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SOME FURTHER PROPERTIES OF THE LIVIATAN'S CONSISTENT ESTIMATOR IN A DISTRIBUTED LAG MODEL

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SOME FURTHER PROPERTIES OF THE LIVIATAN'S CONSISTENT ESTIMATOR IN A DISTRIBUTED LAG MODEL

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1. INTRODUCTION AND SUMMARY

Given a finite number T of time series observations on each of the (economic) variables x and y, one may estimate the well-known Koyck's [2] distributed lag model

$$(1.1)\quad y_t = \alpha x_t + \alpha\beta x_{t-1} + \alpha\beta^2 x_{t-2} + \dots + u_t\ , \quad 0 \leq \beta < 1$$

where x_t and y_t denote the observations on the "independent" and "dependent" variables x and y, respectively, at time t ($t = 1, \dots, T$); by using the liviatan's [3] instrumental variable technique or by any other suitable estimation procedure. Liviatan's method is independent of the autocorrelation properties of the u-disturbances and yields consistent estimates of the parameters α and β i.e. with negligible bias in large samples.

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[1] This paper was written when the author was working as a Research Associate at the Econometric Institute, Netherlands. I am thankful to Professor W.H. Somermeyer for the help in the preparation of this note. Part of this research work was completed at the time of author's registration for his doctoral degree under the supervision of Professor A.L. Nagar at the Delhi School of Economics, Delhi. I am greatful to him for the valuable guidance. However, for the omissions, if any, the author alone is responsible.

However, the results of large sample theory may not hold good if the sample size is not sufficiently large which is the usual situation encountered in econometric applications. Our interest, therefore, lies in examining the behaviour of the estimator in small samples. In an earlier paper Nagar and Gupta [4] analyzed the bias, to order $1/T$ (T being the sample size), of the Liviatan estimator under fairly general conditions. Later, the moment matrix, to order $1/T^2$, of the same estimator was derived under the assumption of inter-temporal independence of the disturbances.[2] The present note is a sequel to the above-mentioned articles [4] and [1]. It deals with the analysis of Karl Pearson's measure of skewness $\sqrt{\beta_1}$, to order $1/T$, of the probability distribution of the Liviatan's consistent estimator. The bias, to order $1/T$, of the residual variance estimator is also derived according to Liviatan's method of estimation.

The next section of the article describes the estimation procedure along with the underlying assumptions. The results on the skewness and the bias of the residual variance estimator are enunciated in Section 3. Finally, Sections 4 and 5 are devoted to the derivation of the results.

2. THE ESTIMATION PROCEDURE[3]

Adopting the notation of the article [4], the model in (1.1) can be written as (cf. [4], Section 2)

$$(2.1)\quad y_t = \alpha x_t + \beta y_{t-1} + w_t\,, \qquad w_t = u_t - \beta u_{t-1}$$

or, in matrix notation

$$(2.2)\quad y_{**} = Z\delta + w_{**}\,, \qquad w_{**} = u_{**} - \beta u_{*}$$

where $Z = (x_{**} \;\; y_{*})$ is a matrix of size (T-1) x 2, x_{**}, y_{**} and u_{**} are column vectors with components x_t, y_t and u_t ($t = 2, \ldots, T$), respectively; x_{*}, y_{*} and u_{*} are column vectors with components x_t, y_t and u_t ($t = 1, \ldots T-1$), respectively, and $\delta = \binom{\alpha}{\beta}$ is a 2 x 1 parameter vector to be estimated.

[2] Cf. Gupta [1]

[3] The notation and part of the exposition given in the earlier paper [4] has been used in presenting the analysis in this and subsequent sections of the article. Accordingly, for explanation of symbols and detailed steps, the reader is advised to refer to the above-mentioned article.

