1 - \frac{2}{N-K} \sigma^2\right)^{-\frac{N-K}{2}}$$

with  $q = \frac{y'My}{\sigma^2}$  and where use has been made of the known moment generating function of a  $\chi^2$ -distribution with  $N-K$  degrees of freedom. From (5.3) it follows that  $\hat{\omega}^2 - 1$  is an asymptotically unbiased estimator of  $\omega^2$ , since

$$(5.4) \quad \lim_{N \rightarrow \infty} E(\hat{\omega}^2) = \lim_{N \rightarrow \infty} \left[1 - \frac{2}{N-K} \sigma^2\right]^{-\frac{N-K}{2}} = e^{\sigma^2} = \omega^2 + 1$$

To determine the variance of  $\hat{\omega}^2$ , we compute the expectation of  $(\hat{\omega}^2)^2$ :

$$(5.5) \quad E[(\hat{\omega}^2)^2] = E[e^{2S^2}] = \int_0^\infty e^{\frac{2\sigma^2}{N-K} q} \chi^2(N-K) dq = \left(1 - \frac{4}{N-K} \sigma^2\right)^{-\frac{N-K}{2}}$$

Hence

$$(5.6) \quad \begin{aligned} \text{var } \hat{\omega}^2 &= E[(\hat{\omega}^2)^2] - [E(\hat{\omega}^2)]^2 \\ &= \left(1 - \frac{4\sigma^2}{N-K}\right)^{-\frac{N-K}{2}} - \left(1 - \frac{2\sigma^2}{N-K}\right)^{-N-K} \end{aligned}$$

From (5.6) it is clear that the variance of  $\hat{\omega}^2$  tends towards zero for large  $N$ . We shall use  $\hat{\omega}^2$  as an estimator of  $e^{\sigma^2}$  in what follows.

## 5.2. The Mean Squared Errors of the Three Estimators

In the case where  $\sigma^2$  is unknown, the three estimators can be written in the following general form:

$$(5.7) \quad y_{\xi p} = e^{\frac{x' \hat{\beta} + \xi}{p} \hat{\sigma}^2} = e^{\frac{x' \beta + \xi}{p} S^2}$$

<sup>7</sup> See Appendix C.

$$\text{with } \xi_p = 0 \quad \text{for } \hat{Y}_p$$

$$\xi_p = \frac{1}{2} \quad \text{for } \hat{\hat{Y}}_p$$

$$\xi_p = \frac{1}{2}(1 - 3\alpha_p) \quad \text{for } \bar{Y}_p$$

The mean squared error of  $Y_{\xi_p}$  follows from  $E(Y_{\xi_p})$  and  $\text{var } Y_{\xi_p}$ , which are derived below. Equation (5.7) can also be written as

$$(5.8) \quad Y_{\xi_p} = e_p^{x' \hat{\beta}} \cdot e_p^{\xi_p S^2}$$

In Appendix C it is shown that  $e_p^{x' \hat{\beta}}$  and  $e_p^{\xi_p S^2}$  are independent random variables, hence

$$(5.9) \quad E(Y_{\xi_p}) = E(e_p^{x' \hat{\beta}} \cdot e_p^{\xi_p S^2}) = E(e_p^{x' \hat{\beta}}) \cdot E(e_p^{\xi_p S^2})$$

$$= e_p^{x' \hat{\beta} - \frac{1}{2} \sigma^2 (1 - \alpha_p)} \left[ 1 - \frac{2\xi_p \sigma^2}{N - K} \right]^{-\frac{N-K}{2}}$$

and

$$(5.10) \quad \text{var } (Y_{\xi_p}) = E(e_p^{2x' \hat{\beta}} e_p^{2\xi_p S^2}) - [E(e_p^{x' \hat{\beta}})]^2 [E(e_p^{\xi_p S^2})]^2$$

$$= e_p^{2x' \hat{\beta} - \sigma^2 (1 - 2\alpha_p)} \left[ 1 - \frac{4\xi_p \sigma^2}{N - K} \right]^{-\frac{N-K}{2}} - e_p^{2x' \hat{\beta} - \sigma^2 (1 - \alpha_p)} \left[ 1 - \frac{2\xi_p \sigma^2}{N - K} \right]^{-(N-K)}$$

So, the mean squared error of  $Y_{\xi_p}$  equals

$$(5.11) \quad \pi'_{\xi_p} = e_p^{2x' \hat{\beta} - \sigma^2 (1 - 2\alpha_p)} \left\{ e_p^{\frac{4\xi_p \sigma^2}{N - K} - \frac{N-K}{2}} - \frac{1}{2} \sigma^2 (1 - \alpha_p) \right. \\ \left. - 2e_p^{-\frac{1}{2} \sigma^2 (1 - \alpha_p)} \left[ 1 - \frac{2\xi_p \sigma^2}{N - K} \right]^{-\frac{N-K}{2}} + 1 \right\}$$

or

$$(5.12) \quad \pi_{\xi_p} = e_p^{-\sigma^2 (1 - 2\alpha_p)} \left[ 1 - \frac{4\xi_p \sigma^2}{N - K} \right]^{-\frac{N-K}{2}} - 2e_p^{-\frac{1}{2} \sigma^2 (1 - \alpha_p)} \left[ 1 - \frac{2\xi_p \sigma^2}{N - K} \right]^{-\frac{N-K}{2}} + 1$$

$$\text{with } \pi_{\xi_p} = \pi'_{\xi_p} \cdot e_p^{-2x' \hat{\beta}}$$

From (5.12) it follows substituting the corresponding values of  $\xi_p$  (see (5.7)) that

$$(5.13) \quad \hat{\pi}_p = e^{-\sigma^2(1-2\alpha_p)} - 2e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} + 1$$

which is obviously identical to (3.13).

$$(5.14) \quad \hat{\pi}_p = e^{-\sigma^2(1-2\alpha_p)} \left[1 - \frac{2\sigma^2}{N-K}\right]^{-\frac{N-K}{2}} - 2e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} \left[1 - \frac{\sigma^2}{N-K}\right]^{-\frac{N-K}{2}} + 1$$

$$(5.15) \quad \bar{\pi}_p = e^{-\sigma^2(1-2\alpha_p)} \left[1 - \frac{2(1-3\alpha_p)\sigma^2}{N-K}\right]^{-\frac{N-K}{2}} - 2e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} \left[1 - \frac{(1-3\alpha_p)\sigma^2}{N-K}\right]^{-\frac{N-K}{2}} + 1$$

These formulae will be used in the next sections.

### 5.3. Large Sample Properties

In Section 3.3 we introduced an assumption about the behaviour of  $X'X$  for large  $N$  (assumption (3.29)). In the sequel we shall investigate the large sample properties of the three estimators discussed above; under assumption (3.29) or (3.30) and under assumption (3.31).

It turns out that under assumption (3.27) the maximum likelihood as well as the new estimator converge in quadratic mean, whereas the least squares estimator does not, as can easily be verified from equations (5.13)-(5.15):

$$(5.16) \quad \left\{ \begin{array}{l} \lim_{N \rightarrow \infty} E[\hat{Y}_p - E(Y | x_p)]^2 = e^{2x_p' \beta} \lim_{N \rightarrow \infty} \hat{\pi}_p = e^{2x_p' \beta} (e^{-\sigma^2} - 2e^{-\frac{1}{2}\sigma^2} + 1) \\ \lim_{N \rightarrow \infty} E[\hat{\hat{Y}}_p - E(Y | x_p)]^2 = e^{2x_p' \beta} \lim_{N \rightarrow \infty} \hat{\pi}_p = 0 \\ \lim_{N \rightarrow \infty} E[\bar{Y}_p - E(Y | x_p)]^2 = e^{2x_p' \beta} \lim_{N \rightarrow \infty} \bar{\pi}_p = 0 \end{array} \right.$$

From (2.22) and (4.10) it can be seen that the maximum likelihood estimator and the new estimator coincide for  $\alpha_p = 0$ .

Let us now replace assumption (3.30) by (3.31). In Section 5.1 it has been shown that the bias and the variance of  $\hat{\omega}^2$  both tend to zero as  $N$  tends to infinity, which is tantamount to saying that  $\hat{\omega}^2$  is a squared error consistent estimator of  $e^{\sigma^2}$ . Hence in the limit there is no longer uncertainty about  $\sigma^2$  and the situation is identical to that where we assumed  $\sigma^2$  or  $e^{\sigma^2}$  to be known. This implies that for  $N \rightarrow \infty$  and non-vanishing  $\alpha_p$  the new estimator  $\bar{Y}_p$  is most efficient in  $C_p$ . It should be noted, however, that in this case  $\bar{Y}_p$  is no longer squared error consistent, due to the fact that  $\alpha_p$  does not vanish:

$$\begin{aligned}
 \lim_{N \rightarrow \infty} E[\hat{Y}_p - E(Y | x_p)]^2 &= (e^{-\sigma^2(1-2\alpha_p)} - 2e^{-\frac{1}{2}\sigma^2(1-\alpha_p)} + 1)e^{2x'_p\beta} \\
 (5.17) \quad \lim_{N \rightarrow \infty} E[\hat{\bar{Y}} - E(Y | x_p)]^2 &= (e^{2\alpha_p\sigma^2} - 2e^{\frac{1}{2}\alpha_p\sigma^2} + 1)e^{2x'_p\beta} \\
 \lim_{N \rightarrow \infty} E[Y - E(Y | x_p)]^2 &= (1 - e^{-\alpha\sigma^2})e^{2x'_p\beta}
 \end{aligned}$$

In the next Section we shall focus our attention on the small sample properties of the three estimators, which are in practice much more interesting than the large sample characteristics.

