Netherlands School of Economics

ECONOMETRIC INSTITUTE

Report 7004

A LIMITING BAYESIAN APPROACH TO SIMULTANEOUS EQUATION SYSTEMS

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February 5, 1970

Preliminary and Confidential

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Contents

1. INTRODUCTION

In a recent paper[1] we proposed a method of incorporating prior information about structural parameters into the statistical analysis of simultaneous economic equation systems. This method is based on the assumption that we have at least some prior information about all structural parameters while none of them is completely known in advance. Frequently, however, we would like to postulate complete certainty about some parameters such as those traditionally set equal to one for normalization purposes. On the other hand, our prior information about parameters such as the constant term is often so vague that we would prefer to specify non-informative[2] prior densities

[1] See Harkema (1969).

[2] See for this concept Jeffreys (1961), p. 179 ff.

for them. This paper is concerned with a sequence of prior densities defined on the space of the structural parameters by means of which the assumption mentioned above can be relaxed so as to make allowance for complete certainty or complete uncertainty about all parameters in the same column of the matrix of structural parameters.

To facilitate reading, the following notational convention is used: parameters of prior distributions can be recognized by a first subscript 0, sample statistics by a first subscript 1, and parameters of posterior distributions by a first subscript 2.

The order of discussion is as follows. We first present a double sequence of joint prior densities so that some marginal densities converge in distribution to an a priori specified known real number, while some other marginal densities can be made increasingly uniform over the entire real axis In Subsection 2.2 the limiting behavior of the marginal prior distributions is examined, while Subsection 2.3 gives the corresponding sequence of posterior distributions on the space of the reduced form parameters. In Subsections 3.1 and 3.2 we determine the limits of the parameters of the sequence of posterior distributions and derive the limiting posterior distribution. Section 4 discusses a modification of the assumptions and summarizes our findings.

2. A SEQUENCE OF PRIOR AND POSTERIOR DISTRIBUTIONS

Let us consider the following system of simultaneous equations

$$\begin{aligned} \Gamma_{11}y_1 + \Gamma_{12}y_2 + \Delta_{11}x_1 + \Delta_{12}x_2 + \Delta_{13}x_3 &= u \\ \Gamma_{21}y_1 + \Gamma_{22}y_2 + \Delta_{21}x_1 + \Delta_{22}x_2 + \Delta_{23}x_3 &= 0 \end{aligned} \tag{2.1}$$

where y_1 is an m-dimensional vector of endogenous variables, y_2 is a p-dimensional vector of endogenous variables, x_1 is an r-dimensional vector of exogenous variables, x_2 is an s-dimensional vector of exogenous variables, x_3 is a t-dimensional vector of exogenous variables, u is an m-dimensional vector of disturbances, and Γ_{11}, Γ_{12}, Δ_{11}, Δ_{12}, Δ_{13}, Γ_{21}, Γ_{22}, Δ_{21}, Δ_{22}, and Δ_{23} are matrices of structural parameters of orders $m \times m$, $m \times p$, $m \times r$, $m \times s$, $m \times t$, $p \times m$, $p \times p$, $p \times r$, $p \times s$, and $p \times t$ respectively.

The system (2.1) is supposed to satisfy the following assumptions:[3]

(i) The matrix Γ defined as

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

has full rank, which implies that an ordering of the endogenous variables exists so that Γ_{22} is non-singular. Without loss of generality we assume that y_1 and y_2 represent such an ordering.

(ii) The matrices Γ_{21}, Γ_{22}, Δ_{21}, Δ_{22}, and Δ_{23}, which pertain to that part of the system representing the identities, consist of known constants.

(iii) The m-dimensional vector of disturbances u is normally distributed with zero mean and unknown variance-covariance matrix Σ^{-1}.

(iv) The vector of random variables u is distributed independently of the vector of exogenous variables x.

2.1. The Sequence of Prior Distributions

We assume that we are willing to express our prior information about the structural parameters which correspond to that part of the system representing the behavioral relations by means of a matrix Normal-Wishart distribution with m degrees of freedom. This means that our prior distribution looks as follows:[4]

$$f(\Lambda, \Sigma \mid \Lambda_0, T_0, M_0) \propto$$

$$|\Sigma|^{\frac{1}{2}(m+p+r+s+t)} \exp\{-\tfrac{1}{2} \operatorname{tr} \Sigma[\Lambda - \Lambda_0]M_0[\Lambda - \Lambda_0]'\} \qquad (2.2)$$

$$|\Sigma|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2} \operatorname{tr} \Sigma\, T_0\} \qquad -\infty < \lambda_{ij} < \infty, \quad \Sigma \ \text{PDS}$$

[3] Note that we do not specify a normalization rule. A detailed discussion as to why we do not normalize our system can be found in Harkema (1969), Section 3.

[4] See, e.g., Harkema (1969), Section 2.

with

$$\Lambda = [\Gamma_{11} \quad \Gamma_{12} \quad \Delta_{11} \quad \Delta_{12} \quad \Delta_{13}]$$

$$\Lambda_0 = [\Gamma_{011} \quad \Gamma_{012} \quad \Delta_{011} \quad \Delta_{012} \quad \Delta_{013}]$$

$$M_0 = \begin{bmatrix} M_{011} & M_{012} & M_{013} & M_{014} & M_{015} \\ M_{021} & M_{022} & M_{023} & M_{024} & M_{025} \\ M_{031} & M_{032} & M_{033} & M_{034} & M_{035} \\ M_{041} & M_{042} & M_{043} & M_{044} & M_{045} \\ M_{051} & M_{052} & M_{053} & M_{054} & M_{055} \end{bmatrix}$$

where the matrices M_{0ij} $(i, j = 1, ..., 5)$ denote the appropriate submatrices of M_0 and where T_0 and M_0 are supposed to be positive definite symmetric.

Frequently, however, we would like to postulate complete certainty about some structural parameters. In many cases,[5] for example, the model is such (or can be made so by adding new definitions) that it seems only natural to specify an exact identity matrix for the matrix Γ_{11}. In other cases[6] small partial reduced-form operations can be used to obtain a revised version of the model so that it seems obvious to postulate an exact identity matrix for the new Γ_{11}-matrix. In addition to this type of certainty, an exogenous variable sometimes shows up only in the identities. Clearly, in this case, we would like to specify an exact zero column in one of the matrices Δ_{1i} $(i = 1, 2, 3)$. On the other hand, our information about the influence of particular exogenous variables such as the constant term is often so vague that a non-informative density seems to represent our prior knowledge fairly well. To incorporate these considerations into the analysis, we shall use the following double sequence of prior densities

[5] See, e.g., the second Haavelmo consumption model and the food example of Haavelmo and Girshick in Hood and Koopmans (1953) or the Klein I model in Klein (1950).

