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NOTE ON CONSISTENT ESTIMATION OF THE VARIANCE OF
THE DISTURBANCES IN THE LINEAR MODEL

by T. Kloek

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Preliminary and Confidential

NOTE ON CONSISTENT ESTIMATION OF THE VARIANCE OF
THE DISTURBANCES IN THE LINEAR MODEL

by T. Kloek¹

We consider the linear model $y = X\beta + \epsilon$ with the classical assumptions:
(i) X is an $n \times r$ matrix of rank r with nonstochastic known real elements;
(ii) β is an unknown real r -vector; (iii) ϵ is an n -vector whose elements are independent, identically distributed real-valued random variables with zero mean and unknown variance σ^2 ; (iv) y is an observed real n -vector which has been generated by $y = X\beta + \epsilon$. The problem is to estimate β and σ^2 , given y and X . The least-squares estimator of β is $b = (X'X)^{-1}X'y$. Well-known estimators of σ^2 are

$$(1) \quad s^2 = (y - Xb)'(y - Xb)/n$$

and

$$(2) \quad \hat{s}^2 = ns^2/(n - r)$$

In this note we shall prove that neither normality of ϵ has to be assumed, nor additional assumptions on the elements of the matrix X have to be made² in order to prove that both s^2 and \hat{s}^2 converge in probability to σ^2 . In addition we shall show that under a rather weak condition the residual variance \bar{s}^2 obtained from the application of least squares to an incorrect model either tends in probability to a number greater than σ^2 or is divergent.

Since it is obvious that s^2 and \hat{s}^2 converge to the same probability limit if they converge at all, we shall concentrate our attention on the probability limit of s^2 when the number of observations n tends to infinity.

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² Proofs which make use either of a normality assumption on ϵ or of certain restrictions on the X -matrix can be found in several textbooks.

We observe that, if X has rank r for some $n = n_0$, it has rank r for all $n > n_0$, so that for any $n \geq n_0$ there exists a real nonsingular $r \times r$ matrix P such that

$$(3) \quad PP' = (X'X)^{-1}$$

or, equivalently,

$$(4) \quad P'X'XP = I$$

With respect to the disturbance vector ϵ , we assume that Assumption (iii) above can be extended for the infinite sequence of random variables $\{\epsilon_n\}_{n=1}^{\infty}$; that is, we assume that $\epsilon_1, \epsilon_2, \dots$ are independent, identically distributed real-valued random variables with zero mean and variance σ^2 . This implies that $\epsilon_1^2, \epsilon_2^2, \dots$ are independent, identically distributed random variables with mean σ^2 . It then follows from the law of large numbers that

$$(5) \quad \text{plim}_{n \rightarrow \infty} \frac{\epsilon' \epsilon}{n} = \sigma^2$$

Proofs can be found in Feller (1957), Section X.2 or Loève (1955), Section 20.1.

We now return to s^2 . It is well known that

$$(6) \quad s^2 = \frac{1}{n} \epsilon' \epsilon - \frac{1}{n} \epsilon' X (X'X)^{-1} X' \epsilon$$

Since we have already considered the first term of this expression in (5), we now turn to the second term. It can be written as $-\eta' \eta$ where η is a random r -vector defined by

$$(7) \quad \eta = P'X' \epsilon \cdot n^{-\frac{1}{2}}$$

compare (3). It is easily seen that $E\eta = 0$ and that

$$(8) \quad E(\eta\eta') = \frac{\sigma^2}{n} P'X'XP = \frac{1}{n} \sigma^2 I$$

Thus, the sequence of η vectors converges in the squared mean to zero, which implies $\text{plim}_{n \rightarrow \infty} \eta = 0$, $\text{plim}_{n \rightarrow \infty} \eta' \eta = 0$, and³

$$(9) \quad \text{plim}_{n \rightarrow \infty} s^2 = \sigma^2$$

Next we consider the consequences of a specification error with respect to X . Suppose that one has replaced the true X -matrix by a nonstochastic $n \times p$ matrix Z with rank p and that one has computed $\bar{b} = (Z'Z)^{-1}Z'y$ and