#### 5.4. Small Sample Properties

In this section we will trace the conditions under which the new estimator  $\bar{Y}_p$  has a lower mean squared error than the maximum likelihood and the least squares estimators. Instead of the mean squared errors we will consider the  $\pi$ -functions, as defined in (5.13)-(5.15). Results based on the comparison of these auxiliary functions equally apply to the mean squared errors functions. Because of the rather complicated character of the  $\pi$ -functions, the comparison has been carried out numerically instead of analytically. This means that the  $\pi$ -functions have been computed and compared for a limited number of argument values only. In the tables I, II and III we find for  $\sigma^2 = .5, 1.0$  and  $2.0$ <sup>8</sup> respectively the logarithms of  $\hat{\pi}_p$ ,  $\hat{\pi}$  and  $\bar{\pi}_p$  as functions of  $\alpha_p$ .<sup>9</sup> For the  $\hat{\pi}$  and the  $\bar{\pi}$  functions which are also affected by the number of degrees of freedom, we distinguished again four cases:  $N - K = 5, 10, 15$  and the case "Known  $\sigma^2$ ".

<sup>8</sup> It is hard to see how these values of  $\sigma^2$  work out in the original multiplicative model (2.1). We therefore computed for a number of  $\sigma^2$ -values the 95%-intervals of  $v_i$ . Let  $b_l$  and  $b_u$  be the lower and upperbound of the 95%-interval of  $v_i$ , then we can construct the following table

$\sigma^2$	$b_l$	$b_u$
.5	.20	3.06
1.0	.09	4.31
1.5	.05	4.95
2.0	.02	5.70

With respect to the interpretation of  $v_i$ , it should be realized that the sample outcome, say  $v_i^* = 2.0$  means that the observed value  $Y_i = Y_i^*$  is as twice as large as its expected value. From this table we see that an economic model of type (2.1) with a  $\sigma^2 = 2.0$  has hardly any explanatory value. We therefore do not consider any values of  $\sigma^2$  larger than 2.0.

<sup>9</sup> We confine ourselves to the interval  $(0, 1]$ ; it should be kept in mind that the true lower bound of  $\alpha_p$  equals  $1/N$ .

The numerical investigation suggests the following properties:

$$(5.18) \quad \bar{\pi}_p < \hat{\pi}_p \quad \text{for} \quad \alpha > 0, \quad \sigma^2 > 0 \quad \text{and} \quad N - K \geq 1$$

$$(5.19) \quad \bar{\pi}_p < \hat{\pi}_p \quad \text{for} \quad \alpha > \frac{1}{3}^{10} \sigma^2 > 0 \quad \text{and} \quad N - K \geq 1$$

These two inequalities show that for  $\alpha > \frac{1}{3}$  the new estimator is superior to the other two for all admissible values of  $\sigma^2$  and  $N - K$ .

Especially for higher values of  $\alpha_p$  the use of the new estimator gives a considerable reduction of the mean squared error as compared with the maximum likelihood estimator and least squares estimators. Let us consider the case where  $\alpha = 1$ . In Table IV (and graph IV) the  $\pi$ -functions have been computed as a function of  $\sigma^2$  for  $N - K = 8, 10$ , and  $12$ . From this table we see that e.g. for  $\sigma^2 = 0.40$  and  $N - K = 8$ :

$$^{10}\log \hat{\pi}_p - ^{10}\log \bar{\pi}_p = 0.3305$$

or

$$\hat{\pi}_p = 2.14 \bar{\pi}_p$$

and

$$^{10}\log \hat{\pi}_p - ^{10}\log \bar{\pi}_p = 0.8396$$

or

$$\hat{\pi}_p = 6.91 \bar{\pi}_p$$

Up till now we have not considered the case  $0 < \alpha < \frac{1}{3}$ . In this interval the new estimator is not always better than the least squares estimator. Whether  $\bar{Y}_p$  has a lower mean squared error than  $\hat{Y}_p$  depends on the values assumed  $\sigma^2$ ,  $N - K$  and  $\alpha$ . As we see from the graphs, where the  $\log \bar{\pi}_p$  intersects the  $\log \hat{\pi}_p$ , it is possible to indicate for any combination of  $\sigma^2$  and  $N - K$  a value of  $\alpha = \alpha_0$ ,  $0 < \alpha_0 < \frac{1}{3}$ , which is the lower bound of the interval  $[\alpha_0, \infty)$  where  $\hat{\pi}_p \geq \bar{\pi}_p$ . These lower bounds  $\alpha_0$  are given for a number of combinations of  $\sigma^2$  and  $N - K$  in Table V. This table is not completely filled, since the  $\pi$ -function is only defined for  $(N - K)/2 > \sigma^2$  (see equation (5.15)). From this table it can be seen that, for a fairly large number of combinations of  $\sigma^2$  and  $N - K$ , the new estimator is better than the least squares estimator for  $\alpha \geq 0$ . What can be said in this

<sup>10</sup> From equations (5.13)-(5.15) it follows that for  $\pi = 1/3 \hat{\pi}_p$  and  $\bar{\pi}_p$  coincide.

context with respect to the use of the new estimator in economic models? If we keep in mind what was said in footnote 8 about the relation between  $\sigma^2$  and the explanatory value of the model, we can safely confine ourselves to those economic models which have a  $\sigma^2 \leq 1.00$ . A look at table V learns us that, for these models, the new estimator  $\bar{Y}_p$  is superior to  $\bar{Y}_p$  for  $N - K \geq 6$ .

Our conclusion can be that for most cases arising in the field of estimation of economic models the new estimator will have a lower mean squared error than the other two. Furthermore, it appears that the maximum likelihood estimator  $\hat{Y}$  is overthrown by the new estimator. Finally, if we replace  $\sigma^2$  in Table V by  $S^2$ , this table can be used to decide whether  $\hat{Y}_p$  or  $\bar{Y}_p$  is the optimal estimator in any particular case.

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TABLE I

$$\sigma^2 = 0.5$$

$\alpha_p$	N - K = 5			N - K = 10		N - K = 15		Known $\sigma^2$	
	$\hat{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$
0.00	-3.0173	-3.4239	-3.4239	-4.2552	-4.2552	-4.7040	-4.7040	-∞	-∞
0.04	-2.8468	-2.8922	-3.1331	-3.3276	-3.4940	-3.4932	-3.6269	-3.8771	-3.9220
0.08	-2.6885	-2.5271	-2.8830	-2.8289	-3.0616	-2.9330	-3.1207	-3.1492	-3.2388
0.12	-2.5405	-2.2448	-2.6667	-2.4787	-2.7577	-2.5558	-2.7868	-2.7092	-2.8432
0.16	-2.4012	-2.0121	-2.4778	-2.2048	-2.5233	-2.2667	-2.5377	-2.3871	-2.5654
0.20	-2.2695	-1.8124	-2.3114	-1.9776	-2.3324	-2.0296	-2.3391	-2.1298	-2.3521
0.24	-2.1444	-1.6365	-2.1634	-1.7818	-2.1717	-1.8270	-2.1744	-1.9134	-2.1796
0.28	-2.0251	-1.4784	-2.0309	-1.6087	-2.0331	-1.6490	-2.0338	-1.7253	-2.0352
0.32	-1.9109	-1.3342	-1.9112	-1.4528	-1.9113	-1.4892	-1.9114	-1.5580	-1.9115
0.36	-1.8013	-1.2011	-1.8026	-1.3103	-1.8030	-1.3437	-1.8031	-1.4065	-1.8034
0.40	-1.6959	-1.0772	-1.7034	-1.1787	-1.7055	-1.2096	-1.7062	-1.2676	-1.7077
0.44	-1.5943	-0.9609	-1.6123	-1.0559	-1.6170	-1.0848	-1.6187	-1.1389	-1.6221
0.48	-1.4960	-0.8511	-1.5283	-0.9407	-1.5361	-0.9678	-1.5389	-1.0186	-1.5447
0.52	-1.4009	-0.7469	-1.4505	-0.8317	-1.4617	-0.8574	-1.4657	-0.9054	-1.4742
0.56	-1.3087	-0.6474	-1.3781	-0.7282	-1.3929	-0.7526	-1.3982	-0.7982	-1.4097
0.60	-1.2190	-0.5522	-1.3106	-0.6294	-1.3290	-0.6527	-1.3357	-0.6962	-1.3502
0.64	-1.1319	-0.4608	-1.2474	-0.5348	-1.2693	-0.5571	-1.2775	-0.5988	-1.2951
0.68	-1.0470	-0.3727	-1.1881	-0.4439	-1.2135	-0.4654	-1.2230	-0.5053	-1.2439
0.72	-0.9641	-0.2875	-1.1322	-0.3562	-1.1611	-0.3769	-1.1720	-0.4155	-1.1962
0.76	-0.8833	-0.2051	-1.0794	-0.2715	-1.1118	-0.2915	-1.1241	-0.3288	-1.1515
0.80	-0.8043	-0.1251	-1.0295	-0.1895	-1.0652	-0.2089	-1.0789	-0.2449	-1.1096
0.84	-0.7269	-0.0474	-0.9821	-0.1098	-1.0211	-0.1286	-1.0361	-0.1636	-1.0701
0.88	-0.6512	0.0283	-0.9371	-0.0324	-0.9793	-0.0507	-0.9957	-0.0847	-1.0329
0.92	-0.5770	0.1021	-0.8942	0.0429	-0.9396	0.0251	-0.9573	-0.0079	-0.9977
0.96	-0.5042	0.1743	-0.8532	0.1165	-0.9018	0.0992	-0.9208	0.0669	-0.9643
1.00	-0.4327	0.2449	-0.8140	0.1884	-0.8657	0.1715	-0.8860	0.1399	-0.9327