[6] See the Klein-Goldberger model in Klein and Goldberger (1956).

$$f_{k,\ell}(\Lambda, \Sigma \mid \Lambda_0, T_0, L_{k\ell}M_0L_{k\ell}) \propto$$

$$(2.3) \qquad |\Sigma|^{\frac{1}{2}(m+p+r+s+t)} \exp\{-\tfrac{1}{2}\mathrm{tr}\, \Sigma\, [\Lambda - \Lambda_0]L_{k\ell}M_0L_{k\ell}[\Lambda - \Lambda_0]'\}$$

$$|\Sigma|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\mathrm{tr}\, \Sigma\, T_0\} \qquad k, \ell = 1, 2, \ldots$$

with

$$L_{k\ell} = \begin{bmatrix} kI & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & kI & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\ell}I \end{bmatrix}$$

where the identity matrices are of orders $m \times m$, $p \times p$, $r \times r$, $s \times s$, and $t \times t$ respectively.[7] In the next subsection we investigate the consequences of this specification for the limiting behavior of the marginal prior densities.

2.2. Limiting Behavior of the Marginal Prior Distributions

We begin our investigations with the limiting behavior of the marginal prior distributions of the elements of Γ_{11}. Denoting the i-th row of Λ with λ_i', it can be proved[8] that the marginal prior distribution of λ_i' is Student with parameters λ_{0i}', $L_{k\ell}M_0L_{k\ell}/t_{0ii}$, and 1, where λ_{0i}' and t_{0ii} denote the i-th row of Λ_0 and the (i, i)-th element of T_0 respectively. If λ_{ij} (i, j = 1, ..., m) denotes the (i, j)-th element of Γ_{11}, it can be verified[9] that the marginal prior distribution of λ_{ij} is also Student with parameters λ_{0ij}, $k^2/(M_{0jj}^{11}t_{0ii})$ and 1, where λ_{0ij} and M_{0jj}^{11} denote the (i, j)-th element of Γ_{011} and the (j, j)-th element of M_0^{11} respectively.[10] Hence,

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[7] Note that we exclude the case of no information about the elements of Γ_{12}. This situation will be discussed briefly in Section 4.

[8] See Dickey (1967) or Harkema (1969), Section 4.

[9] See Raiffa and Schlaifer (1961), p. 259.

[10] As usual the M_0^{ij} (i, j = 1, ..., 5) denote the appropriate submatrices of M_0^{-1}.

(2.4) $$f_{k\ell}(\lambda_{ij} \mid \lambda_{0ij}, k^2/(M^{11}_{0jj}t_{0ii}), 1) \propto$$

$$[1 + \frac{k^2}{M^{11}_{0jj}t_{0ii}} (\lambda_{ij} - \lambda_{0ij})^2]^{-1}$$

As is well-known, the corresponding distribution function is given by

(2.5) $$F_{k\ell}(x) = \int_{-\infty}^{x} f_{k\ell}(\lambda_{ij} \mid \lambda_{0ij}, k^2/(M^{11}_{0jj}t_{0ii}), 1)\, d\lambda_{ij}$$

$$= \frac{1}{\pi} [\text{arc tan } \{\frac{k(x - \lambda_{0ij})}{\sqrt{M^{11}_{0jj}t_{0ii}}}\} + \frac{\pi}{2}]$$

On taking limits, we obtain

(2.6) $$\lim_{k,\ell\to\infty} F_{k\ell}(x) = 0 \qquad \text{if} \qquad x < \lambda_{0ij}$$

$$\lim_{k,\ell\to\infty} F_{k\ell}(x) = 1 \qquad \text{if} \qquad x > \lambda_{0ij}$$

Evidently, the marginal prior distributions of the elements of Γ_{11} converge in distribution to the corresponding medians. This implies that we can specify an exact identity matrix for Γ_{11} by simply postulating $\Gamma_{011} = I$. It goes without saying[11] that the elements of Δ_{12} converge in distribution to their medians in the same way.

If $\lambda_{i,m+j}$ ($i = 1, \ldots, m$; $j = 1, \ldots, p$) denotes the (i, j)-th element of Γ_{12}, it is easily verified that its marginal prior distribution is Student with parameters $\lambda_{0i,m+j}$, $(t_{0ii}M^{22}_{0jj})^{-1}$, and 1, where $\lambda_{0i,m+j}$ and M^{22}_{0jj} denote the (i, j)-th element of Γ_{012} and the (j,j)-th element of M^{22}_0, respectively. This means that the marginal prior distributions of the elements of Γ_{12} and, of course, also of those of Δ^{11}_{11} are independent of the values of k and ℓ.

Finally, we have to examine the limiting behavior of the marginal prior distributions of the elements of Δ_{13}. Let $\lambda_{i,z+j}$ ($i = 1, \ldots, m$; $j = 1, \ldots, t$; $z = m + p + r + s$) denote the (i, j)-th element of Δ_{13}; then its marginal prior distribution is Student with parameters $\lambda_{0i,z+j}$, $(\ell^2 M^{55}_{0jj}t_{0ii})^{-1}$, and 1, where $\lambda_{0i,z+j}$ and M^{55}_{0jj} denote the (i, j)-th element of Δ_{013} and the (j,j)-th element of M^{55}_0, respectively. Hence, the corresponding distribution function is given by

$$F_{k\ell}(x) = \frac{1}{\pi} [\text{arc tan } \{\frac{x - \lambda_{0i,z+j}}{\ell\sqrt{M^{55}_{0jj}t_{0ii}}}\} + \frac{\pi}{2}]$$

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11 Compare the definition of $L_{k\ell}$ in (2.3)

On taking limits, we obtain

(2.7) $$\lim_{k,\ell\to\infty} F_{k\ell}(x) = \tfrac{1}{2} \qquad \text{all } x$$

Obviously, when ℓ approaches infinity, the marginal prior distributions of the elements of Δ_{13} become increasingly uniform over the entire real axis.

Apparently, the sequence of prior densities introduced in Subsection 2.1 enables us to handle very precise and very vague prior ideas about structural parameters simultaneously.

