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ESTIMATING DEMAND EQUATIONS

by A.P. Barten

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1. INTRODUCTION AND SUMMARY

In this article an estimation procedure is formulated which is designed to handle a complete system of demand equations for all commodities of a consumer's budget simultaneously. This method is based not only on the "systematic factors" which are explicitly taken into consideration in classical demand theory (income and prices) but also on the "unsystematic" or "random" components of consumer behavior represented by the disturbance terms of empirical econometrics. In the approach followed in this article these random components are an essential part of the explanation of consumer demand and are not introduced merely to let the theory fit the facts.

The organization of this article is as follows. In Section 2 a summary of demand theory, in the form in which it was developed by the author [1] and Theil [14,16], will be given. This theory uses as its

¹ The author wishes to thank Professor Henri Theil of the Netherlands School of Economics and the University of Chicago for his guidance and co-operation. The research project leading up to this article actually has been a joint venture by Professor Theil and the author. Some of the results of this project have already been published or are in the process of publication. Reference is made to [1, 2, 5, 14, 16, 17]. Of course, the responsibility for any defects in this article remains with the author alone.

basic ingredients the utility function and the budget constraint. It results in a set of demand equations, which express demand in income and prices. In Section 3 random shocks in the linear part of the utility function are introduced to take into account the effect of all other factors than income and prices which are relevant for the consumer's decision. These random shocks affect the quantities demanded in roughly the same way as changes in prices and cause the presence of random disturbance terms in the demand equation. Under plausible assumption with respect to the covariance matrix of the random fluctuations in utility a specification of the covariance matrix of these disturbance terms can be formulated, which - apart from a scalar - is equal to the matrix of substitution coefficients of the set of demand equations. Frequently, empirical demand studies are concerned with the "strategy" of demand analysis, as Stone [11] puts it. That is, attention is concentrated on the explanation of demand for large groups of commodities. Section 4 is devoted to the formulation of a system of demand equations for these large composites of commodities where use is made of the assumption of want independence between commodity groups or block additivity of the utility function. The resulting demand equations for want independent composite commodities are, with suitable definitions of the aggregate variables, analogous to those for want independent elementary goods. With the theoretical foundations laid the rest of the article is devoted to the estimation of a system of (composite) demand equations under conditions of want independence or additive utility. The system in its theoretical formulation is non-linear in the unknown coefficients. To facilitate estimation it is first linearized in these coefficients, which is done in Section 5. The following section contains a matrix formulation of the system of linearized demand equations, which is extremely helpful in this case because of the linear dependence of the equations and the presence of all coefficients to be estimated in each equation. Section 7 and 8 deal with the formulation of the estimator and its covariance matrix. This estimator is under certain conditions a best linear unbiased estimator. The proof of the minimum variance property of the estimator is delegated to Appendix A. Section 9 contains a brief description of the data used for the actual estimation of the system. These data themselves, referring to consumer demand for four major groups (food, pleasure goods, durables and other commodities and services) in the Netherlands in the periods 1922 - 1939 and 1949 - 1963, are presented in two tables in Appendix B. In the last three sections numerical results are given and briefly discussed. These results offer no reason to challenge the hypotheses which have been used to obtain them.

No doubt, at several instances brevity of exposition has been achieved at the costs of clarity. Also at more than one occasion results could have been generalized without too many additional complications,

while also some implications could have been more elaborated than is done here. To keep this article within reasonable limits many interesting aspects of the system and its estimation have not been considered explicitly and are left for further research.

2. DEMAND EQUATIONS IN INCOME AND PRICES

The demand equations which are the subject of this section are of the following form:

$$(2.1) \quad \frac{w_{it} + w_{i,t-1}}{2} \Delta (\ln q_{it}) = \mu_i \Delta (\ln q_t) + \sum_{j=1}^n v_{ij} \Delta (\ln \bar{p}'_{jt})$$

with $i = 1, \dots, n$ and where w_{it} is the value share in period t of the i^{th} commodity:

$$(2.2) \quad w_{it} = \frac{p_{it} q_{it}}{m_t} \quad \sum_i w_{it} = 1$$

(m_t being income or total expenditure, p_{it} the price per unit and q_{it} the quantity of commodity i). Furthermore, $w_{i,t-1}$ is the value share of the preceding period, $\Delta (\ln q_t)$ the sum over i of the variables on the left-hand side of (2.1) and $\bar{p}'_{jt}$ is the relative price of the j^{th} commodity. The price variable is defined as follows:

$$(2.3) \quad \Delta (\ln \bar{p}'_{jt}) = \Delta (\ln p_{jt}) - \sum_{k=1}^n \mu_k \Delta (\ln p_{kt})$$