TABLE II

$$\sigma^2 = 1.0$$

$\alpha_p$	N - K = 5			N - K = 10		N - K = 15		Known $\sigma^2$	
	$\hat{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$
0.00	-1.8655	-1.6088	-1.6088	-2.6831	-2.6831	-3.1988	-3.1988	-∞	-∞
0.04	-1.8266	-1.3198	-1.7657	-2.1173	-2.4353	-2.4156	-2.6792	-3.1492	-3.2388
0.08	-1.7770	-1.0713	-1.8225	-1.7186	-2.2076	-1.9348	-2.3295	-2.3871	-2.5654
0.12	-1.7170	-0.8511	-1.7990	-1.4030	-2.0049	-1.5758	-2.0664	-1.9134	-2.1796
0.16	-1.6475	-0.6517	-1.7242	-1.1374	-1.8262	-1.2834	-1.8563	-1.5580	-1.9115
0.20	-1.5693	-0.4682	-1.6231	-0.9052	-1.6687	-1.0329	-1.6824	-1.2676	-1.7077
0.24	-1.4836	-0.2974	-1.5122	-0.6969	-1.5294	-0.8114	-1.5347	-1.0186	-1.5447
0.28	-1.3915	-0.1369	-1.4012	-0.5068	-1.4056	-0.6112	-1.4070	-0.7982	-1.4097
0.32	-1.2941	0.0149	-1.2947	-0.3307	-1.2949	-0.4272	-1.2950	-0.5988	-1.2951
0.36	-1.1924	0.1597	-1.1948	-0.1660	-1.1955	-0.2561	-1.1957	-0.4155	-1.1962
0.40	-1.0873	0.2984	-1.1020	-0.0106	-1.1057	-0.0955	-1.1069	-0.2449	-1.1096
0.44	-0.9798	0.4317	-1.0163	0.1369	-1.0241	0.0565	-1.0269	-0.0847	-1.0329
0.48	-0.8706	0.5604	-0.9372	0.2780	-0.9498	0.2012	-0.9544	0.0669	-0.9643
0.52	-0.7602	0.6851	-0.8641	0.4134	-0.8817	0.3398	-0.8882	0.2113	-0.9026
0.56	-0.6492	0.8062	-0.7963	0.5438	-0.8190	0.4731	-0.8276	0.3496	-0.8467
0.60	-0.5379	0.9240	-0.7334	0.6700	-0.7612	0.6017	-0.7718	0.4826	-0.7958
0.64	-0.4268	1.0390	-0.6746	0.7924	-0.7076	0.7262	-0.7203	0.6110	-0.7492
0.68	-0.3160	1.1514	-0.6196	0.9114	-0.6577	0.8471	-0.6726	0.7353	-0.7064
0.72	-0.2058	1.2615	-0.5679	1.0274	-0.6112	0.9648	-0.6282	0.8561	-0.6669
0.76	-0.0963	1.3694	-0.5191	1.1407	-0.5677	1.0796	-0.5868	0.9736	-0.6304
0.80	0.0123	1.4754	-0.4729	1.2516	-0.5269	1.1919	-0.5480	1.0883	-0.5966
0.84	0.1201	1.5796	-0.4289	1.3602	-0.4884	1.3017	-0.5117	1.2004	-0.5651
0.88	0.2269	1.6822	-0.3869	1.4668	-0.4522	1.4095	-0.4776	1.3102	-0.5357
0.92	0.3327	1.7833	-0.3466	1.5716	-0.4178	1.5153	-0.4455	1.4178	-0.5083
0.96	0.4375	1.8831	-0.3079	1.6748	-0.3852	1.6194	-0.4152	1.5236	-0.4827
1.00	0.5413	1.9815	-0.2705	1.7763	-0.3542	1.7219	-0.3865	1.6275	-0.4586

TABLE III

$$\sigma^2 = 2.0$$

$\alpha_p$	$\hat{\pi}$	N - K = 5		N - K = 10		N - K = 15		Known <sup>2</sup>	
		$\hat{\pi}$	$\bar{\pi}$	$\hat{\pi}$	$\bar{\pi}$	$\hat{\pi}$	$\bar{\pi}$	$\hat{\pi}$	$\bar{\pi}$
0.00	-0.9173	1.7795	1.7795	-0.7030	-0.7030	-1.4540	-1.4540	$-\infty$	$-\infty$
0.04	-0.9338	1.9645	0.7257	-0.3489	-0.9831	-0.9513	-1.4797	-2.3871	-2.5654
0.08	-0.9433	2.1471	-0.0356	-0.0362	-1.1284	-0.5499	-1.4163	-1.5580	-1.9115
0.12	-0.9438	2.3274	-0.5495	0.2476	-1.1582	-0.2071	-1.3073	-1.0186	-1.5447
0.16	-0.9333	2.5056	-0.8254	0.5102	-1.1132	0.0975	-1.1833	-0.5988	-1.2951
0.20	-0.9099	2.6820	-0.9155	0.7565	-1.0308	0.3752	-1.0604	-0.2449	-1.1096
0.24	-0.8719	2.8567	-0.8975	0.9899	-0.9353	0.6329	-0.9458	0.0669	-0.9643
0.28	-0.8180	3.0299	-0.8310	1.2129	-0.8394	0.8752	-0.8420	0.3496	-0.8467
0.32	-0.7475	3.2017	-0.7485	1.4273	-0.7489	1.1052	-0.7490	0.6110	-0.7492
0.36	-0.6604	3.3724	-0.6647	1.6346	-0.6658	1.3254	-0.6662	0.8561	-0.6669
0.40	-0.5573	3.5419	-0.5851	1.8358	-0.5905	1.5374	-0.5924	1.0883	-0.5966
0.44	-0.4397	3.7105	-0.5115	2.0320	-0.5225	1.7426	-0.5266	1.3102	-0.5357
0.48	-0.3091	3.8782	-0.4439	2.2238	-0.4612	1.9421	-0.4678	1.5236	-0.4827
0.52	-0.1675	4.0451	-0.3816	2.4118	-0.4057	2.1367	-0.4150	1.7299	-0.4361
0.56	-0.0169	4.2113	-0.3239	2.5965	-0.3553	2.3271	-0.3674	1.9303	-0.3949
0.60	0.1407	4.3768	-0.2699	2.7784	-0.3092	2.5138	-0.3244	2.1257	-0.3583
0.64	0.3040	4.5418	-0.2188	2.9577	-0.2669	2.6975	-0.2852	2.3167	-0.3257
0.68	0.4713	4.7062	-0.1701	3.1349	-0.2277	2.8784	-0.2493	2.5041	-0.2966
0.72	0.6415	4.8702	-0.1230	3.3102	-0.1912	3.0569	-0.2165	2.6882	-0.2704
0.76	0.8135	5.0338	-0.0771	3.4837	-0.1570	3.2333	-0.1861	2.8695	-0.2468
0.80	0.9868	5.1969	-0.0318	3.6558	-0.1246	3.4079	-0.1579	3.0483	-0.2255
0.84	1.1606	5.3598	0.0131	3.8265	-0.0938	3.5809	-0.1317	3.2251	-0.2062
0.88	1.3345	5.5223	0.0581	3.9961	-0.0643	3.7524	-0.1071	3.3999	-0.1887
0.92	1.5083	5.6846	0.1036	4.1646	-0.0358	3.9226	-0.0839	3.5731	-0.1729
0.96	1.6817	5.8467	0.1498	4.3321	-0.0081	4.0917	-0.0619	3.7448	-0.1585
1.00	1.8545	6.0085	0.1970	4.4989	0.0189	4.2599	-0.0410	3.9152	-0.1454