2.3. The Sequence of Posterior Distributions

In order to obtain the reduced-form equations system, we begin by eliminating the identities from (2.1). As Γ_{22} is assumed to be non-singular,

(2.8) $$y_2 = -\Gamma_{22}^{-1}\Gamma_{21}y_1 - \Gamma_{22}^{-1}\Delta_{21}x_1 - \Gamma_{22}^{-1}\Delta_{22}x_2 - \Gamma_{22}^{-1}\Delta_{23}x_3$$

Substitution of (2.8) into (2.1) gives

(2.9) $$Ay_1 + B_1x_1 + B_2x_2 + B_3x_3 = u$$

where

(2.10) $$A = \Gamma_{11} - \Gamma_{12}\Gamma_{22}^{-1}\Gamma_{21}$$
$$B_i = \Delta_{1i} - \Gamma_{12}\Gamma_{22}^{-1}\Delta_{2i} \qquad (i = 1, 2, 3)$$

The reduced form corresponding with (2.9) is given by

(2.11) $$y_1 = C^{(1)}x_1 + C^{(2)}x_2 + C^{(3)}x_3 + v$$

where

$$[C^{(1)} \quad C^{(2)} \quad C^{(3)}] = -A^{-1}[B_1 \quad B_2 \quad B_3] \equiv C$$

$$v = A^{-1}u$$

which implies that v is normally distributed with zero mean and variance-covariance matrix $\Omega^{-1} = (A' \; \Sigma \; A)^{-1}$.

To determine the sequence of posterior densities on the space of the reduced-form parameters corresponding with the sequence of prior densities defined in (2.3), we make extensive use of the results arrived at in a previous paper and restate them here without further proof.[12] Using the same notation, we introduce the following definitions

$$K = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ \Gamma_{22}^{-1}\Gamma_{21} & I & \Gamma_{22}^{-1}\Delta_{21} & \Gamma_{22}^{-1}\Delta_{22} & \Gamma_{22}^{-1}\Delta_{23} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

(2.12) $$Q_{0k\ell} = KL_{k\ell}M_0L_{k\ell}K'$$

and

$$P_{0k\ell}^{-1} = \begin{bmatrix} Q_{0k\ell}^{11} & Q_{0k\ell}^{13} & Q_{0k\ell}^{14} & Q_{0k\ell}^{15} \\ Q_{0k\ell}^{31} & Q_{0k\ell}^{33} & Q_{0k\ell}^{34} & Q_{0k\ell}^{35} \\ Q_{0k\ell}^{41} & Q_{0k\ell}^{43} & Q_{0k\ell}^{44} & Q_{0k\ell}^{45} \\ Q_{0k\ell}^{51} & Q_{0k\ell}^{53} & Q_{0k\ell}^{54} & Q_{0k\ell}^{55} \end{bmatrix}$$

where the $Q_{0k\ell}^{ij}$ (i, j = 1, 3, 4, 5) denote the appropriate submatrices of $Q_{0k\ell}^{-1}$. Before we can proceed to the sequence of posterior densities of the reduced-form parameters, we still need a second set of definitions, namely,

[12] See Harkema (1969), Section 3.

$$A_0 = \Gamma_{011} - \Gamma_{012}\Gamma_{22}^{-1}\Gamma_{21}$$

(2.13)

$$B_{0i} = \Delta_{01i} - \Gamma_{012}\Gamma_{22}^{-1}\Delta_{2i} \qquad (i = 1, 2, 3)$$

$$D_0 = \begin{bmatrix} A_0 & B_{01} & B_{02} & B_{03} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

where the identity matrices are of orders r × r, s × s, and t × t respectively,

(2.14)

$$P^*_{0k\ell} = D_0 P_{0k\ell} D'_0$$

$$T^*_0 = \begin{bmatrix} T_0 & 0 \\ 0 & 0 \end{bmatrix}$$

where the lower right-hand zero matrix is of order (r + s + t) × (r + s + t)

(2.15)

$$G_{0k\ell} = P^*_{0k\ell} + T^*_0 = D_0 P_{0k\ell} D'_0 + T^*_0$$

$$R_{0k\ell} = [R_{01k\ell} \quad R_{02k\ell} \quad R_{03k\ell} \quad R_{04k\ell}]$$

$$= [I \quad 0 \quad 0 \quad 0] D_0^{-1} P^*_{0k\ell} G^{-1}_{0k\ell}$$

and

$$S_{0k\ell} = R_{0k\ell}[T^*_0 + T^*_0 P^{*\,-1}_{0k\ell} T^*_0] R'_{0k\ell}$$

Defining

(2.16)

$$G^{-1}_{0k\ell} = \begin{bmatrix} G^{11}_{0k\ell} & G^{12}_{0k\ell} \\ G^{21}_{0k\ell} & G^{22}_{0k\ell} \end{bmatrix}$$

where $G^{11}_{0k\ell}$ and $G^{22}_{0k\ell}$ are of orders $(m \times m)$ and $(r + s + t) \times (r + s + t)$ respectively, it can be proved that the sequence of prior densities on the space of the reduced-form coefficients is also of the matrix Normal-Wishart form with parameters $C_{0k\ell}$, $S_{0k\ell}$, $N_{0k\ell}$, and m degrees of freedom, where $S_{0k\ell}$ is defined in (2.15) and $C_{0k\ell}$ and $N_{0k\ell}$ are defined by

$$(2.17) \qquad C_{0k\ell} = [C^{(1)}_{0k\ell} \quad C^{(2)}_{0k\ell} \quad C^{(3)}_{0k\ell}] = [R_{02k\ell} \quad R_{03k\ell} \quad R_{04k\ell}]$$

$$N_{0k\ell} = \left[G^{22}_{0k\ell}\right]^{-1}$$

We then suppose that there is a sample of T observations on the endogenous variables y_1 and the exogenous variables x_1, x_2, x_3 available and define

$$Y = \begin{bmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & & \vdots \\ y_{T1} & \cdots & y_{Tm} \end{bmatrix}$$

$$(2.18) \qquad X = [X^{(1)} \quad X^{(2)} \quad X^{(3)}]$$

$$= \begin{bmatrix} x^{(1)}_{11} & \cdots & x^{(1)}_{1r} & x^{(2)}_{11} & \cdots & x^{(2)}_{1s} & x^{(3)}_{11} & \cdots & x^{(3)}_{1t} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x^{(1)}_{T1} & \cdots & x^{(1)}_{Tr} & x^{(2)}_{T1} & \cdots & x^{(2)}_{Ts} & x^{(3)}_{T1} & \cdots & x^{(3)}_{Tt} \end{bmatrix}$$

If we now let $C_1 = [C^{(1)}_1 \quad C^{(2)}_1 \quad C^{(3)}_1]$ denote any solution of the "normal equations"[13]

$$C_1 X'X = Y'X$$

and define

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13 It must be stressed that we do not require the matrix X to have full column rank. This is important in the case of large models where the number of observations is generally smaller than the number of exogenous variables.