Under the assumptions

(a) $\text{Cov}(x_t, w_t) = \text{Cov}(x_{t-1}, w_t) = 0$

(b) $Eu_t = 0$, $\text{var}\, u_t = \sigma_u^2$, for all t

x_t's being non-stochastic, it is well known that the Liviatan estimator

$$(2.3) \qquad \hat{\delta} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (X'Z)^{-1} X'y_{**} ,$$

of the parameter δ, where $X = (x_{**} \quad x_{*})$ is a $(T - 1) \times 2$ matrix of observations on the instruments, is consistent.

3. STATEMENT OF THEOREMS

For presenting the results on the skewness and the bias of the residual variance estimator, we further assume that

(c) $\text{Cov}(u_t, u_{t'}) = 0$, $t \neq t'$

(d) u_t's are normally distributed with zero mean and constant variance σ_u^2.

The following results will now be proved in the subsequent sections.

THEOREM 1. Under the Assumptions (a), (b), (c) and (d) given above, Karl Pearson's coefficient of skewness, to order 1/T in probability, of the probability distribution of the Liviatan estimator $\hat{\delta}$ is given by

$$(3.1) \qquad \sqrt{\beta_1} = 6\, \sigma_u^4 (g'Wg)^{-\frac{3}{2}} (g'Hg - \beta . g'g) . [\beta(n'Hg + g'Hn) - (1 + \beta^2) . g'n],$$

where

$$(3.2) \qquad H = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & 0 & 0 & & 1 \\ \hline 0 & 0 & 0 & & 0 \end{array}\right]$$

is an auxiliary unit matrix of size $(T - 1) \times (T - 1)$,

$$(3.3) \qquad n = \begin{bmatrix} n_2 \\ \cdot \\ \cdot \\ \cdot \\ n_T \end{bmatrix} = XQ'\iota$$

is a vector of T − 1 elements of the second column of the (T − 1) x 2 matrix XQ', the 2 x 2 matrix Q consisting of non-stochastic elements and the 2 x 1 column vector ι are given by

$$(3.4) \qquad Q = \begin{bmatrix} x'_{**}\, x_{**} & \alpha x'_{**}\, Bx_{*} \\ x'_{*}\, x_{**} & \alpha x'_{*}\, Bx_{*} \end{bmatrix}^{-1} \quad \text{and} \quad \iota = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively; the square matrix

$$(3.5) \qquad B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \beta & 1 & \cdots & 0 & 0 \\ & \vdots & & & & \\ 0 & \beta^{T-3} & & & \beta & 1 \end{bmatrix}$$

is of size T − 1, the matrix W of the variances and covariances of the disturbances and the column vector g of the non-stochastic elements are defined in Section 4 of the text.

Before we state the next theorem, we define

$\frac{1}{T-1}\, \hat{w}'_{**}\, \hat{w}_{**}$ as an estimate of the residual variance σ_w^2

in (2.1) where $\hat{w}_{**} = y_{**} - Z\hat{\delta}$

THEOREM 2. Under the assumptions of Theorem 1, the bias, to order 1/T in probability, of the estimator of the residual variance σ_w^2 is given by

$$(3.6) \qquad \frac{1}{T-1}\, E\, \hat{w}'_{**}\, \hat{w}_{**} - \sigma_w^2 = \sigma_u^4[(1 + 3\beta^2)n'n - 4\beta\, n'Hn]$$
$$+ \frac{1}{T}\, \mathrm{tr}\, [\bar{Z}_1 QX'WXQ'\bar{Z}_1' - 2\, \bar{Z}_1 QX'W],$$

where

$$(3.7) \qquad \bar{Z}_1 = (x_{**} \quad \alpha Bx_{*})$$

is a (T − 1) x 2 matrix of non-stochastic elements.