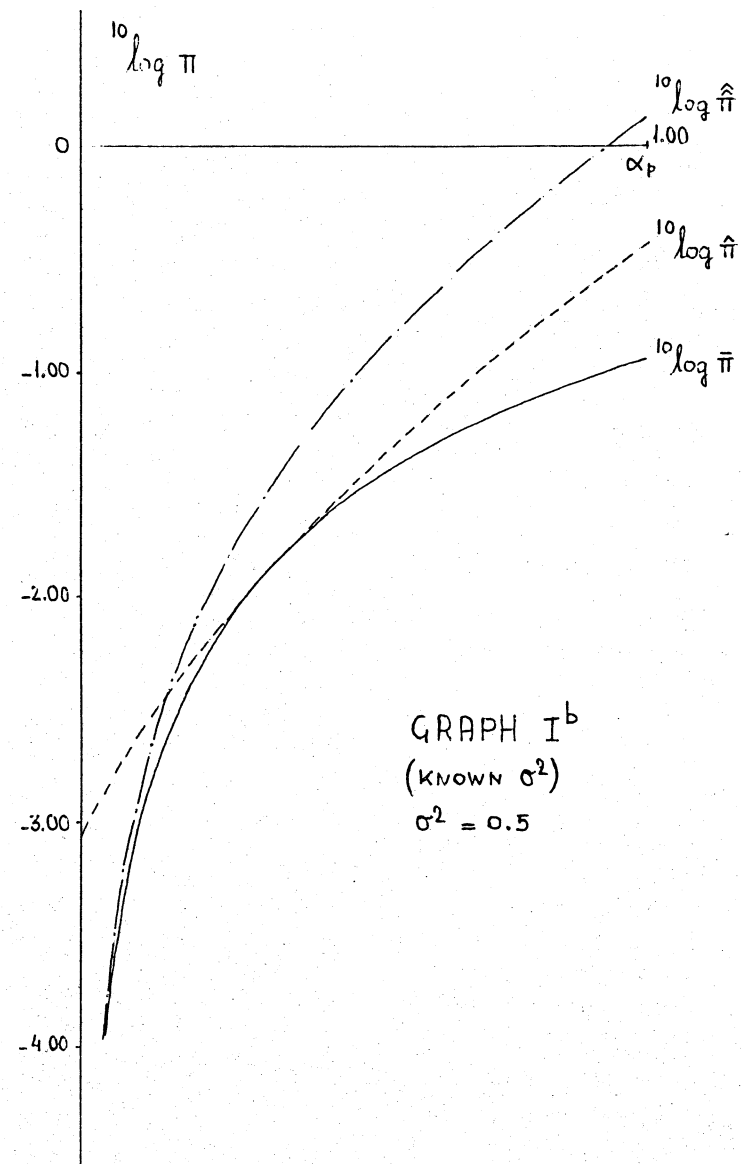
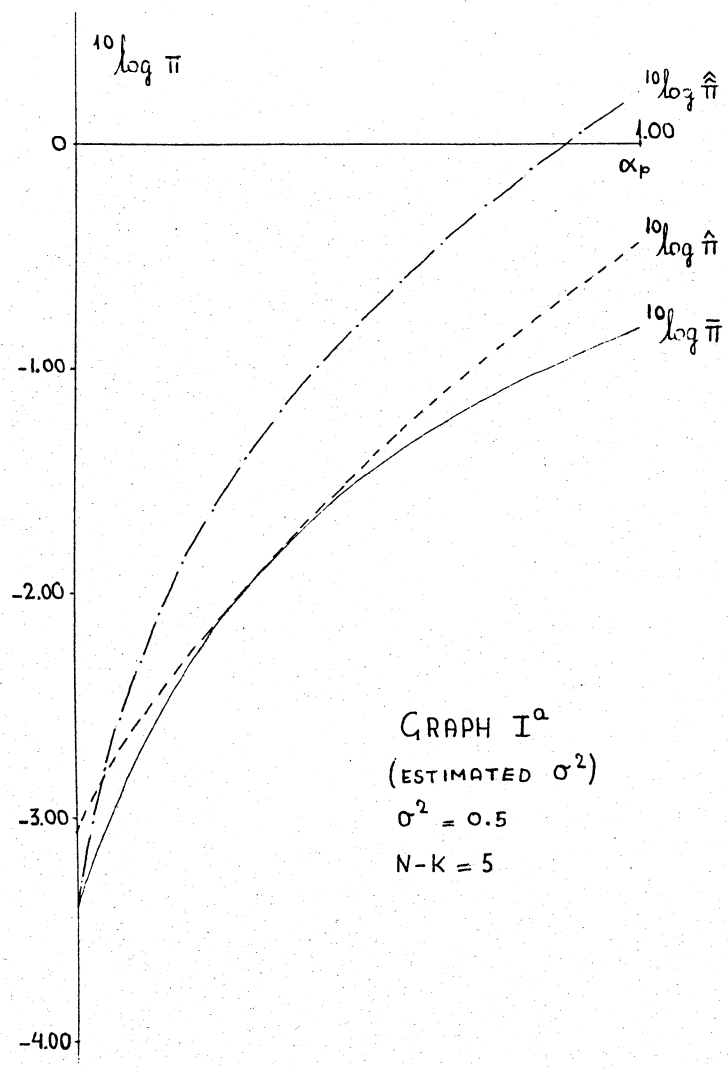
TABLE IV

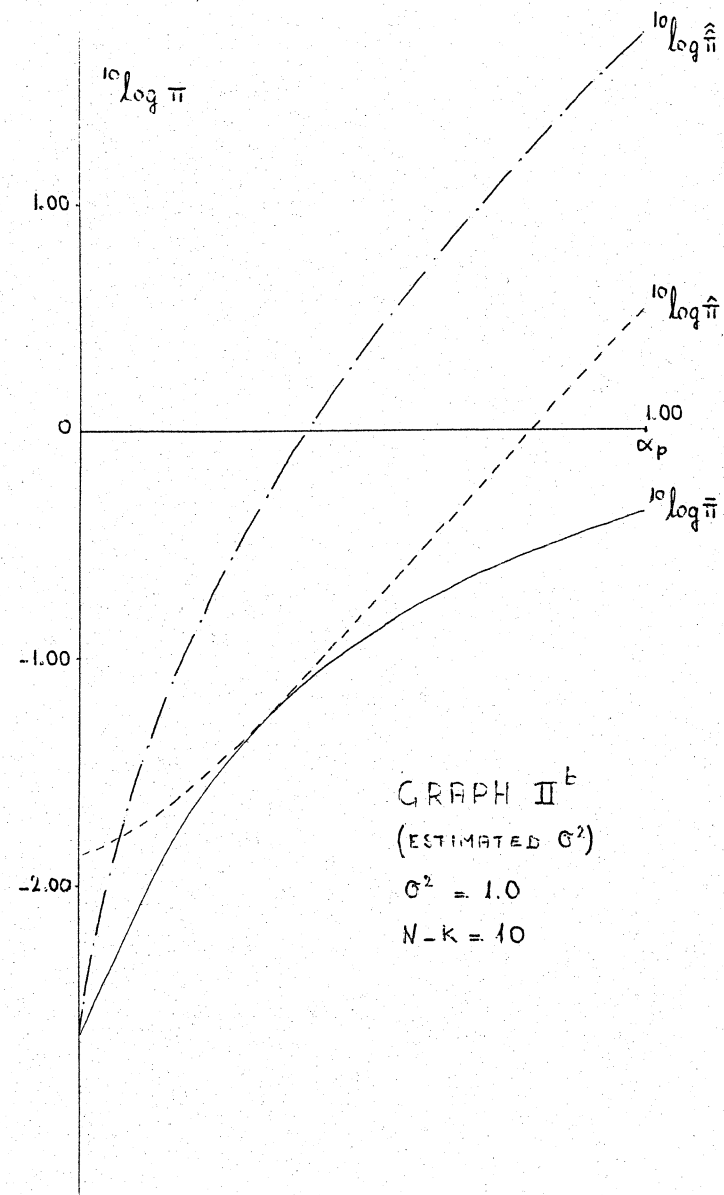
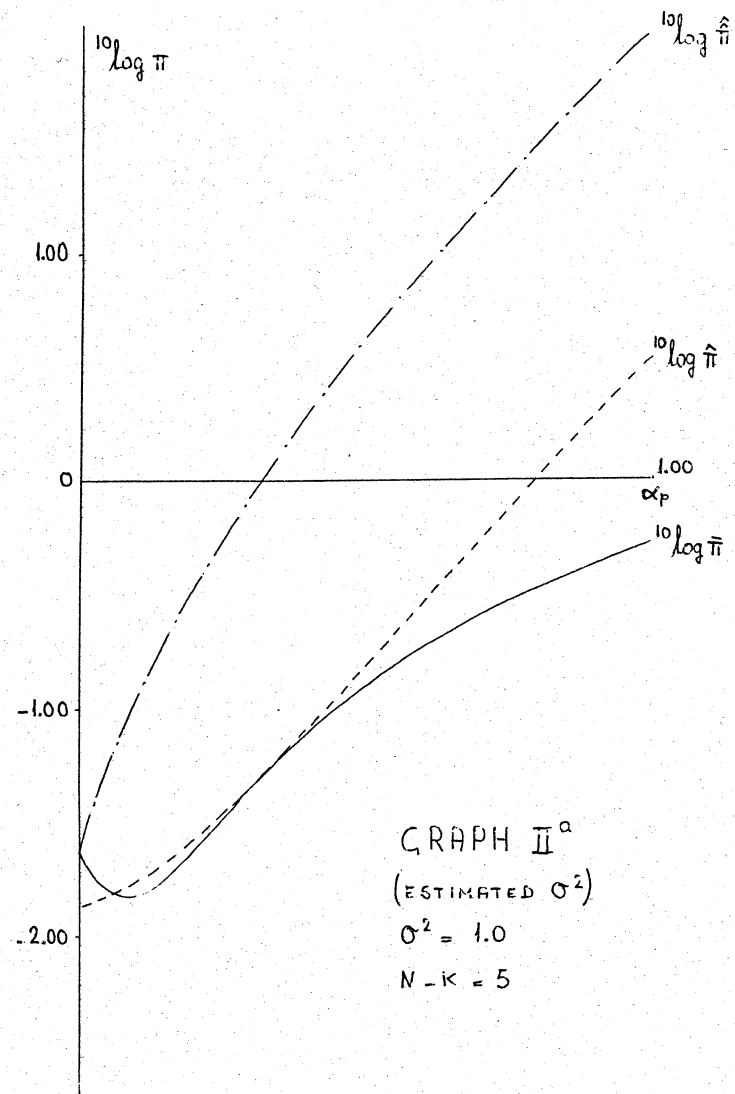
$$\alpha_p = 1.0$$