$$N_1 = X'X$$

(2.19) $$p_1 = \text{rank }(X) = \text{rank }(N_1)$$

$$\lambda_1 = T - p_1$$

$$S_1 = [Y - XC_1']'[Y - XC_1']$$

it can be proved that the sequence of posterior densities on the space of the reduced-form coefficients is also of the matrix Normal-Wishart form with parameters $C_{2k\ell}$, $S_{2k\ell}$, $N_{2k\ell}$, and λ_2 degrees of freedom, where

$$N_{2k\ell} = N_{0k\ell} + N_1$$

(2.20) $$C_{2k\ell} = [C_{0k\ell}N_{0k\ell} + C_1N_1]N_{2k\ell}^{-1}$$

$$\lambda_2 = m + T$$

and

$$S_{2k\ell} = S_{0k\ell} + S_1 + C_{0k\ell}N_{0k\ell}C_{0k\ell}' + C_1N_1C_1'$$
$$- C_{2k\ell}N_{2k\ell}C_{2k\ell}'$$

Hence, the sequence of posterior densities of the reduced-form coefficients can be represented by

$$f_{k\ell}(C, \Omega \mid C_{2k\ell}, S_{2k\ell}, N_{2k\ell}, \lambda_2) =$$

(2.21) $$\frac{|\Omega|^{\frac{1}{2}(r+s+t)} \; |N_{2k\ell}|^{\frac{1}{2}m}}{(2\pi)^{\frac{1}{2}m(r+s+t)}} \exp\{-\tfrac{1}{2}\text{ tr }\Omega[C - C_{2k\ell}]N_{2k\ell}[C - C_{2k\ell}]'\}$$
$$\frac{|S_{2k\ell}|^{\frac{1}{2}\lambda_2} \; |\Omega|^{\frac{1}{2}(\lambda_2-m-1)} \quad \exp\{-\tfrac{1}{2}\text{ tr }\Omega S_{2k\ell}\}}{2^{\frac{1}{2}\lambda_2 m} \; \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma[\tfrac{1}{2}(\lambda_2 + 1 - i)]}$$

In the next section we determine the limiting distribution of the sequence of densities defined in (2.21).

3. THE LIMITING POSTERIOR DISTRIBUTION OF THE REDUCED-FORM PARAMETERS

In order to exhibit the limiting distribution of the sequence of densities (2.21), we use a theorem due to Scheffé[14] which says that the density of the limiting distribution of a sequence of random variables with densities $\{p_n(x)\}_{n=1}^{\infty}$ is equal to $\lim_{n\to\infty} p_n(x)$, if this limit is a proper density. In the next two subsections we shall focus our attention on the evaluation of $\lim_{k,\ell\to\infty} f_{k\ell}(C, \Omega \mid C_{2k\ell}, S_{2k\ell}, N_{2k\ell}, \lambda_2)$. In order to simplify the notation, we shall delete the indices k and ℓ from now on.

3.1. The Limiting Precision Matrix of the Reduced Form Coefficients

In this subsection, we are concerned with the determination of $\lim_{k,\ell\to\infty} N_2$ or, using (2.20), $\lim_{k,\ell\to\infty} (N_0 + N_1)$. On combining (2.16) and (2.17), we discover that N_0 can be rewritten as follows

$$N_0 = (E_A G_0^{-1} E_A')^{-1} = E_B E^{-1} E_B \tag{3.1}$$

with

$$E = E_B E_A G_0^{-1} E_A' E_B \tag{3.2}$$

and

$$E_A = \begin{bmatrix} 0 & I_{(r)} & 0 & 0 \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & I_{(t)} \end{bmatrix} \qquad E_B = \begin{bmatrix} I_{(r)} & 0 & 0 \\ 0 & I_{(s)} & 0 \\ 0 & 0 & \frac{1}{\ell} I_{(t)} \end{bmatrix} \tag{3.3}$$

where $I_{(j)}$ denotes the identity matrix of order $j \times j$. Evidently, then

$$\lim_{k,\ell\to\infty} N_0 = \lim_{k,\ell\to\infty} E_B \lim_{k,\ell\to\infty} E^{-1} \lim_{k,\ell\to\infty} E_B \tag{3.4}$$

provided that these limits exist. In order to evaluate G_0^{-1}, we notice that (2.14) and (2.15) lead to

[14] See Scheffé (1947).

(3.5)
$$G_0^{-1} = [P_0^* + T_0^*]^{-1} = [D_0P_0D_0' + T_0^*]^{-1}$$
$$= [D_0(P_0 + F_0)D_0']^{-1}$$
$$= (D_0')^{-1}(P_0 + F_0)^{-1}D_0^{-1}$$

where

$$F_0 = \begin{bmatrix} F_{011} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_0^{-1}T_0(A_0')^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

the submatrices on the diagonal being of orders $m \times m$ and $(r + s + t) \times (r + s + t)$ respectively.

Substituting (3.5) into (3.2) we can rewrite E as follows

(3.6)
$$E = E_BE_A(D_0')^{-1}(P_0 + F_0)^{-1}D_0^{-1}E_A'E_B$$
$$= E_BE_A(D_0')^{-1}E_C[E_C(P_0 + F_0)E_C]^{-1}E_CD_0^{-1}E_A'E_B$$

with

(3.7)
$$E_C = \begin{bmatrix} I_{(m)} & 0 & 0 & 0 \\ 0 & I_{(r)} & 0 & 0 \\ 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & \ell I_{(t)} \end{bmatrix}$$

In order to determine the limiting value of the matrix E, we start by evaluating the limit of $E_CD_0^{-1}E_A'E_B$. By inverting the matrix D_0 defined in (2.13) and using the definitions (3.3) and (3.7), we easily find

(3.8)
$$\lim_{k,\ell\to\infty} E_CD_0^{-1}E_A'E_B = \begin{bmatrix} -A_0^{-1}B_{01} & -A_0^{-1}B_{02} & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \equiv D^*$$

It should be noted that the matrix B_{03}, and hence the matrix of location parameters Δ_{013} (see 2.13) does not play any rôle in D*.

