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A NOTE ON A CLASS OF UTILITY AND PRODUCTION FUNCTIONS YIELDING EVERYWHERE DIFFERENTIABLE DEMAND FUNCTIONS

by A.P. Barten, T. Kloek and F.B. Lempers

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1. INTRODUCTION AND SUMMARY

In a recent issue of The Review of Economic Studies (October 1967), Dhrymes discusses some conditions for the differentiability of demand functions in the case where one does not require knowledge of derivatives beyond those of the second order to determine whether a given stationary point is an extremum. Starting from production and utility functions with non-singular Hessian matrices, he develops the well-known salient propositions of the theories of the firm and consumer behaviour (see, e.g., Samuelson (1953)), using convenient matrix notation to avoid cumbersome properties of determinants.

On the one hand it is well-known (see, e.g., Theil (1967)) that the case of constant returns to scale in the theory of the firm involves a singular Hessian. On the other hand the assumption of a Hessian with rank less than n - 1 must be ruled out, since it would contradict the traditional second-order concavity conditions. In this note we shall show that for some propositions of Dhrymes (1967) we are still able to derive the same properties if the Hessian is singular with rank n - 1. Extension to the other propositions and their conclusions is obvious and hence deleted. We have adopted the notation used by Dhrymes (1967) throughout.

2. THE RANK OF THE HESSIAN

To obtain a unique maximum of u(.) subject to a budgetary constraint $p'x = I$ (income), it is sufficient that

(1) $$y'U^{**}y < 0 \quad \text{if} \quad \underline{u}'y = 0 \quad \text{and} \quad y \neq 0$$

(see Dhrymes (1967), Theorem 2), where U^{**} is the $n \times n$ (symmetric) Hessian matrix of u(.) and $\underline{u}$ its (positive) gradient vector. The condition does not require that

U^{**} be definite or even semi-definite, as can be illustrated by the simple example

$$U^{**} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is an indefinite matrix. Now, we define the $(n + 1) \times (n + 1)$ matrix

$$U^{*} = \begin{bmatrix} U^{**} & \underline{u} \\ \underline{u}' & 0 \end{bmatrix} \tag{2}$$

and prove the following lemmas.

LEMMA 1. If U^{**} and $\underline{u}$ satisfy condition (1), U^* is a non-singular matrix.

Proof: Suppose U^* has rank $\leq n$. Then there exist an n-vector c and a scalar γ such that $[{c \atop \gamma}] \neq 0$, $U^{**}c + \underline{u}\gamma = 0$, and $\underline{u}'c = 0$. It follows that $c'U^{**}c + c'\underline{u}\gamma = c'U^{**}c = 0$, which contradicts (1), unless $c = 0$. But $c = 0$ implies $\underline{u}\gamma = 0$ and, hence, $\gamma = 0$, which is not allowed either. This completes the proof.

LEMMA 2. In the $(n + 1) \times (n + 1)$ non-singular matrix

$$X^{*} = \begin{bmatrix} X^{**} & x \\ x' & \xi \end{bmatrix}$$

the rank of the $n \times n$ matrix X^{**} is at least $n - 1$.

Proof. Suppose X^{**} has rank $< n - 1$ and $x \neq 0$. Then there exists a pair of non-zero n-vectors c_1 and c_2, such that $c_1'c_2 = 0$ and $X^{**}c_1 = X^{**}c_2 = 0$. Postmultiply X^* by the $(n + 1)$-vectors

$\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} c_2 \\ 0 \end{bmatrix}$, respectively. Non-singularity of X^* implies that $x'c_1 \neq 0$ and $x'c_2 \neq 0$. Let $c = (x'c_2)c_1 - (x'c_1)c_2$ so that $x'c = 0$ and $c \neq 0$, while obviously $X^{**}c = 0$. Postmultiplication of X^* by $[{c \atop 0}]$ then yields a zero vector which contradicts the non-singularity of X^*. Consequently $r(X^{**}) \geq n - 1$. If $x = 0$, it is a trivial matter to show that $r(X^{**}) = n$ and $\xi \neq 0$. This completes the proof.

On combining Lemmas 1 and 2 we find that $r(U^{**}) \geq n - 1$.

3. PROPERTIES OF THE SUBSTITUTION MATRIX

We partition the inverse matrix of U^* as

$$(U^*)^{-1} = B^* = \begin{bmatrix} B^{**} & b \\ b' & \beta \end{bmatrix} \tag{3}$$

By definition this matrix has the following properties:

$$B^{**}U^{**} + b\underline{u}' = I_n \tag{4}$$

$$B^{**}\underline{u} = 0 \tag{5}$$

$$b'U^{**} + \beta\underline{u}' = 0 \tag{6}$$

$$b'\underline{u} = 1 \tag{7}$$

where I_n is the $n \times n$ unit matrix. Property (7) implies $b \neq 0$.

Now we state some propositions and prove them under the condition that the rank of U^{**} is at least $n - 1$.

Proposition 1.

The substitution matrix is singular of rank $n - 1$; its null-space is spanned by $\underline{u}$.

Proof: The substitution matrix is given by λB^{**}; however, since $\lambda > 0$, it is simpler to deal with B^{**}. Applying Lemma 2 to B^* we find that $r(B^{**}) \geq n - 1$. Since $B^{**}\underline{u} = 0$, see (5), we conclude that

$$r(B^{**}) = n - 1 \qquad \text{q.e.d.} \tag{8}$$

Proposition 2.

The substitution matrix is negative semi-definite.

Proof: Because of (4) and (5) we have

$$B^{**}U^{**}B^{**} = B^{**} \tag{9}$$

Consider the quadratic form

(10) $$z'B^{**}z = z'B^{**}U^{**}B^{**}z = g'U^{**}g$$

with $z \neq 0$ and $g = B^{**}z$. Note that, because of (5), $\underline{u}'g = 0$. As a consequence we have for $g \neq 0$

(11) $$g'U^{**}g < 0$$

in view of (1). If $g = 0$, we have $B^{**}z = 0$. Since B^{**} has rank $n - 1$, $B^{**}z = 0$ holds only if $z = \mu\underline{u}$, see Proposition 1, for some real scalar μ. Hence

(12) $$\begin{cases} z'B^{**}z = 0 & \text{if} \quad z = \mu\underline{u} \\ z'B^{**}z < 0 & \text{otherwise} \end{cases}$$ q.e.d.

REFERENCES

Barten, A.P. (1964), "Consumer Demand Functions under Conditions of Almost Additive Preferences", Econometrica, Vol. 32, 1-38.

Dhrymes, P.J. (1967), "On a Class of Utility and Production Functions Yielding Everywhere Differentiable Demand Functions", Review of Economic Studies, Vol. XXXIV (4) pp. 399-408.

Samuelson, P.A. (1953), Foundations of Economic Analysis, Harvard University Press, Cambridge, Mass.

Theil, H. (1967), Economics and Information Theory, North-Holland Publishing Company, Amsterdam.