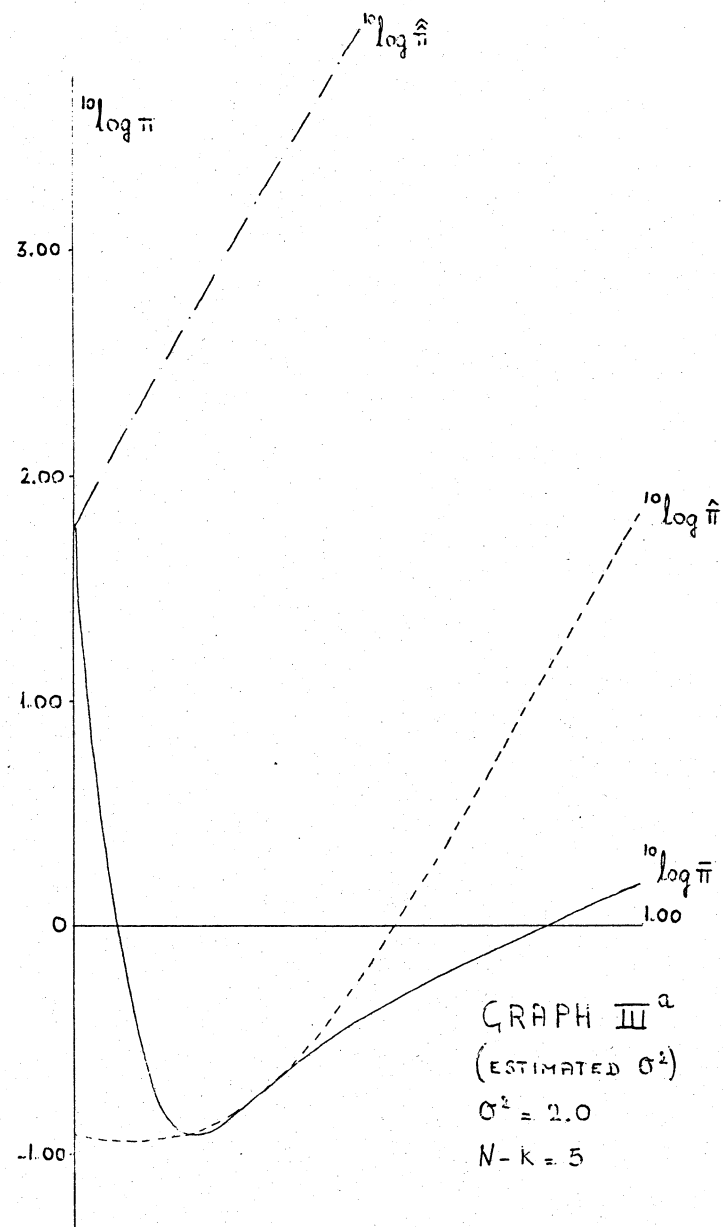
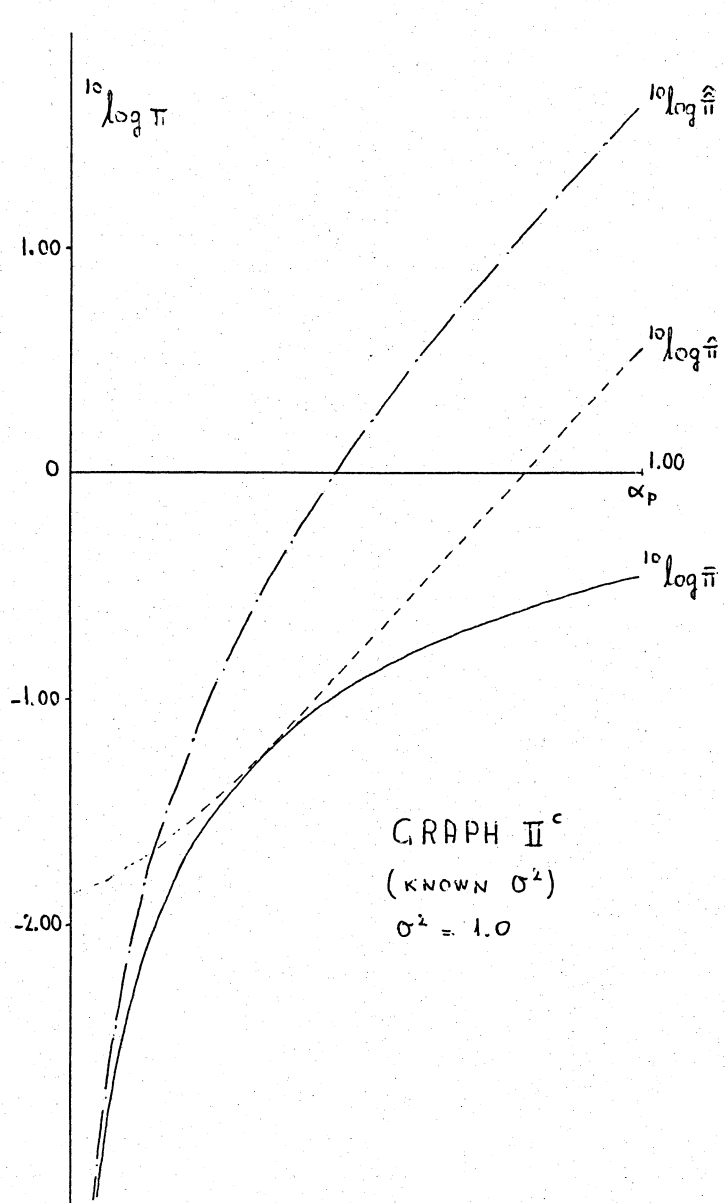
$\sigma^2$	N - K = 8				N - K = 10		N - K = 12	
	$\hat{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$		$\hat{\hat{\pi}}$	$\bar{\pi}$	$\hat{\hat{\pi}}$	$\bar{\pi}$
0.08	-2.4854	-2.3813	-2.5471		-2.3825	-2.5506	-2.3833	-2.5530
0.16	-1.7515	-1.5449	-1.8775		-1.5475	-1.8839	-1.5493	-1.8882
0.24	-1.3047	-0.9966	-1.4973		-1.0012	-1.5060	-1.0042	-1.5119
0.32	-0.9751	-0.5664	-1.2360		-0.5732	-1.2466	-0.5776	-1.2540
0.40	-0.7096	-0.2005	-1.0401		-0.2099	-1.0523	-0.2160	-1.0609
0.48	-0.4843	0.1250	-0.8854		0.1125	-0.8992	0.1045	-0.9089
0.56	-0.2867	0.4231	-0.7592		0.4072	-0.7744	0.3969	-0.7851
0.64	-0.1092	0.7018	-0.6537		0.6819	-0.6702	0.6691	-0.6819
0.72	0.0530	0.9660	-0.5639		0.9416	-0.5818	0.9261	-0.5944
0.80	0.2033	1.2193	-0.4864		1.1899	-0.5056	1.1713	-0.5191
0.88	0.3442	1.4645	-0.4187		1.4295	-0.4392	1.4074	-0.4536
0.96	0.4772	1.7035	-0.3589		1.6621	-0.3809	1.6362	-0.3962
1.04	0.6038	1.9378	-0.3058		1.8894	-0.3291	1.8593	-0.3454
1.12	0.7250	2.1688	-0.2581		2.1125	-0.2830	2.0778	-0.3003
1.20	0.8416	2.3975	-0.2152		2.3325	-0.2416	2.2927	-0.2598
1.28	0.9542	2.6249	-0.1762		2.5502	-0.2042	2.5047	-0.2235
1.36	1.0633	2.8518	-0.1406		2.7663	-0.1703	2.7146	-0.1906
1.44	1.1695	3.0788	-0.1080		2.9813	-0.1393	2.9229	-0.1608
1.52	1.2731	3.3067	-0.0779		3.1959	-0.1110	3.1301	-0.1336
1.60	1.3744	3.5361	-0.0500		3.4106	-0.0850	3.3367	-0.1088
1.68	1.4737	3.7676	-0.0240		3.6257	-0.0611	3.5430	-0.0860
1.76	1.5712	4.0018	0.0001		3.8417	-0.0388	3.7493	-0.0651
1.84	1.6670	4.2393	0.0229		4.0590	-0.0182	3.9560	-0.0457
1.92	1.7614	4.4806	0.0443		4.2779	0.0009	4.1635	-0.0278
2.00	1.8545	4.7265	0.0646		4.4989	0.0189	4.3719	-0.0112