We then have to determine the limit of $[E_C(P_0 + F_0)E_C]^{-1}$. A convenient way of evaluating this limit is to rewrite this matrix as:

$$(3.9) \qquad [E_C(P_0 + F_0)E_C]^{-1} = [E_CP_0E_C + E_CF_0E_C]^{-1}$$

$$= [I + E_C^{-1}P_0^{-1}E_C^{-1}E_CF_0E_C]^{-1}\; E_C^{-1}P_0^{-1}E_C^{-1}$$

From (2.12) it can easily be seen that P_0^{-1} can be rewritten as follows

$$(3.10) \qquad P_0^{-1} = E_DQ_0^{-1}E_D' = E_D(K')^{-1}L^{-1}M_0^{-1}L^{-1}K^{-1}E_D'$$

with

$$E_D = \begin{bmatrix} I_{(m)} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{(r)} & 0 & 0 \\ 0 & 0 & 0 & I_{(s)} & 0 \\ 0 & 0 & 0 & 0 & I_{(t)} \end{bmatrix}$$

Hence,

$$(3.11) \qquad E_C^{-1}P_0^{-1}E_C^{-1} = E_C^{-1}E_D(K')^{-1}L^{-1}M_0^{-1}L^{-1}K^{-1}E_D'E_C^{-1}$$

Inverting the matrices L and K defined in (2.3) and (2.12) and using (3.10) and (3.7), we obtain

$$(3.12) \qquad \lim_{k,\ell\to\infty} L^{-1}K^{-1}E_D'E_C^{-1} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\Gamma_{22}^{-1}\Gamma_{21} & -\Gamma_{22}^{-1}\Delta_{21} & -\Gamma_{22}^{-1}\Delta_{22} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \equiv F^*$$

Clearly,

$$(3.13) \qquad \lim_{k,\ell\to\infty} E_C^{-1}P_0^{-1}E_C^{-1} = (F^*)'M_0^{-1}F^* \equiv W_0$$

It should be noted that the matrices $M_0^{1j} = (M_0^{j1})'$ $(j = 1, \ldots, 5)$ and $M_0^{4j} = (M_0^{j4})'$ $(j = 1, \ldots, 5)$ do not show up in the matrix W_0.

From the definitions of F_0 and E_C given in (3.5) and (3.7) respectively, we easily obtain

$$(3.14) \qquad \lim_{k,\ell\to\infty} E_C F_0 E_C = F_0$$

Combining (3.13) and (3.14), we find

$$(3.15) \qquad \lim_{k,\ell\to\infty} (I + E_C^{-1}P_0^{-1}E_C^{-1}E_C F_0 E_C) = I + W_0F_0$$

Using the definition of F_0 in (3.5), we observe that $(I + W_0F_0)$ can be partitioned as follows

$$(3.16) \qquad I + W_0F_0 = \begin{bmatrix} I + W_{011}F_{011} & 0 & 0 & 0 \\ W_{021}F_{011} & I_{(r)} & 0 & 0 \\ W_{031}F_{011} & 0 & I_{(s)} & 0 \\ W_{041}F_{011} & 0 & 0 & I_{(t)} \end{bmatrix}$$

the leading submatrix being of order $m \times m$. In order to prove that $(I + W_{011}F_{011})$ is non-singular, we proceed as follows. As F_{011} is positive definite (see 3.5), $I + W_{011}F_{011} = (F_{011}^{-1} + W_{011})F_{011}$. From (3.12) and (3.13) we find $W_{011} = \Gamma_{21}'(\Gamma_{22}')^{-1}M_0^{22}\Gamma_{22}^{-1}\,\Gamma_{21}$. As M_0^{22} is positive definite, W_{011} is at least positive semi-definite. Hence, $F_{011}^{-1} + W_{011}$ is equal to the sum of a positive definite and a positive semi-definite matrix. This implies that $I + W_{011}F_{011}$ is equal to the product of two positive definite and hence non-singular matrices. Apparently, $I + W_{011}F_{011}$ and hence $I + W_0F_0$ is non-singular. From (3.15) we now easily find

$$(3.17) \qquad \lim_{k,\ell\to\infty} (I + E_C^{-1}P_0^{-1}E_C^{-1}E_C F_0 E_C)^{-1} = (I + W_0F_0)^{-1}$$

Combining (3.9), (3.13), and (3.17) we obtain

(3.18) $$\lim_{k,\ell\to\infty} [E_C(P_0 + F_0)E_C]^{-1} = (I + W_0F_0)^{-1}W_0 \equiv L^*$$

Taking (3.6), (3.8), and (3.18) together we find

(3.19) $$\lim_{k,\ell\to\infty} E = (D^*)'L^*D^* \equiv E^*$$

In order to discover the conditions under which the matrix E^* is non-singular, we observe, using (3.18) and (3.13), that

$$\text{rank } (L^*) = \text{rank } (W_0) = \text{rank } (F^*)$$

From the definition of F^* given in (3.12), we find rank $(F^*) = (q + r + t)$ where

(3.20) $$q = \text{rank } [\Gamma_{21} \quad \Delta_{22}] \leq p$$

As L^* is the limit of a sequence of positive definite symmetric matrices,[15] we conclude that L^* is a positive semi-definite matrix of order $(m + r + s + t)$ and rank $(q + r + t)$. Observing that the rank of the matrix D^* which has been defined in (3.8) is equal to $(r + s + t)$, it appears that a necessary condition for the matrix E^* to be positive definite is that $q \geq s$. Generally, the only exogenous variables of which the influences are precisely known in advance are those which show up in the identities only.[16] This implies that the rank of Δ_{22} and hence q at least will be equal to the number of exogenous variables whose influences are precisely known in advance. Therefore, the condition will almost always be met. A sufficient condition for the matrix E^* to be positive definite can be established by requiring that the columns of D^* do not lie within the nullspace of the matrix L^*. Henceforward we shall assume this requirement has been met, implying that the matrix E^* as well as $(E^*)^{-1}$ is positive definite.

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[15] See (3.18) and notice that P_0 is positive definite (see 2.12) and F_0 positive semi-definite (see 3.4).

[16] An example of such an exogenous variable is given by the variable G (goods demanded by the government and foreigners) in the Klein I model; see Klein (1950), pp. 62-66.

Finally, taking (3.1), (3.3) and (3.19) together, we obtain

(3.21)
$$\lim_{k,\ell\to\infty} N_0 = E_F(E^*)^{-1}E_F \equiv N_0^*$$

where

$$E_F = \lim_{k,\ell\to\infty} E_B = \begin{bmatrix} I_{(r)} & 0 & 0 \\ 0 & I_{(s)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Substituting (3.21) into (2.20) we find

(3.22)
$$\lim_{k,\ell\to\infty} N_2 = N_0^* + N_1 \equiv N_2^*$$

and, thus

$$\lim_{k,\ell\to\infty} |N_2| = |N_2^*|$$

In concluding this subsection we want to stress three points. Firstly, it should be noted that the matrices N_0^* and N_1 are not required to have full rank. The only restriction we have to impose to obtain a full rank posterior precision matrix N_2^* is that the nullspaces of N_0^* and N_1 be disjoint. Secondly, we want to draw attention to the fact that the matrices N_0^* and N_2^* are independent of the matrix of location parameters Δ_{013} and the matrices $M_0^{ij} = (M_0^{ji})'$ (i = 1, 4; j = 1, ..., 5) as is easily verified from the comments following (3.8) and (3.13). Finally, it can be proved[17] that N_0^* and N_2^* do not depend on the absolute size of the values inserted into the $M_0^{i5} \equiv (M_0^{5i})'$ (i = 2, 3, 5) but only on the value of the "correlations"

(3.23)
$$\rho_{ab}^{i} = \frac{M_{0ab}^{i5}}{\sqrt{M_{0aa}^{ii} M_{0bb}^{55}}} \qquad (i = 2, 3, 5)$$

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[17] See Appendix A.

