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CALCULATING VON NEUMANN TRAJECTORIES
BY SIMULATED MARKET ADJUSTMENTS

by S.P. Burley

June 1, 1970

Preliminary and Confidential

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1. INTRODUCTION

This report is concerned with the linear model of an expanding economy proposed by von Neumann [7]. As in Sections 2 and 3 of his paper, we define A and B as the $m \times n$ input and output matrices respectively for n goods with semi-positive prices $y(j)$, $j = 1, \dots, n$, which can be produced by m processes operating with the semi-positive intensities $x(i)$, $i = 1, \dots, m$, such that the whole complex expands proportionally with a growth factor α and interest factor β . Thus, in long-run economic equilibrium we have:

$$(1.1) \quad xB \geq \alpha xA$$

$$By \leq \beta Ay$$

where, if any of the inequalities hold, the corresponding x or y values are zero; though not all x and y can be zero in a meaningful problem.

In his original treatment, von Neumann concentrated on a constrained form of the model in which every good is involved in every process either as input or as output. With this restriction, he was able to show that the whole model can expand uniformly with α equal to β and unique. Less constrained forms of the model have also been discussed in the literature, notably by Kemeny, Morgenstern and Thompson [4] (KMT) in which, however,

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the economy can separate into independent sub-economies, each with its own growth path. The KMT approach has been further developed by Hamberger, Thompson and Weil [3] to yield operational algorithms for computing the numerical values of x , y , α and β . Here, we shall indicate another computational approach which appears to have some interesting new features worth exploring.

As von Neumann pointed out at the time, his results imply that "the normal price mechanism brings about - if our assumptions are valid - the technically most efficient intensities of production". This immediately suggests the mathematical possibility of an alternative approach to the fixed point, based on the solution of a system of differential equations describing the market adjustment processes (though this formulation is only possible on the basis of von Neumann's proof that a solution of that particular form does in fact exist).

In this problem, there are further analytical complications to standard differential equation techniques due to the non-negativity restrictions on x and y ; but the possibility of a steady state solution is clear if we bear in mind:

- (a) the analogy between the von Neumann model and a certain fair, zero sum, 2-person game indicated by KMT, and
- (b) the analogy between games and systems of differential equations pointed out by Brown and von Neumann [2].

An advantage of this somewhat indirect approach is that it is "constructive" in the sense that it readily leads itself to direct utilization when actually computing numerical solutions for economies with specific input and output matrices. For practical computations carried out on a digital machine, a difference equation formulation seems to be even more convenient. We shall show that this can be set up along the lines suggested above by taking advantage of a further suggestion made by Brown indicating a possible difference equation extension of the analogy between games and differential equations [1]. This can be used to explore the convergence of the market adjustment process described in the next section.

2. DESCRIPTION OF THE ALGORITHM

Consider the following iterative market learning process:

(1) Let the "Growth Planner" choose some initial value for the "waiting factor" α . This is intended to serve as both a growth factor and an interest factor for the whole system.

(2) Set all prices and intensities initially at zero.

(3) Set one process going at unit intensity.

(4a) Note the good which, at the current intensity levels and growth factor, has the minimum "excess growth" i.e. excess over α . Let the market increase the relative price of this scarcest good (c.f. Walrasian market adjustment).

(4b) Check that the minimum excess growth in (4a) is ≤ 0 ; if it is > 0 , the Growth Planner must raise his estimate of the overall growth possible and go back to 2.

(5a) Note the process which, at the current prices and interest factor, has maximum "excess return" i.e. excess over α . Let the entrepreneurs increase the relative intensity of this most profitable process (c.f. Marshallian market adjustment).

(5b) Check that the maximum excess return in 5a is ≥ 0 ; if it is < 0 , the Growth Planner must reduce his estimate of the return possible and go back to 2.

(6) Go back to (4a).

We shall show that, if these adjustments are carried out in a suitable manner, the procedure iterates towards an equilibrium value of α corresponding to a (possibly non-unique) set of competitive prices and intensities. Further, this value of α is not:

(a) lower than the expansion factor of the slowest growing good, nor

(b) greater than the return payable by the most profitable process.

Under von Neumann's original conditions, there is only one value of which satisfies all these conditions, while a range of values is permitted by the weaker KMT conditions.