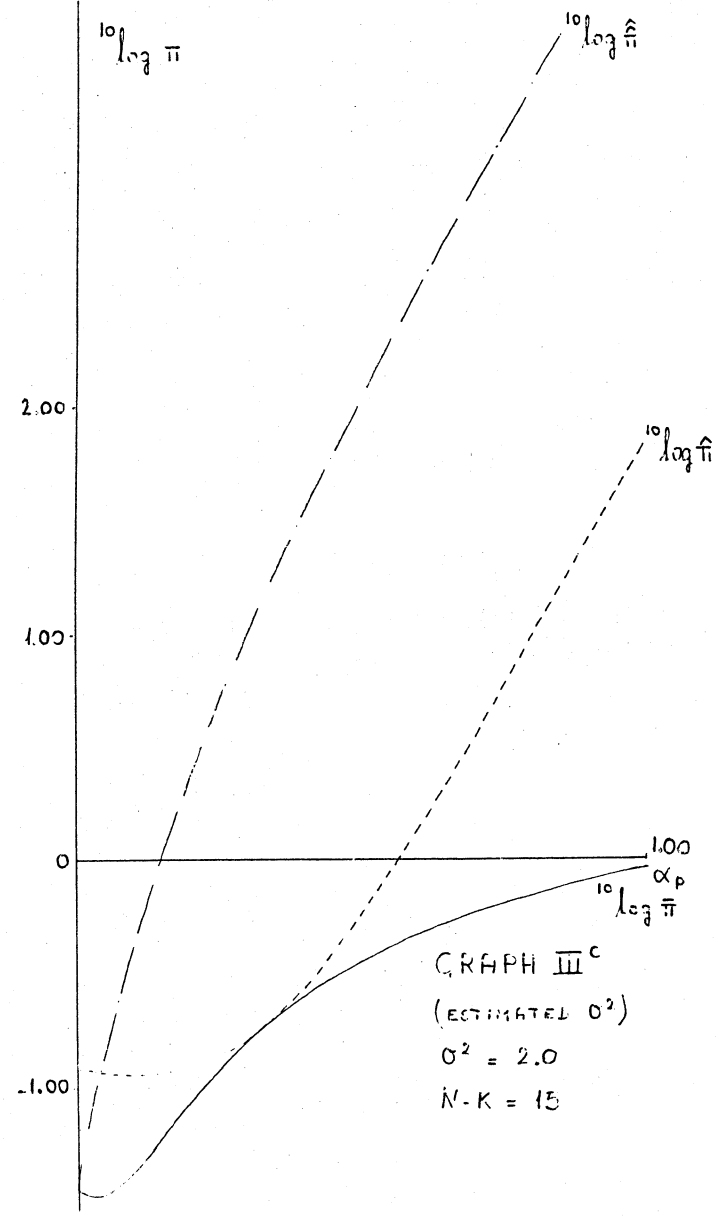
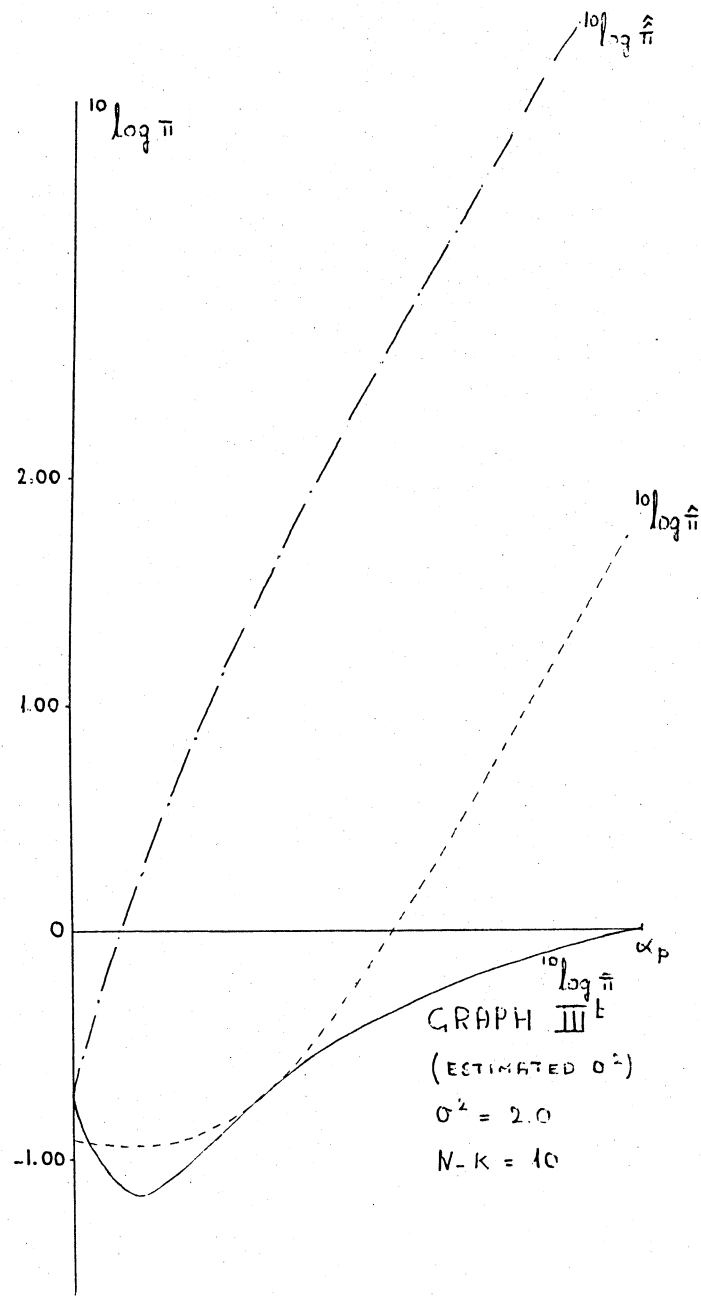
Lower bounds  $\alpha_0$ 