3.2. Limiting Values of the Other Parameters of the Posterior Distribution of the Reduced Form Coefficients

After evaluating the limit of N_0, the determination of the limits of C_2 and S_2 becomes a rather simple affair. Combining (2.15) and (2.17) we obtain

$$R_0 = [R_{01} \quad C_0] = E_G D_0^{-1} P_0^* G_0^{-1} \tag{3.24}$$

where

$$E_G = [I_{(m)} \quad 0 \quad 0 \quad 0]$$

the zero matrices being of orders $m \times r$, $m \times s$, and $m \times t$, respectively. Substituting (2.14) and (2.15) into (3.24) we find

$$\begin{aligned} R_0 &= E_G D_0^{-1} P_0^* [P_0^* + T_0^*]^{-1} \\ &= E_G D_0^{-1} D_0 P_0 D_0' [D_0 P_0 D_0' + T_0^*]^{-1} \\ &= E_G P_0 [P_0 + F_0]^{-1} D_0^{-1} \\ &= E_G [I + F_0 P_0^{-1}]^{-1} D_0^{-1} \end{aligned} \tag{3.25}$$

where F_0 has been defined in (3.5). Next, we rewrite R_0 as follows

$$\begin{aligned} R_0 &= E_G E_C^{-1} [I + E_C F_0 E_C E_C^{-1} P_0^{-1} E_C^{-1}]^{-1} \; E_C D_0^{-1} \\ &= E_G [I + E_C F_0 E_C E_C^{-1} P_0^{-1} E_C^{-1}]^{-1} E_C D_0^{-1} \end{aligned} \tag{3.26}$$

where E_C has been defined in (3.7). From (3.24) we easily obtain

$$C_0 N_0 = E_G [I + E_C F_0 E_C E_C^{-1} P_0^{-1} E_C^{-1}]^{-1} \; E_C D_0^{-1} E_A' N_0 \tag{3.27}$$

where E_A has been defined in (3.3). In order to determine $\lim_{k,\ell\to\infty} C_0 N_0$, we notice that substitution of (3.1) into $E_C D_0^{-1} E_A' N_0$ leads to

(3.28) $$E_C D_0^{-1} E_A' N_0 = E_C D_0^{-1} E_A' E_B E^{-1} E_B$$

From (3.8), (3.19), and (3.21) we then obtain

(3.29) $$\lim_{k,\ell\to\infty} E_C D_0^{-1} E_A' N_0 = D^*(E^*)^{-1} E_F$$

Combining (3.27), (3.17), and (3.29) we find

(3.30) $$\lim_{k,\ell\to\infty} C_0 N_0 = E_G[I + F_0 W_0]^{-1} D^*(E^*)^{-1} E_F \equiv Z^*$$

and thus from (2.20) and (3.22)

(3.31) $$\lim_{k,\ell\to\infty} C_2 = [Z^* + C_1 N_1](N_2^*)^{-1} \equiv C_2^*$$

Again it should be noted that Z^* does not depend on Δ_{013} and the $M_0^{ij} = (M_0^{ji})'$ (i = 1, 4; j = 1, ..., 5); moreover it can be proved[18] that Z^* only depends on the prior "correlations" defined in (3.23) and not on the absolute size of the values inserted into the $M_0^{i5} = (M_0^{5i})'$ (i = 2, 3, 5). Finally, it is interesting to observe that the last t columns of Z^* are zero columns which follows from the fact that the last t columns of E_F are zero columns.

We still have to evaluate the limit of S_2 defined in (2.20). In order to determine this limit, we first turn to S_0 and observe that, after substituting (2.13) and (2.14) into (2.15), this matrix can be written as follows

(3.32) $$S_0 = R_{01}[T_0 + T_0(A_0')^{-1} P_0^{11} A_0^{-1} T_0] R_{01}'$$

From (3.26), (3.17), (3.7), and (2.13), we obtain

(3.33) $$\lim_{k,\ell\to\infty} R_{01} = (I + F_{011} W_{011})^{-1} A_0^{-1}$$

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[18] See Appendix B.

F_{011} and W_{011} denoting the leading (m × m) submatrices of F_0 and W_0 respectively. Moreover, it is clear from (3.13) and the definition of E_C in (3.7) that

$$\lim_{k,\ell\to\infty} P_0^{11} = W_{011} \tag{3.34}$$

Combining (3.32), (3.33), and (3.34) we obtain

$$\lim_{k,\ell\to\infty} S_0 =$$

$$= (I + F_{011}W_{011})^{-1}A_0^{-1}[T_0 + T_0(A_0')^{-1}W_{011}A_0^{-1}T_0](A_0')^{-1}(I + W_{011}F_{011})^{-1}$$

By substituting $F_{011} = A_0^{-1}T_0(A_0')^{-1}$ (see 3.5) this expression can be simplified to

$$\lim_{k,\ell\to\infty} S_0 = [A_0'T_0^{-1}A_0 + W_{011}]^{-1} \equiv S_0^* \tag{3.35}$$

As is easily verified, we can repeat the comments accompanying (3.22) with respect to N_0^* also with respect to S_0^*.