3. CONVERGENCE CONSIDERATIONS

One useful way of exploring the convergence of the above procedure is based on some mathematical techniques which have grown out of Game Theory. It can be shown that, if we delete steps (4b) and (5b) from our algorithm, we have a difference equation process for iterating towards the solution of a game with payoff matrix $B - \alpha A$. We propose to carry out these adjustments by a method similar to that suggested by Brown for solving games [1]. Thus, at the N^{th} iteration, we add on an increment of order $1/N$ to that intensity (and price) on which there is most "economic pressure" (in terms of steps (4a) and (5a)) using as data the cumulated effect of all past price (and intensity) changes. Specifically, after N iterations, the intensity of the i^{th} process is given by:

$$(3.1) \quad x(i) = \frac{N(i)}{N}$$

where $N(i)$ is the number of times the i^{th} process has been stepped up. The same rule holds for prices. Under similar conditions, Robinson [8] has proved that the corresponding iterative game process has the property

$$(3.2) \quad \min x(B - \alpha A) \leq x(B - \alpha A)y \leq \max (B - A)y$$

and converges to give the value of the game and good mixed strategies x and y . To complete the analogy, we may note that KMT have shown that equilibrium solutions to the von Neumann model correspond to α 's such that the value of this game is zero, i.e.

$$(3.3) \quad v(\alpha) = x(B - \alpha A)y = 0$$

So, if during our iteration we find evidence that this is not so, i.e. if we see that either

$$(3.4) \quad \min x(B - \alpha A) > 0,$$

or

$$\max (B - \alpha A)y < 0$$

then we would have to think of adapting our trial value of α so as to steer the value of the game onto zero, c.f. steps (4b) and (5b).

We might consider carrying out this latter correction process by means of an iterative procedure corresponding to Newton's method. Thus, given the first order approximation

$$(3.5) \quad v(\alpha'') = v(\alpha') + (\alpha'' - \alpha') \left. \frac{\partial v}{\partial \alpha} \right|_{\alpha'}$$

we have, for convergence towards a root of $v(\alpha) = 0$, the standard adjustment

$$(3.6) \quad \alpha'' = \alpha' - v(\alpha') / \left. \frac{\partial v}{\partial \alpha} \right|_{\alpha'}$$

(We might also think of including a conventional economic adjustment factor $k \neq 1$ in the second term on the right-hand side.)

Under von Neumann's original conditions, $v(\alpha)$ is a strictly decreasing function and so our convergence onto the unique equilibrium α is well behaved. If, however, the economy contains independent subeconomies, then the equilibrium α is not unique, and we can also have difficulties associated with zeros of $\frac{\partial v}{\partial \alpha}$. So, if we are going to use this particular adjustment process, we must ensure that our economy cannot be decomposed into subsectors, or we must deal with each subsector separately.² However, in the more general case, other (more complicated) adjustment procedures can be used to lead to the various equilibrium values of α , c.f. [3].

In our simple case, to estimate $\left. \frac{\partial v}{\partial \alpha} \right|_{\alpha'}$ we shall use the computed values of x and y given by Equation 3.1 in Mills' formula for the marginal value of a game [5]. This gives

$$(3.7) \quad \left. \frac{\partial v}{\partial \alpha} \right|_{\alpha'} = x(-A)y$$

Similarly, as an estimate of $v(\alpha')$ we shall use

$$(3.8) \quad v(\alpha') = x(B - \alpha'A)y$$

Hence, for any necessary corrections at steps (4b) and (5b), we propose the difference equation

² Morgenstern and Thompson have pointed out [6] that we can often expect real economies to be held together by public and private consumption effects if not by technology. Aggregation of either processes or goods has a similar effect.

$$(3.9) \quad \alpha'' = \alpha' + x(B - \alpha'A)y/xAy$$

In order to find a stopping rule, we could e.g. terminate this whole process when we find that after say N time periods (each with its own price and intensity adjustment), the value of α is still feasible from the point of view of requirements (4b) and (5b). At this stage, we also have convergence of order $1/N$ on the corresponding zero value game condition [8].

In a similar way we could demonstrate the convergence of other processes e.g. the possibly "more realistic" one which sends the process back to (4a) (instead of (2)) at steps (4b) and (5b).

4. DISCUSSION OF THE METHOD

As an illustration of the properties of the algorithm described in Sections 2 and 3, we can consider the calculated solution of the 16×12 production process shown below.