$$(10) \quad \bar{s}^2 = \frac{1}{n}(y - Z\bar{b})'(y - Z\bar{b})$$

while, in fact, y has been generated by $y = X\beta + \epsilon$. We assume that there does not exist a p -vector α such that $Z\alpha = X\beta$; otherwise the specification with $Z\alpha$ would be a correct alternative to that with $X\beta$. Upon defining

$$(11) \quad M = I - Z(Z'Z)^{-1}Z'$$

we can write

$$(12) \quad \bar{s}^2 = \frac{1}{n}y'My = \frac{1}{n}\beta'X'MX\beta + \frac{2}{n}\epsilon'MX\beta + \frac{1}{n}\epsilon'M\epsilon$$

or

$$(13) \quad \bar{s}^2 = \frac{1}{n}u'u + \frac{2}{n}\epsilon'u + \frac{1}{n}\epsilon'M\epsilon = v + \frac{1}{n}\epsilon'M\epsilon$$

where u and v are defined by

$$(14) \quad u = MX\beta \quad v = \frac{1}{n}u'u + \frac{2}{n}\epsilon'u$$

³ Note that $n^{\frac{1}{2}}\eta$ need not be asymptotically normally distributed, since under the assumptions made the elements of $P'X'$ are not necessarily uniformly asymptotically negligible; compare Loève (1955). A counter-example is the case $X' = [I \ 0]$, where the identity matrix is of order $r \times r$ and the zero matrix is of order $r \times (n - r)$.

Note that u can be interpreted as the vector of least-squares residuals which results when $X\beta$ is "explained" by the columns of Z . So, our assumption $Z\alpha \neq X\beta$ for all α implies that $u \neq 0$. Now we introduce the additional assumption that, when the sample size increases, from some n_0 onward the mean square of the elements of u is bounded below by a certain positive number g . More precisely, we assume that g exists such that

$$(15) \quad \frac{u'u}{n} > g > 0 \quad (n \geq n_0)$$

The random variable v , defined in (14) has mean $u'u/n$ and variance $\frac{1}{n} u'u/n^2$. Its standard deviation is $2\sigma(u'u)^{1/2}/n$. From Chebyshev's inequality we have for any positive k

$$(16) \quad P[v > \frac{u'u}{n} - k \frac{2\sigma(u'u)^{1/2}}{n}] > 1 - \frac{1}{k^2}$$

or

$$(17) \quad P[v > \frac{u'u}{n} \{1 - \frac{2\sigma k}{(u'u)^{1/2}}\}] > 1 - \frac{1}{k^2}$$

It follows from (15) that $u'u \rightarrow \infty$ as $n \rightarrow \infty$, so that the factor in braces tends to unity. Hence, for any given $\delta > 0$

$$(18) \quad \lim_{n \rightarrow \infty} P[v \geq g - \delta] = 1$$

The last term of (13) $\varepsilon'M\varepsilon/n$ converges in probability to σ^2 [the proof is analogous to that of (9)] so that for any given $\delta > 0$

$$(19) \quad \lim_{n \rightarrow \infty} P[\bar{s}^2 \geq g + \sigma^2 - \delta] = 1$$

This result includes both the case that $\text{plim } \bar{s}^2 > \sigma^2$ and the case that \bar{s}^2 does not converge at all. We conclude that, as the sample size increases, we may have more confidence that the model with the greater s^2 (that is with the smaller squared correlation coefficient R^2) was incorrectly specified. This

⁴ It follows from (13) and (14) that $n\bar{s}^2/(n-p)$ has expectation $\sigma^2 + u'u/(n-p) > \sigma^2$. This result has already been given in Theil (1958), Section 6.2.4.

positive result on R^2 for large samples contrasts with recent negative results for small samples found by Koerts and Abrahamse (1969).

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