Next, we observe that, after substituting (3.1), $C_0N_0C_0'$ can be rewritten as follows

$$C_0N_0C_0' = C_0E_BE^{-1}E_BC_0' \tag{3.36}$$

From (3.24) and (3.26) we find

$$C_0E_B = E_G[I + E_CF_0E_CE_C^{-1}P_0^{-1}E_C^{-1}]^{-1}E_CD_0^{-1}E_A'E_B \tag{3.37}$$

Combining (3.8) and (3.17) we then obtain

$$\lim_{k,\ell\to\infty} C_0E_B = E_G[I + F_0W_0]^{-1}D^* \equiv C_0^* \tag{3.38}$$

and hence, using (3.19),

$$\lim_{k,\ell\to\infty} C_0N_0C_0' = C_0^*(E^*)^{-1}(C_0^*)' \tag{3.39}$$

Once again we can repeat our comments accompanying (3.22) with respect to N_0^* for $\lim_{k,\ell\to\infty} C_0N_0C_0'$ (see Appendix C). Upon substituting (3.35), (3.39), (3.31), and (3.22) into (2.20), we obtain

(3.40) $$\lim_{k,\ell\to\infty} S_2 = S_0^* + S_1 + C_0^*(E^*)^{-1}(C_0^*)' + C_1N_1C_1' - C_2^*N_2^*(C_2^*)' \equiv S_2^*$$

In order to prove that S_2^* is positive definite we consider the following sequence of matrices

(3.41) $$\Theta_{k\ell} = [C_{2k\ell} - C_{0k\ell}]N_{0k\ell}[C_{2k\ell} - C_{0k\ell}]' + [C_{2k\ell} - C_1]N_1[C_{2k\ell} - C_1]' + S_1$$

Clearly, $\Theta_{k\ell}$ is at least positive semi-definite for all k and ℓ. Next, we rewrite $\Theta_{k\ell}$ as follows

(3.42) $$\Theta_{k\ell} = C_{2k\ell}[N_{0k\ell} + N_1]C_{2k\ell}' + C_{0k\ell}N_{0k\ell}C_{0k\ell}' + C_1N_1C_1' - [C_{0k\ell}N_{0k\ell} + C_1N_1]C_{2k\ell}' - C_{2k\ell}[N_{0k\ell}C_{0k\ell}' + N_1C_1'] + S_1$$

Substituting $N_{2k\ell} = N_{0k\ell} + N_1$ and $C_{2k\ell}N_{2k\ell} = C_{0k\ell}N_{0k\ell} + C_1N_1$ into (3.42) ,we obtain

(3.43) $$\Theta_{k\ell} = C_{0k\ell}N_{0k\ell}C_{0k\ell}' + C_1N_1C_1' - C_{2k\ell}N_{2k\ell}C_{2k\ell}' + S_1$$

As all the terms of the sequence are at least positive semi-definite, the limit is at least positive semi-definite also. Hence,

(3.44) $$\Theta^* \equiv \lim_{k,\ell\to\infty} \Theta_{k\ell} = C_0^*(E^*)^{-1}(C_0^*)' + C_1N_1C_1' - C_2^*N_2^*(C_2^*)' + S_1$$

is at least positive semi-definite. Observing that S_0^* is positive definite,[19] we conclude that S_2^* must be positive definite.

From (3.22), (3.31), and (3.40) we easily find that the limit of the sequence of posterior densities defined in (2.21) is given by

$$(3.45) \qquad f^*(C, \Omega \mid C_2^*, S_2^*, N_2^*, \lambda_2) =$$

$$\frac{|\Omega|^{\frac{1}{2}(r+s+t)} |N_2^*|^{\frac{1}{2}m}}{(2\pi)^{\frac{1}{2}m(r+s+t)}} \exp\{-\tfrac{1}{2} \operatorname{tr} \Omega [C - C_2^*] N_2^* [C - C_2^*]'\}$$

$$\frac{|S_2^*|^{\frac{1}{2}\lambda_2} |\Omega|^{\frac{1}{2}(\lambda_2 - m - 1)} \exp\{-\tfrac{1}{2} \operatorname{tr} \Omega S_2^*\}}{2^{\frac{1}{2}\lambda_2 m} \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma[\tfrac{1}{2}(\lambda_2 + 1 - i)]}$$

As this limit is a proper Normal-Wishart density, we conclude, using Scheffé's theorem,[20] that the density of the limiting posterior distribution of the reduced-form coefficients is given by (3.45).

4. CONCLUSION

From the viewpoint of specification of information it may be interesting to summarize which parameters of the prior distribution of the structural parameters show up in the limiting posterior distribution of the reduced-form coefficients. Turning to the location parameters first, the comments under (3.23), (3.31), (3.35), and (3.39) clearly show that all location parameters of the prior distribution appear in the limiting posterior distribution except for Δ_{013}. However, as Δ_{013} relates to parameters about which no prior information is available, this result is in accordance with

[19] Compare (3.35) and note that $A_0' T_0^{-1} A_0$ is positive definite and W_{011} positive semi-definite.

[20] See Scheffé (1947).

our expectations. By tracing the rôle which the scale parameters of the prior distribution play in the limiting posterior distribution, we find from the same comments that only the M_0^{ij} (i, j = 2, 3, 5) show up. As the $M_0^{ij} = (M_0^{ji})'$ (i = 1, 4; j = 2, 3, 5) have to do with "correlations" between prior ideas about parameters which are completely known in advance and parameters about which only probability statements can be made, it seems only natural to postulate the $M_0^{ij} = (M_0^{ji})'$ (i = 1, 4; j = 2, 3, 5) to be zero. This implies that the M_0^{ij} (i, j = 2, 3, 5) are solely dependent on the M_{0ij} (i, j = 2, 3, 5). However, the M_{0ij} (i, j = 2, 3, 5) constitute the precision matrix of the conditional prior distribution of the structural parameters given that $\Gamma_{11} = \Gamma_{011}$ and $\Delta_{12} = \Delta_{012}$.[21] But this is precisely the situation to which our prior information relates. Hence, we only need to specify our information with respect to Γ_{12}, Δ_{12}, and Δ_{13} and calculate from these specifications the values to be inserted into the M_{0ij} (i, j = 2, 3, 5). As regards the parameters Δ_{13} about which we should like to be non-informative, it is proved in the Appendices to this paper that the limiting posterior distribution does not depend on the absolute size of the values inserted into the M_{0i5} (i = 2, 3, 5) but only on the value of the "correlations" defined in (3.23). Indeed, in many cases such as the case of the constant term in a consumption function, our information is so vague that we are not able or willing to express that information by a probability statement although it is clear that a strong negative correlation must exist between the value of the constant term and the value of the marginal propensity to consume.

A second point we want to stress is that the sample precision matrix N_1 has not been required to have full rank. The only restriction we have to impose in order to obtain a proper limiting posterior distribution is that the nullspaces of the limiting prior precision matrix N_0^* and the sample precision matrix N_1 are disjoint. This is important for large models where, generally, the number of exogenous variables is larger than the number of observations, implying that N_1 is a matrix of order $(r + s + t) \times (r + s + t)$ while rank $(N_1) = T < r + s + t$.

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[21] See Dickey (1967) or Harkema (1969).

Finally, we have to consider the case of no information about the elements of Γ_{12} which has been excluded in Subsection 2.1. Unfortunately, it seems impossible to treat this case by means of the present analysis because of the fact that the behavior of some sequences of matrices such as $E_C^{-1}P_0^{-1}E_C^{-1}$ in (3.13) is not clear in this case. A rigorous analysis of the technical problems involved in deriving the limiting posterior distribution under these conditions, however, is outside the scope of this paper.