4. PROOF OF THEOREM 1[4]

Denoting the estimator of the j - th (j = 1,2) parameter in the 2 x 1 column parameter vector δ by $\hat{\delta}(j)$, we may obtain an expression for its sampling error $\hat{\delta}(j) - \delta(j)$ by premultiplying $(\hat{\delta} - \delta)$ by the 1 x 2 row vector e' which has its j - th element equal to 1 and the other element equal to zero. This implies that $e' = (1 \quad 0)$ if j = 1 and $e' = (0 \quad 1)$ if j = 2. In the latter case $e = \iota$. Accordingly, by using the analysis given in Section 3 of article [4] and retaining terms to $O(T^{-3/2})$ only we can write

$$\hat{\delta}(j) - \delta(j) = e'(\hat{\delta} - \delta) = e'A_{-1/2} + e'A_{-1} + e'A_{-3/2} \tag{4.1}$$

where

$$A_{-1/2} = QX'w_{**}$$

$$A_{-1} = - QD_{1/2}QX'w_{**} \tag{4.2}$$

$$A_{-3/2} = QD_{1/2}QD_{1/2}QX'w_{**} - QD_0QX'w_{**} ,$$

suffixes of A indicating the order of magnitude in probability,

$$D_{\frac{1}{2}} = X' \;(\underline{0} \quad u_*) \quad \text{and} \quad D_0 = X' \;(\underline{0} \quad (y_1 - u_1)a) \tag{4.3}$$

are 2 x 2 matrices, the elements of which are $O(T^{1/2})$ and $O(1)$ respectively, and the column vector

$$a = \begin{bmatrix} 1 \\ \beta \\ \cdot \\ \cdot \\ \cdot \\ \beta^{T-2} \end{bmatrix} \tag{4.4}$$

is of size T - 1. The third non-central moment μ_3', to order $T^{-5/2}$ in probability, of the estimator $\hat{\delta}(j)$ about its true value $\delta(j)$ is then obtained as

$$\mu_3' = E[\hat{\delta}(j) - \delta(j)]^3 = E\,(e'A_{-1/2})^3 + 3E[(e'A_{-1/2})^2 (e'A_{-1})] + 3E\,[(e'A_{-1/2})^2 (e'A_{-3/2}) + (e'A_{-1/2})(e'A_{-1})^2] \tag{4.5}$$

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[4] For proving the theorems 1 and 2 we assume y_1, the first element of the vector y_* to be fixed and non-stochastic.

In order to evaluate (4.5), we introduce

$$(4.6) \qquad XQ' = (m \quad n) = \begin{bmatrix} m_2 & n_2 \\ \vdots & \vdots \\ m_T & n_T \end{bmatrix}$$

and

$$(4.7) \qquad g' = e'QX' = (g_2 \ldots g_T)$$

where m and n denote the first and the second column of the $(T-1) \times 2$ matrix XQ', respectively, and g is a $(T-1) \times 1$ vector of non-stochastic elements. It should be noted that

$$(4.8) \qquad \begin{aligned} g &= m \qquad \text{for } j = 1 \\ &= n \qquad \text{for } j = 2 \end{aligned}$$

Thus, under the assumptions (a) through (d) stated in the preceding sections and using (4.2), (4.5) - (4.7) it is easy to verify that

$$(4.9) \qquad \begin{aligned} &E(e'A_{-1/2})^3 = 0 \\ &E(e'A_{-1/2})(e'A_{-1})^2 = 0 \end{aligned}$$

and

$$\begin{aligned} (4.10) \qquad & E(e'A_{-1/2})^2(e'A_{-3/2}) \\ &= E\; e'QX'w_{**}\; e'QX'w_{**}\; e'QD_{\frac{1}{2}}\; QD_{\frac{1}{2}}\; QX'w_{**} - \\ &- E\; e'QX'w_{**}\; e'QX'w_{**}\; e'QD_0\; QX'w_{**} \\ &= 0 \end{aligned}$$

because the first term on the right hand side of the first equality sign in (4.10) and the terms in (4.9) involve only odd order moments of the normal distribution while the second term on the right of the equality sign in (4.10) yields an expression of $O(1/T^3)$ and is, therefore, neglected. Hence, up to order $1/T^{5/2}$, we have

$$(4.11) \qquad \mu_3' = 3E[(e'A_{-1/2})^2(e'A_{-1})]$$

and using (4.2), it can be written as

$$(4.12) \qquad \mu_3' = -3E(e'QX'w_{**}e'QX'w_{**}\; w_{**}'\; XQ'D_{1/2}'\; Q'e).$$