The symbol Δ stands for first (backward) differences, while "ln" denotes natural logarithms.

Demand equation (2.1) is derived from the following equation in infinitesimal changes:

$$(2.4) \quad w_i d (\ln q_i) = \mu_i [d (\ln m) - \sum_k w_k d (\ln p_k)] \\ + \sum_j v_{ij} [d (\ln p_j) - \sum_k \mu_k d (\ln p_k)]$$

which in turn is based on the classical theory of consumer demand. That is, we take as our starting point a consumer who maximizes a differentiable utility function $u(q)$ subject to a budget constraint $p'q = m$, where p and q are n - element column vectors of strictly positive prices and quantities, respectively. The first-order derivatives of the utility function, $u_i = \partial u / \partial q_i$, also known as marginal utilities, are assumed to be all positive, while the matrix of second-order derivatives of the utility function, U , is assumed

to be symmetric and negative definite. The well-known first-order conditions for utility maximization under the budget - constraint are

$$(2.5) \quad u_i - \lambda p_i \quad (i = 1, \dots, n)$$

Here λ , the Langrangean multiplier, can be interpreted as the marginal utility of income (or total expenditure). Using appropriate differentiation procedures one then obtains a number of restrictions on the demand equations, which are the expressions of the quantities in terms of income and prices (see [1]). Theil [14] has proposed the functional forms (2.1) and (2.4) which satisfy these restrictions completely. The following points summarize the highlights of these equations.

(i) The effect of price changes on demand is split up into three parts: the specific substitution effect, the general substitution effect (together forming the substitution effect in the terminology of Hicks and Allen [7]) and the income effect. The general substitution effect and the income effect are handled by deflation. The income effect is represented by the term

$$- \sum_k w_k d(\ln p_k)$$

within the first pair of square brackets of (2.4). The composite term

$$d(\ln m) - \sum_k w_k d(\ln p_k)$$

can be interpreted as the infinitesimal relative change in price deflated or real income. The general substitution effect, a designation introduced by Houthakker [8], is represented by the price deflator

$$- \sum_k \mu_k d(\ln p_k)$$

within the second pair of square brackets of (2.4)

(ii) Thus we see that the two deflators use different weights. The income deflator is based on value shares, the price deflator on μ_k which are the same as the income coefficients in (2.4). These are the marginal value shares (or the marginal propensities to spend on the various commodities):

$$(2.6) \quad \mu_i = p_i \frac{\partial q_i}{\partial m} \quad \sum_i \mu_i = 1$$

In the case of inferior commodities the corresponding weights of the price deflator are negative. From (2.6) it can be easily derived that the income elasticity of demand for the i^{th} commodity, η_i , is related to μ_i by means of $\eta_i = \mu_i / w_i$.

(iii) The specific substitution effect is represented by the term $v_{ij}^d (\ln p_j)$ in (2.4). The coefficient v_{ij} is defined as follows

$$(2.7) \quad v_{ij} = \frac{\lambda p_i p_j u^{ij}}{m}$$

where u^{ij} is the $(i, j)^{\text{th}}$ element of U^{-1} , the inverse of the matrix of second order derivatives of the utility function. The assumption that U and hence U^{-1} is symmetric and negative definite implies the symmetry condition:

$$(2.8) \quad v_{ij} = v_{ji} \quad \text{for all pairs } (i, j)$$

and the negative condition $v_{ii} < 0$.