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APPENDIX A

In order to prove that N_0^* does not depend on the absolute size of the values inserted into the $M_0^{i5} = (M_0^{5i})'$ (i = 2, 3, 5) but only on the "correlations" defined in (3.23), we start by defining the following matrices

$$H_A = \begin{bmatrix} I_{(m+p+r+s)} & 0 \\ 0 & H_0 \end{bmatrix} \qquad H_B = \begin{bmatrix} I_{(m+r+s)} & 0 \\ 0 & H_0 \end{bmatrix} \tag{A.1}$$

$$H_C = \begin{bmatrix} I_{(r+s)} & 0 \\ 0 & H_0 \end{bmatrix}$$

$I_{(j)}$ denoting the identity matrix of order $j \times j$ and H_0 denoting an arbitrary nonsingular matrix of order $t \times t$. From (3.5), (3.8), (3.12), and (3.21), we then easily obtain the following equalities

$$H_B F_0 H_B' = F_0 \tag{A.2}$$

$$H_B D^* = D^* H_C \tag{A.3}$$

$$H_A F^* = F^* H_B \tag{A.4}$$

$$E_F H_C^{-1} = E_F = (H_C')^{-1} E_F \tag{A.5}$$

Let us now consider the following matrix

$$\hat{M}^{-1} = H_A' M_0^{-1} H_A \tag{A.6}$$

Analogous to (3.13), we then obtain

$$\hat{W}_0 = (F^*)' \hat{M}^{-1} F^* = (F^*)' H_A' M_0^{-1} H_A F^* \tag{A.7}$$

or, after substituting (A.4) into (A.7),

(A.8) $$\hat{W}_0 = H_B'(F^*)'M_0^{-1}F^*H_B = H_B'W_0H_B$$

Next, we define, as in (3.18),

(A.9) $$\begin{aligned}\hat{L}^* &= (I + \hat{W}_0F_0)^{-1}\hat{W}_0 \\ &= (I + H_B'W_0H_BF_0)^{-1}H_B'W_0H_B \\ &= H_B'(I + W_0H_BF_0H_B')^{-1}W_0H_B\end{aligned}$$

or, after substituting (A.2) into (A.9),

(A.10) $$\hat{L}^* = H_B'L^*H_B$$

By defining the analogue of (3.19) and substituting (A.3), we obtain

(A.11) $$\begin{aligned}\hat{E}^* &= (D^*)'\hat{L}^*D^* = (D^*)'H_B'L^*H_BD^* \\ &= H_C'(D^*)'L^*D^*H_C = H_C'E^*H_C\end{aligned}$$

Hence, substituting (A.5) into the analogue of (3.21), we find

$$\begin{aligned}\hat{N}_0^* &= E_F(\hat{E}^*)^{-1}E_F = E_FH_C^{-1}(E^*)^{-1}(H_C')^{-1}E_F \\ &= E_F(E^*)^{-1}E_F = N_0^*\end{aligned}$$

Clearly, the matrix N_0^* is insensitive to transformations of the type (A.6). Hence, if we take the matrix H_0 to be diagonal with diagonal elements $H_{0ii} = (1/\sqrt{M_{0ii}^{55}})$ $(i = 1, \ldots, t)$, our statement at the beginning of this Appendix easily follows.

APPENDIX B

In this Appendix we prove that Z^*, defined in (3.30), also depends only on the "correlations" defined in (3.23) and not on the absolute size of the values inserted into the $M_0^{i5} = (M_0^{5i})'$ (i = 2, 3, 5). From (3.5), (A.1), (3.24), and (3.8), we easily obtain the following equalities

(B.1) $$F_0 H_B' = F_0 = H_B F_0$$

(B.2) $$E_G H_B^{-1} = E_G$$

(B.3) $$H_B D^* H_C^{-1} = D^*$$

Analogous to (3.30) we now consider the following matrix

(B.4) $$\hat{Z}^* = E_G[I + F_0\hat{W}_0]^{-1} D^*(\hat{E}^*)^{-1} E_F$$

Substituting (A.11) and (A.5), we obtain

(B.5) $$D^*(\hat{E}^*)^{-1}E_F = D^* H_C^{-1}(E^*)^{-1}(H_C')^{-1}E_F$$

$$= D^* H_C^{-1}(E^*)^{-1}E_F$$

In the same way we find, after substituting (A.8) and (B.1),

(B.6) $$[I + F_0\hat{W}_0]^{-1} = [I + F_0 H_B' W_0 H_B]^{-1}$$

$$= [I + F_0 W_0 H_B]^{-1}$$

$$= [H_B^{-1}(I + H_B F_0 W_0) H_B]^{-1}$$

$$= H_B^{-1}(I + F_0 W_0)^{-1} H_B$$

Hence, by combining (B.4), (B.5),and (B.6) and substituting (B.2) and (B.3), we find

$$\text{(B.7)} \qquad \hat{Z}^* = E_G H_B^{-1}(I + F_0 W_0)^{-1} H_B D^* H_C^{-1}(E^*)^{-1} E_F$$

$$= E_G(I + F_0 W_0)^{-1} D^*(E^*)^{-1} E_F = Z^*$$

which proves the introductory statement in this Appendix.

APPENDIX C

In order to prove that $\lim_{k,\ell\to\infty} C_0 N_0 C_0'$ depends on the "correlations" in (3.23) only and not on the absolute size of the values inserted into the $M_0^{i5} = (M_0^{5i})'$ $(i = 2, 3, 5)$, we start by considering the analogue of (3.38)

(C.1) $$\hat{C}_0^* = E_G[I + F_0\hat{W}_0]^{-1}D^*$$

By substituting (B.6), (B.2), and (A.3) into (C.1), it appears that $\hat{C}_0^*$ can be rewritten as follows

(C.2) $$\hat{C}_0^* = E_G H_B^{-1}(I + F_0 W_0)^{-1} H_B D^*$$
$$= E_G(I + F_0 W_0)^{-1} D^* H_C = C_0^* H_C$$

Hence, combining (A.11) and (C.2), we obtain for the analogue of (3.39)

(C.3) $$\hat{C}_0^*(\hat{E}^*)^{-1}(\hat{C}_0^*)' =$$
$$= C_0^* H_C H_C^{-1}(E^*)^{-1}(H_C')^{-1} H_C'(C_0^*)'$$
$$= C_0^*(E^*)^{-1}(C_0^*)'$$

which proves the first statement of this Appendix.