Replacing $D_{\frac{1}{2}}$ by its right hand expression in (4.3) and using (4.6), (4.7) we can rewrite (4.12) as

$$(4.13) \qquad \mu_3' = -\ 3E\ (g' w_{**} g' w_{**} w_{**}' \ n\ u_*'\ g).$$

Or, alternatively

$$(4.14) \qquad \mu_3' = -\ 3\ g'[E u_{**} g' u_{**} u_{**}'\ n\ u_*' - \beta E u_{**} g' u_{**} u_*' n\ u_*'$$
$$-\ \beta E\ u_{**} g' u_* u_{**}' n\ u_*' + \beta^2 E\ u_{**} g'\ u_* u_*'\ n\ u_*'$$
$$-\ \beta E\ u_* g'\ u_{**} u_{**}'\ n\ u_*' + \beta^2\ E\ u_* g'\ u_{**} u_*'\ n\ u_*'$$
$$+ \beta^2\ E\ u_* g'\ u_* u_{**}'\ n\ u_*' - \beta^3\ E\ u_* g'\ u_* u_*'\ n\ u_*'\ ]g$$

where use has been made of the relationship $w_{**} = u_{**} - \beta u_*$ as defined in (2.2). If we pre and post multiply each of the terms within square brackets in (4.14) by g' and g, respectively, it immediately follows that third and fourth terms are respectively equal to fifth and sixth terms. The value of the third expectation can be obtained simply by interchanging n and g in the value of the second term. Moreover, the value of the last expectation within brackets can easily be derived from that of fourth by symmtry. Accordingly, we are required to evaluate first, second, fourth and seventh terms only and the value of the remaining terms then follows.

Let us now consider the first expectation within square brackets in (4.14).

$$(4.15) \qquad E\ u_{**} g'\ u_{**}\ u_{**}'\ n\ u_*' = E[(u_{**} u_*')(g' u_{**})(u_{**}' n)]$$

$$= E \begin{bmatrix} u_2 u_1 & u_2^2 & \cdots & u_2 u_{T-1} \\ u_3 u_1 & u_3 u_2 & & u_3 u_{T-1} \\ \vdots & & & \\ u_{T-1} u_1 & u_{T-1} u_2 & & u_{T-1}^2 \\ u_T\ u_1 & u_T\ u_2 & & u_T u_{T-1} \end{bmatrix} \left(\sum_2^T g_t\ u_t\right)\left(\sum_2^T n_t\ u_t\right)$$

$$= \sigma_u^4 \begin{bmatrix} 0 & (2g_2n_2 + \sum_2^T g_tn_t) & (g_2n_3 + g_3n_2) & \cdots & (g_2n_{T-1} + g_{T-1}n_2) \\ 0 & (g_2n_3 + g_3n_2) & (2g_3n_3 + \sum_2^T g_tn_t) & & (g_3n_{T-1} + g_{T-1}n_3) \\ \vdots & & & & \\ 0 & (g_2n_{T-1} + g_{T-1}n_2) & (g_3n_{T-1} + g_{T-1}n_3) & & (2g_{T-1}n_{T-1} + \sum_2^T g_tn_t) \\ 0 & (g_2n_T + g_T n_2) & (g_3n_T + g_Tn_3) & & (g_Tn_{T-1} + g_{T-1}n_T) \end{bmatrix}$$

$$= \sigma_u^4 [(g'n).H + gn'H + ng'H],$$

where n, g and H have been defined in (4.6), (4.7) and (3.2), respectively. Similarly, it is easy to verify that

(4.16) Second: $E\, u_{**}g'\, u_{**}u_*'\; n\, u_*' = \sigma_u^4[g\, n' + Hng'H + (g'Hn).H]$