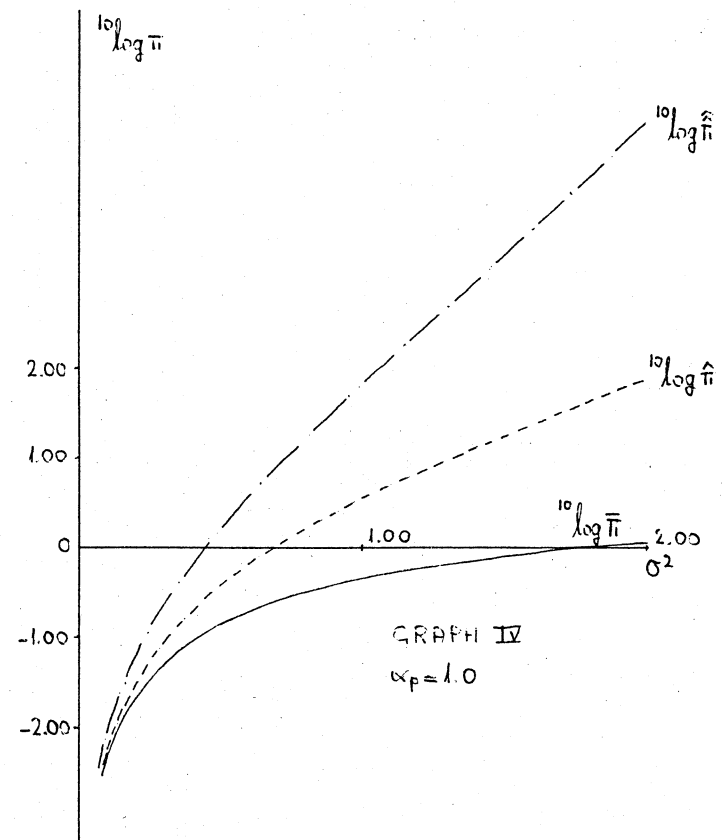
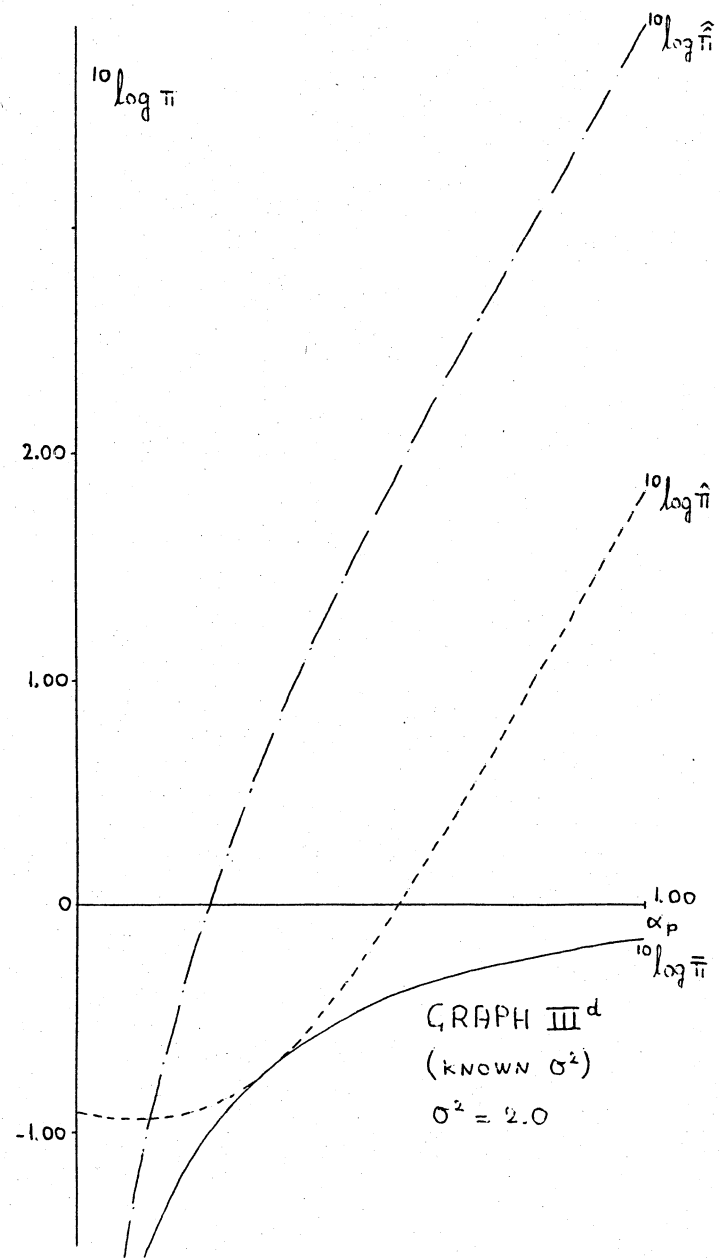
VARIANCE ( $\sigma^2$ )	DEGREES OF FREEDOM											
	12	11	10	9	8	7	6	5	4	3	2	1
2.00	0.00	0.01	0.04	0.06	0.10	0.13	0.16	0.20				
1.95	0.00	0.00	0.03	0.06	0.09	0.12	0.16	0.20	0.24			
1.90	0.00	0.00	0.02	0.05	0.08	0.12	0.15	0.19	0.23			
1.85	0.00	0.00	0.01	0.04	0.08	0.11	0.15	0.19	0.23			
1.80	0.00	0.00	0.00	0.03	0.07	0.10	0.14	0.18	0.23			
1.75	0.00	0.00	0.00	0.03	0.06	0.10	0.14	0.18	0.23			
1.70	0.00	0.00	0.00	0.02	0.05	0.09	0.13	0.18	0.22			
1.65	0.00	0.00	0.00	0.01	0.04	0.08	0.13	0.17	0.22			
1.60	0.00	0.00	0.00	0.00	0.04	0.08	0.12	0.17	0.22			
1.55	0.00	0.00	0.00	0.00	0.03	0.07	0.11	0.16	0.21			
1.50	0.00	0.00	0.00	0.00	0.02	0.06	0.10	0.15	0.21			
1.45	0.00	0.00	0.00	0.00	0.00	0.05	0.10	0.15	0.20	0.26		
1.40	0.00	0.00	0.00	0.00	0.00	0.04	0.09	0.14	0.20	0.26		
1.35	0.00	0.00	0.00	0.00	0.00	0.03	0.08	0.13	0.19	0.26		
1.30	0.00	0.00	0.00	0.00	0.00	0.02	0.07	0.13	0.19	0.26		
1.25	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.12	0.18	0.25		
1.20	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.11	0.18	0.25		
1.15	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.10	0.17	0.25		
1.10	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.09	0.16	0.24		
1.05	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.08	0.15	0.24		
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.14	0.23		
0.95	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.13	0.23	0.33	
0.90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.12	0.22	0.33	
0.85	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.11	0.22	0.33	
0.80	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.10	0.21	0.33	
0.75	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.08	0.20	0.33	
0.70	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.19	0.33	
0.65	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.18	0.33	
0.60	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.17	0.33	
0.55	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.15	0.33	
0.50	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.13	0.33	
0.45	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.11	0.33	0.33
0.40	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.08	0.33	0.33
0.35	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.33	0.33
0.30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.33	0.33
0.25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.33	0.33
0.20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.33	0.33
0.15	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.33	0.33
0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.33	0.33
0.05	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.33	0.33











## APPENDIX A

Let the random variable  $x$  be normally distributed with mean  $\mu$  and variance  $\sigma^2$ , then the variable

$$(A.1) \quad y = e^x$$

has a so-called lognormal distribution which is denoted by

$$(A.2) \quad y \sim \Lambda(\mu, \sigma^2) .$$

Thus, the distribution of  $y$  is characterized by the mean and the variance of the underlying normal distribution. The distribution of  $y$  can be derived as follows

$$(A.3) \quad F(y) = P[Y \leq y] = P[e^X \leq y] = P[X \leq \ln y] \\ = \int_{-\infty}^{\ln y} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x - \mu)^2} dx$$

The density function of  $y$  follows by differentiation of the distribution function (A.3) with respect to  $y$ :

$$(A.4) \quad f(y) = \frac{dF(y)}{dy} = \frac{1}{\sigma y\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\ln y - \mu)^2}$$

It turns out that it is very useful to derive the mean and variance of  $y$  as functions of  $\mu$  and  $\sigma^2$ . The following computations provide us with the mean  $E(y)$  and the variance  $\text{var}(y)$ , which are presented in (A.5) and (A.6) respectively.

$$(A.5) \quad E(y) = \int_{-\infty}^{\infty} e^x f(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\ = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu + \frac{1}{2}\sigma^2} e^{-\frac{1}{2}\left(\frac{x-(\mu+\sigma^2)}{\sigma}\right)^2} dx$$

and

$$\begin{aligned}
 E(y^2) &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{2x - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2\mu - 2\sigma^2} e^{-\frac{1}{2}\left(\frac{x-(\mu-2\sigma^2)}{\sigma}\right)^2} dx \\
 &= e^{2\mu - 2\sigma^2}
 \end{aligned}$$

hence

$$(A.6) \quad \text{var}(y) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu} [e^{2\sigma^2} - e^{\sigma^2}]$$

Next, we establish the inverse functions, i.e. express  $\mu$  and  $\sigma^2$  as functions of  $E(y)$  and  $\text{var}(y)$ . From (A.5) it follows that

$$\mu + \frac{1}{2}\sigma^2 = \ln [E(y)]$$

or

$$(A.7) \quad \mu = \ln [E(y)] - \frac{1}{2}\sigma^2$$

Now, we substitute (A.7) into (A.6) and get

$$(A.8) \quad [E(y)]^2 (e^{\sigma^2} - 1) = \text{var}(y)$$

From (A.8) it follows that:

$$(A.9) \quad \sigma^2 = \ln \frac{\text{var}(y)}{[E(y)]^2} + 1$$

and substitution of (A.9) into (A.7) yields

$$(A.10) \quad \mu = \ln [E(y)] - \frac{1}{2} \ln \left[ \frac{\text{var}(y)}{[E(y)]^2} + 1 \right]$$

In Section 2 it has been assumed that  $E(y) = 1$ ; substitution of this assumption into (A.9) and (A.10) gives

$$(A.11) \quad \sigma^2 = \ln [\text{var}(y) + 1]$$

and

$$(A.12) \quad \mu = -\frac{1}{2} \ln [\text{var}(y) + 1]$$

Hence a lognormal distribution with the mean equal to unity corresponds to a normal distribution with  $\mu = -\frac{1}{2}\sigma^2$ .

## APPENDIX B

In Section 3.1.2 the least squares estimator of  $E(Y|x_p)$  has been defined as

$$(B.1) \quad \hat{Y}_p = e^{x'_p \hat{\beta}} = e^{\hat{y}_p}$$

According to Appendix A the variance of this estimator equals

$$(B.2) \quad \text{var } \hat{Y}_p = e^{2E(\hat{y}_p)} [e^{2\text{var } \hat{y}_p} - e^{\text{var } \hat{y}_p}]$$

since  $\hat{y}_p$  is normally distributed. Hence, to determine  $\text{var } \hat{Y}_p$ , we have to find  $E(\hat{y}_p)$  and  $\text{var } \hat{y}_p$ .

The mean of  $\hat{y}_p$  can be established as follows:

$$\begin{aligned} (B.3) \quad E(\hat{y}_p) &= E(x'_p \hat{\beta}) = E[x'_p (X'X)^{-1} X'y] \\ &= E[x'_p \beta + x'_p (X'X)^{-1} X'u] = x'_p \beta - \frac{1}{2} \sigma^2 x'_p (X'X)^{-1} X'1 \\ &= x'_p \beta - \frac{1}{2} \sigma^2 \end{aligned}$$

where use has been made of the fact that the first element of  $x_p$  equals unity.