A MATRIX

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 0 | 9 | 1 | 9 | 0 | 3 | 1 | 1 | 1 | 2 | 9 |
| 0 | 1 | 1 | 0 | 6 | 3 | 1 | 5 | 8 | 1 | 6 | 9 |
| 1 | 3 | 4 | 1 | 0 | 1 | 1 | 2 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 3 | 9 | 1 | 8 | 2 | 3 | 4 | 3 | 3 | 0 | 0 | 6 |
| 2 | 4 | 5 | 7 | 9 | 0 | 0 | 0 | 0 | 4 | 2 | 3 |
| 0 | 1 | 0 | 1 | 0 | 3 | 1 | 0 | 0 | 1 | 8 | 2 |
| 1 | 2 | 1 | 2 | 1 | 2 | 9 | 9 | 9 | 0 | 1 | 5 |
| 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 3 | 5 | 0 | 4 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 |
| 0 | 0 | 5 | 6 | 0 | 8 | 0 | 1 | 3 | 5 | 7 | 9 |
| 0 | 0 | 8 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 8 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | 1 | 2 |
| 1 | 0 | 2 | 5 | 8 | 7 | 1 | 2 | 0 | 1 | 0 | 1 |
| 0 | 2 | 0 | 3 | 0 | 4 | 0 | 5 | 0 | 6 | 0 | 7 |
| 2 | 0 | 1 | 0 | 3 | 0 | 2 | 0 | 1 | 0 | 5 | 0 |

B MATRIX

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 1 | 3 | 0 | 1 | 2 | 2 | 0 | 0 | 6 | 3 | 5 |
| 1 | 0 | 2 | 1 | 5 | 0 | 6 | 3 | 1 | 3 | 9 | 2 |
| 5 | 2 | 0 | 8 | 8 | 5 | 0 | 7 | 5 | 2 | 8 | 9 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 4 | 3 | 0 | 0 | 3 | 0 | 5 | 4 | 4 | 1 | 2 | 3 |
| 3 | 5 | 6 | 7 | 8 | 1 | 2 | 1 | 2 | 5 | 1 | 3 |
| 1 | 0 | 1 | 0 | 1 | 1 | 4 | 2 | 1 | 0 | 9 | 4 |
| 2 | 1 | 1 | 1 | 0 | 0 | 9 | 9 | 1 | 2 | 0 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 4 | 0 | 6 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 1 |
| 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 4 | 6 | 8 | 8 |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |
| 0 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 2 | 3 |
| 0 | 1 | 0 | 1 | 9 | 9 | 2 | 2 | 1 | 0 | 1 | 0 |
| 3 | 0 | 4 | 0 | 5 | 0 | 6 | 0 | 7 | 0 | 8 | 0 |
| 0 | 3 | 0 | 1 | 0 | 2 | 0 | 4 | 0 | 2 | 0 | 6 |

TRIAL ALFA VALUES

1.000

1.126

1.176

EQUILIBRIUM GROWTH RATE

1.176

INTENSITY VECTOR

0.000 0.081 0.107 0.000 0.000 0.327 0.080 0.000 0.000 0.070 0.000 0.000
0.084 0.000 0.107 0.143

PRICE VECTOR

0.025 0.045 0.444 0.000 0.052 0.097 0.000 0.000 0.000 0.234 0.052 0.050

We may note that there are only 8 active processes and 8 scarce goods in the final solution. We can also see that the growth factor converges in only two steps from an initial value of 1.000 to a value of 1.176 which proved viable over 1,000 competitive price and intensity reactions. Execution time was 85 seconds on IBM 1130 computer.³

A useful feature of this algorithm is that, for a given degree of convergence (corresponding to the assigned iteration limit N for competitive adjustments), the main computational effort is related to the sum of the number of rows and columns of the input and output matrices, i.e. $m + n$. So, for large matrices, this is much faster than the linear programming approach, where computer time is a higher order function of m and n , e.g. something like $m^2 \times n$ for one common algorithm.

On the other hand, our adjustment approach leads to only approximate equilibrium solutions - containing the cumulative effect of all decisions made in getting there (even bad ones). Thus, unprofitable processes having been set up are never completely retrenched, and overproduced goods which were once scarce never become completely free. However, the effect of such bad decisions on the computed growth path becomes negligible as time goes on.⁴

Finally, it should be noted that the historical adjustment approach suggested here lends itself naturally to studying the reaction of a von Neumann system to external shocks or cycles. There are obvious extensions to richer dynamics than those described here.

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³ Listings of the program used are available on request to the author at La Trobe University.

⁴ It may be recollected that the poor quality of actual input-output data always makes the calculation of "completely accurate" solutions a specious exercise in practice.

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