(4.17) Third : $E\, u_{**}g'\, u_*u_{**}'n\, u_*' = \sigma_u^4[n\, g' + Hgn'H + (n'Hg).H]$

(4.18) Fourth: $E\, u_{**}\, g'\, u_*u_*'\; n\, u_*' = \sigma_u^4[(g'n).H + Hgn' + Hng']$

(4.19) Seventh: $E\, u_*\, g'\, u_*\, u_{**}'\; n\, u_*' = \sigma_u^4[(n'Hg).I + H'ng' + gn'H]$

and

(4.20) Eighth : $E\, u_*\, g'\, u_*u_*'\; n\, u_*' = \sigma_u^4[(g'\underline{n}).I + gn' + ng']$.

Hence, combining (4.14) through (4.20) we get

(4.21) $$\mu_3' = -3\sigma_u^4 [2\{(1 + 2\beta^2)(g'n) - \beta(g'Hn) - 2\beta\, (n'Hg)\}\, (g'Hg) + \\ + \{(1 + 3\beta^2)(n'Hg) + 2\beta^2(g'Hn) - 3\beta\, (1 + \beta^2)(g'n)\}(g'g)].$$

The first two moments, to order 1/T, of the Liviatan estimator $\hat{\delta}(j)$ around its true parameter value $\delta(j)$ have already been evaluated in [4] (cf. equations (2.13) and (2.9)) which may be written as

(4.22) $\mu_1' = E[\hat{\delta}(j) - \delta(j)] = -\, g'\Omega\;\; n + O(1/T^2)$

and

(4.23) $\mu_2' = E[\hat{\delta}(j) - \delta(j)]^2 = g'Wg + O(1/T^2)$

where

(4.24) $\Omega = Eu_*\, w_{**}' = \sigma_u^2\, (H' - \beta.I)$

and

(4.25) $W = Ew_{**}\, w_{**}' = \sigma_u^2\, [1 + \beta^2).I - \beta(H + H')]$.

Using the well known relationship between moments

(4.26) $\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\,\mu_1'^3$,

we obtain the third central moment μ_3 of $\hat{\delta}(j)$, to order $T^{-5/2}$, as

(4.27) $$\mu_3 = \mu_3' + 3\,(g'Wg)(g'\Omega\, n),$$

$$= 6\,\sigma_u^4\,(g'Hg - \beta . g'g)[\,\beta(n'Hg + g'Hn) - (1 + \beta^2).g'n],$$

where use has been made of (4.21).

Similarly, the second central moment μ_2 of the estimator $\hat{\delta}(j)$ is given by the relationship

(4.28) $$\mu_2 = \mu_2' - \mu_1'^2 = g'Wg + O(1/T^2).$$

Finally, using (4.27), (4.28) and applying the formula

(4.29) $$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} ,$$

we get the result stated in Theorem 1.

5. PROOF OF THEOREM 2

Using (2.2), we can write

(5.1) $$\hat{w}_{**} = y_{**} - Z\hat{\delta} = w_{**} - Z(\hat{\delta} - \delta)$$

and, thus

(5.2) $$E\,\hat{w}_{**}'\,\hat{w}_{**} = E\,w_{**}'\,w_{**} - 2\,E\,w_{**}'\,Z(\hat{\delta} - \delta) +$$

$$+ E(\hat{\delta} - \delta)'Z'Z(\hat{\delta} - \delta).$$

Since we are required to evaluate $\frac{1}{T-1}\,E\hat{w}_{**}'\,\hat{w}_{**}$ to $O(1/T)$, it is, therefore, sufficient if we retain terms to $O(1)$ only in $E\,\hat{w}_{**}'\,\hat{w}_{**}$. Accordingly, we evaluate (5.2) term by term and obtain for the leading term on the right of the equality sign as

(5.3) $$E\,w_{**}'\,w_{**} = (T - 1)\sigma_w^2 ,$$

where var $w_t = \sigma_w^2$ for $t = 2, \ldots, T$.