The variance of  $\hat{y}_p$  equals

$$\begin{aligned} (B.4) \quad \text{var } \hat{y}_p &= E(\hat{y}_p^2) - [E(\hat{y}_p)]^2 = E(x'_p \hat{\beta} \hat{\beta}' x_p) - (x'_p \beta - \frac{1}{2} \sigma^2)^2 \\ &= x'_p \{E[(\beta + (X'X)^{-1} X'u)(\beta + (X'X)^{-1} X'u)']\} x_p \\ &\quad - (x'_p \beta - \frac{1}{2} \sigma^2)^2 \\ &= x'_p [\beta \beta' + 2\beta E(u') X (X'X)^{-1} + (X'X)^{-1} X' E(uu') X (X'X)^{-1}] x_p \\ &\quad - (x'_p \beta - \frac{1}{2} \sigma^2)^2 \\ &= (x'_p \beta)^2 - \sigma^2 x'_p \beta + \sigma^2 x'_p (X'X)^{-1} x_p + \frac{1}{4} \sigma^2 - (x'_p \beta - \frac{1}{2} \sigma^2)^2 \\ &= \sigma^2 x'_p (X'X)^{-1} x_p \\ &= \sigma^2 \alpha_p \end{aligned}$$

Substitution of (B.3) and (B.4) into (B.2) gives us the variance of  $\hat{Y}_p$ :

$$(B.5) \quad \text{var } \hat{Y}_p = e^{2x'_p\beta - \sigma^2} [e^{2\sigma^2_{\alpha p}} - e^{\sigma^2_{\alpha p}}]$$

## APPENDIX C

In this appendix it will be proved that the random variables  $x_p'\hat{\beta}$  and  $e_p'\xi_p S^2$  are independently distributed. To that end we first consider  $x_p'\hat{\beta}$  and  $\xi_p S^2$ . These two variables can both be written as functions of the disturbance vector  $u$ :

$$(C.1) \quad x_p'\hat{\beta} = x_p'\beta + x_p'(X'X)^{-1}X'u$$

and

$$(C.2) \quad \xi_p S^2 = \frac{\xi_p}{N-K} y'My = \frac{\xi_p}{N-K} u'Mu$$

Now we shall make use of the following theorem: "Let  $e$  be a vector of  $n$  independent normal variables each with zero mean and unit variance. Then the linear and quadratic forms

$$L = a'e \quad \text{and} \quad Q = e'Be$$

are independently distributed if  $Ba = 0$ .<sup>1)</sup>"

Since

$$u \sim N(-\frac{1}{2}\sigma^2_1, \sigma^2 I)$$

we apply the following transformation on  $u$

$$(C.3) \quad e = \frac{1}{\sigma} (u + \frac{1}{2}\sigma^2_1)$$

so that the vector  $e$  is normally distributed with zero mean and unit variance.

From (C.3) it follows that

$$(C.4) \quad u = \sigma e - \frac{1}{2}\sigma^2_1$$

---

<sup>1)</sup> For a proof see Kendall & Stuart [ 7 ] pp 356-359

Substitution of (C.4) into (C.1) and (C.2) yields

$$(C.5) \quad x_p' \hat{\beta} = x_p' \beta - \frac{1}{2} \sigma^2 + \sigma x_p' (X'X)^{-1} X'e$$

and

$$(C.6) \quad \xi_p S^2 = \frac{\xi_p \sigma^2}{N-K} e'Me$$

where use has been made of the property  $M1 = 0$ . The linear and quadratic forms

$$x_p' (X'X)^{-1} X'e \quad \text{and} \quad e'Me$$

are independently distributed, since

$$MX(X'X)^{-1} x_p = 0$$

Hence  $x_p' \hat{\beta}$  and  $\xi_p S^2$  are also independently distributed and so are  $\exp(x_p' \hat{\beta})$  and  $\exp(\xi_p S^2)$ .

## A P P E N D I X D

The matrix  $(X'X)^{-1}$  can be partitioned as follows:

$$(D.1) \quad (X'X)^{-1} = \begin{bmatrix} N & 1 & 1'X_1 \\ \hline X_1'1 & & X_1'X_1 \end{bmatrix}^{-1}$$

and it can easily be verified that

$$(D.2) \quad \begin{bmatrix} N & 1 & 1'X_1 \\ \hline X_1'1 & & X_1'X_1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{N}(1 + \frac{1}{N}1'X_1(X_1'EX_1)^{-1}X_1'1) & -\frac{1}{N}1'X_1(X_1'EX_1)^{-1} \\ \hline -\frac{1}{N}(X_1'EX_1)^{-1}X_1'1 & (X_1'EX_1)^{-1} \end{bmatrix}$$

where

$$(D.3) \quad E = I - \frac{1}{N}11'$$

We prove that the matrix  $X_1'EX_1$  has full rank by assuming the contrary.

If the matrix  $EX_1$  has not full column rank then there is a vector  $p \neq 0$  such that

$$(D.4) \quad EX_1p = 0$$

or

$$(D.5) \quad Eq = 0$$

where

$$(D.6) \quad q = X_1p$$

Since  $Eq = 0$  if and only if  $q = \ell \mathbf{1}$  where  $\ell$  is an arbitrary scalar, the following relation should hold

$$(D.7) \quad X_1 p = \mathbf{1}$$

This, however, is violated by the assumption that  $X = [\mathbf{1} \quad X_1]$  has full column rank. Thus,  $EX_1$  has full column rank and

$$(D.8) \quad X'EX = (EX_1)'(EX)$$

is a non-singular matrix.

We apply the result obtained in (D.2) and write  $\alpha_p$  as

$$(D.9) \quad \alpha_p = x_p'(X'X)^{-1}x_p = \begin{bmatrix} 1 & x_{p_1}' \end{bmatrix} \begin{bmatrix} \frac{1}{N} + \frac{1}{N^2} \mathbf{1}' X_1 (X_1' EX_1)^{-1} X_1' \mathbf{1} & -\frac{1}{N} \mathbf{1}' X_1 (X_1' EX_1)^{-1} \\ -\frac{1}{N} (X_1' EX_1)^{-1} X_1' \mathbf{1} & (X_1' EX_1)^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ x_{p_1} \end{bmatrix}$$

$$= \frac{1}{N} + \frac{1}{N} (X_1' \mathbf{1} - x_{p_1})' (X_1' EX_1)^{-1} (\frac{1}{N} X_1' \mathbf{1} - x_{p_1})$$

with

$$x_p' = \begin{bmatrix} 1 & x_{p_1}' \end{bmatrix}$$

And since  $(X'EX)^{-1}$  is a positive definite matrix (see (D.8))

$$(D.10) \quad \alpha_p \geq \frac{1}{N}$$

If  $x_p$  is a row vector of  $X$ , say  $x_p = x_i$ , then  $\alpha_p$  can be written as

$$\alpha_p = e_i' X (X'X)^{-1} X' e_i = \frac{e_i' X (X'X)^{-1} X' e_i}{e_i' e_i} \leq 1$$

Since  $E\mathbf{q} = 0$  if and only if  $\mathbf{q} = \ell \mathbf{1}$ , where  $\ell$  is an arbitrary scalar, the following relation should hold

$$(D.7) \quad X_1 \mathbf{p} = \mathbf{1}$$

This, however, is violated by the assumption that  $X = [\mathbf{1} \quad X_1]$  has full column rank. Thus,  $EX_1$  has full column rank and

$$(D.8) \quad X'EX = (EX_1)'(EX)$$

is a non-singular matrix.

We apply the result obtained in (D.2) and write  $\alpha_p$  as

$$(D.9) \quad \alpha_p = \mathbf{x}'_p (X'X)^{-1} \mathbf{x}_p = \begin{bmatrix} 1 & \vdots & \mathbf{x}'_{p_1} \end{bmatrix} \begin{bmatrix} \frac{1}{N} + \frac{1}{N^2} \mathbf{1}' X_1 (X_1' EX_1)^{-1} X_1' \mathbf{1} & -\frac{1}{N} \mathbf{1}' X_1 (X_1' EX_1)^{-1} \\ \hline -\frac{1}{N} (X_1' EX_1)^{-1} X_1' \mathbf{1} & (X_1' EX_1)^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ \hline \mathbf{x}_{p_1} \end{bmatrix}$$

$$= \frac{1}{N} + \frac{1}{N} (\mathbf{x}'_{p_1} - \mathbf{1}' \mathbf{x}_{p_1})' (X_1' EX_1)^{-1} (\frac{1}{N} \mathbf{x}'_{p_1} - \mathbf{x}_{p_1})$$

with

$$\mathbf{x}'_p = \begin{bmatrix} 1 & \mathbf{x}'_{p_1} \end{bmatrix}$$

And since  $(X'EX)^{-1}$  is a positive definite matrix (see (D.8))

$$(D.10) \quad \alpha_p \geq \frac{1}{N}$$

If  $\mathbf{x}_p$  is a row vector of  $X$ , say  $\mathbf{x}_p = \mathbf{x}_i$ , then  $\alpha_p$  can be written as

$$\alpha_p = \mathbf{e}'_i X (X'X)^{-1} X' \mathbf{e}_i = \frac{\mathbf{e}'_i X (X'X)^{-1} X' \mathbf{e}_i}{\mathbf{e}'_i \mathbf{e}_i} \leq 1$$

where we made use of the knowledge that  $X(X'X)^{-1}X'$  is a idempotent matrix, and that the largest characteristic root of such a matrix is equal to unity.