In order to evaluate the next two terms, we first write

$$(5.4) \qquad Z = \bar{Z} + V$$

where the systematic part of Z is given by

$$(5.5) \qquad \bar{Z} = \bar{Z}_1 + \bar{Z}_0 = (x_{**} \quad \alpha B x_{*}) + (\underline{0} \quad a y_1)$$

and

$$(5.6) \qquad V = (\underline{0} \quad \bar{v}_{*})$$

is the stochastic part of Z; $\bar{y}_*$ and $\bar{v}_*$ being the systematic and non-systematic parts of the vector y_*, respectively, defined as

$$(5.7) \qquad \begin{aligned} y_* &= \bar{y}_* + \bar{v}_* \\ \bar{y}_* &= a y_1 + \alpha B x_* \\ \bar{v}_* &= u_* - a u_1 \end{aligned} \qquad \text{(cf.[4], Section 3)}$$

Using (5.4), (5.5) and the result (3.8) of article [4], we find the second expectation on the right of the equality sign in (5.2) as

$$(5.8) \qquad E\, w'_{**} Z(\hat{\delta} - \delta) = E\, w'_{**} \bar{Z}_1 A_{-1/2} + E\, w'_{**} V A_{-1/2} + E\, w'_{**} V A_{-1} ,$$

where terms of order smaller than 1 have been neglected and $A_{-1/2}$, A_{-1} have been defined in (4.2). It is now easy to see that[6]

$$(5.9) \qquad E\, w'_{**} \bar{Z}_1 A_{-1/2} = \mathrm{tr}\, (\bar{Z}_1 Q X' W),$$

W being defined in (4.25) and

$$(5.10) \qquad E\, w'_{**} V A_{-1/2} = E\, w'_{**} V Q X' w_{**} = 0$$

because it involves third order moments of the normal distribution which are zero according to assuption (c) given in Section 2. With the help of the results given in (4.2), (4.6) and (5.6), we write the last term in (5.8) as

$$\begin{aligned} (5.11) \qquad E\, w'_{**} V A_{-1} &= -E\, w'_{**} (\underline{0} \quad \bar{v}_*) \begin{pmatrix} m' \\ n' \end{pmatrix} (\underline{0} \quad u_*) \begin{pmatrix} m' \\ n' \end{pmatrix} w_{**} \\ &= -E\, w'_{**} \bar{v}_* n' u_* n' w_{**} \\ &= -n'[E(u_* w'_{**}) . (w'_{**} u_*)] n , \end{aligned}$$

[6] "tr" stands for "trace of".

where we have used the relationship $\bar{v}_* = u_* - au_1$ and the term involving au_1 has been neglected due to its smaller order than 1 in probability. Now the value of the expression within square brackets on the right of the third equality in (5.11) is simple to obtain after replacing w_{**} by $(u_{**} - \beta u_*)$. Thus, we may directly write

$$(5.12) \qquad E\, w'_{**}\, VA_{-1} = -\sigma_u^4 \cdot T\, n'\, (\beta^2 . I - \beta . H)n + o(1).$$

Hence, substituting (5.9), (5.10) and (5.12) in (5.8) we get

$$(5.13) \qquad E\, w'_{**}\, Z(\hat{\delta} - \delta) = tr\,(\bar{Z}_1 QX'W) - T\,\sigma_u^4\, n'\,(\beta^2 . I - \beta . H)n,$$

to order '1' only.

Finally, we evaluate the last term in (5.2). For this purpose we first split up the $(T - 1) \times 2$ matrix V as

$$(5.14) \qquad V = V_{1/2} + V_0$$

where

$$(5.15) \qquad V_{1/2} = (\underline{0} \quad u_*) \text{ and } V_0 = (\underline{0} \qquad -a\,u_1),$$

the suffices of V indicating the order of magnitude in probability and then by using (5.4) - (5.6) and (5.14) we obtain

$$(5.16) \qquad Z'Z = (\bar{Z}_1 + \bar{Z}_0 + V_{1/2} + V_0)'(\bar{Z}_1 + \bar{Z}_0 + V_{1/2} + V_0).$$

Now combining (3.8) of article [4] and (5.16) given above, we get the value of the last term on the right of the equality sign in (5.2) as

$$(5.17) \qquad E\,(\hat{\delta} - \delta)'Z'Z(\hat{\delta} - \delta) = EA'_{-1/2}\bar{Z}'_1\bar{Z}_1 A_{-1/2} +$$
$$+ EA'_{-1/2}\, V'_{1/2}\, V_{1/2}\, A_{-1/2} ,$$

where terms of order smaller than '1' have been neglected and $A_{-1/2}$ is given in (4.2). The value of the second member in (5.17) can immediately be written as

$$(5.18) \qquad E\, A'_{-1/2}\, \bar{Z}'_1\, \bar{Z}_1 A_{-1/2} = tr\, \bar{Z}_1\, E(A_{-1/2}\, A'_{-1/2})\, \bar{Z}'_1$$
$$= tr\,(\bar{Z}_1 QX'WXQ'\bar{Z}'_1).$$

The last member in (5.17) can easily be evaluated by substituting the values of $A_{-1/2}$ and $V_{1/2}$ from (4.2) and (5.15), respectively, therein. Thus, it follows that

$$(5.19) \qquad E\, A'_{-1/2}\, V'_{1/2}\, V_{1/2}\, A_{-1/2} = E\, w'_{**}\, XQ' \begin{bmatrix} 0 & 0 \\ 0 & u'_{*}u_{*} \end{bmatrix} QX' w_{**}$$

$$= E\, w'_{**}\, n\, u'_{*}\, u_{*}\, n'\, w_{**} \quad ,$$

where use has been made of the relationship in (4.6). Substituting $(u_{**} - \beta u_{*})$ for w_{**}, we can rewrite (5.19) as

$$(5.20) \qquad E\, A'_{-1/2}\, V'_{1/2}\, V_{1/2}\, A_{-1/2} = n' [E\, u_{**}\, u'_{*}\, u_{*}\, u'_{**} \; -$$

$$- 2\beta\, E\, u_{**}\, u'_{*}\, u_{*}\, u'_{*} \; + \beta^2\, E\, u_{*}\, u'_{*}\, u_{*}\, u'_{*}\,]n.$$

The right hand expression can be evaluated as in the preceding section and so we write

$$(5.21) \qquad \begin{cases} E\, u_{**}\, u'_{*}\, u_{*}\, u'_{**} = T\, \sigma_u^4 \,.\, I + O(1) \\ E\, u_{**}\, u'_{*}\, u_{*}\, u'_{*} = T\, \sigma_u^4 \,.\, H + O(1) \\ E\, u_{*}\, u'_{*}\, u_{*}\, u'_{*} = T\, \sigma_u^4 \,.\, I + O(1), \end{cases}$$

and, hence,

$$(5.22) \qquad E\, A'_{-1/2}\, V'_{1/2}\, V_{1/2}\, A_{-1/2} = T\, \sigma_u^4\, n'[(1 + \beta^2).I - 2\beta\, .H]n,$$

where terms of smaller order than O(1) in probability have been neglected.

Substituting (5.18) and (5.22) in (5.17), we get the value of the last term in (5.2).

$$(5.23) \qquad E\, (\hat{\delta} - \delta)'Z'Z(\hat{\delta} - \delta) = tr\, (\overline{Z}_1 QX'WXQ'\overline{Z}'_1) \; +$$

$$+ T\, \sigma_u^4\, [(1 + \beta^2)n'n - 2\beta\, .n'H\, n].$$

Finally, combining (5.2), (5.3), (5.13) and (5.23) and rearranging the terms we get the result enunciated in Theorem 2.

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