(iv) When the preference ordering on the n - dimensional commodity space can be described by a utility function which is additive in the utilities of the elementary commodities, the matrix U and its inverse are both diagonal, so that $v_{ij} = 0$ for all $i \neq j$ in that special case. This condition of want independence (Frisch [6]) or additive utilities (Houthakker [8]) is rather restrictive, but it can be shown under fairly general conditions that many coefficients can be put equal to zero, in particular in the case of "almost additive preferences" and "utility trees" (see [1, 10, 12, 13]). A specific useful assumption is that of groupwise independence or block additivity, which implies block diagonality of U and U^{-1} . This assumption will be used in the empirical part of this article. Note that the property of additivity or its variant, block additivity, is not invariant under monotonic non-linear transformations of the utility function which is characterized by it. In this sense these properties are not ordinal. Note also that if the preference ordering can be represented by an additive utility function demand equations with the $v_{ij} = 0$ for $i \neq j$ are consistent with this ordering and contain our knowledge with respect to this ordering in the most compact way.

(v) The coefficients v_{ij} are related to the marginal value shares. We have

$$(2.9) \quad \sum_{j=1}^n v_{ij} = \phi \mu_i \quad i = 1, \dots, n$$

where ϕ is defined as the reciprocal of the income elasticity of the marginal utility of income:

$$(2.10) \quad \frac{1}{\phi} = \frac{\partial(\ln \lambda)}{\partial(\ln m)}$$

This coefficient plays a major role in the studies by Frisch [6] and Houthakker [8]. By taking the sum over i in (2.9) and using (2.6) we find that ϕ is the sum of all price coefficients:

$$(2.11) \quad \phi = \sum_i \sum_j v_{ij}$$

When utilities are additive we can replace (2.9) and (2.11) by

$$(2.12) \quad v_{ii} = \phi \mu_i \quad \phi = \sum_i v_{ii}$$

(vi) Just as the demand equation (2.4) expresses quantity changes in terms of price and income changes the change in λ , the marginal utility of income, can be expressed in terms of the latter changes. This equation (see [14]) is as follows:

$$(2.13) \quad d(\ln \lambda) = \frac{1}{\phi} [d(\ln m) - \sum_k w_k d(\ln p_k)] - \sum_k \mu_k d(\ln p_k)$$

The term between square brackets is the infinitesimal relative change in real income, while the other term represents the price deflator. It is easily verified that by using (2.9) an alternative formulation of demand equation (2.4) is:

$$(2.14) \quad w_i d(\ln q_i) = \phi \mu_i d(\ln \lambda) + \sum_j v_{ij} d(\ln p_j) \\ = \sum_j v_{ij} [d(\ln p_j) + d(\ln \lambda)]$$

Apparently, $d(\ln \lambda)$, the relative change in the marginal utility of income, acts as another type of price deflator. In view of (2.5) the term between square brackets can be interpreted as the relative change in the marginal utility of commodity j .

The demand equation in finite changes corresponding to (2.4) can be formulated as follows:

$$(2.15) \quad w_{it}^* \Delta (\ln q_{it}) = \mu_i [\Delta (\ln m_t) - \sum_k w_{kt}^* \Delta (\ln p_{kt})] \\ + \sum_j v_{ij} [\Delta (\ln p_{jt}) - \sum_k \mu_k \Delta (\ln p_{kt})]$$

where

$$(2.16) \quad w_{it}^* = \frac{w_{it} + w_{i,t-1}}{2}$$

This transition implies errors of approximation which are of the third order in the changes in the logarithms - to be named log-changes in the sequel- in income and prices. Writing for the change in the value share of commodity i:

$$(2.17) \quad \Delta w_{it} = \Delta \left(\frac{p_{it} q_{it}}{m_t} \right) = w_{it}^* \Delta (\ln q_{it}) + w_{it}^* \Delta (\ln p_{it}) - w_{it}^* \Delta (\ln m_t)$$

which is correct up to the same order of approximation, the dependent variable of the demand equation is seen to be that part of the change in the value share of commodity i that can be attributed to the change in the quantity demanded of this commodity. The sum over i of Δw_{it} in (2.17) is by definition equal to zero, since the value shares for both periods apart add up to one. Hence,

$$(2.18) \quad \sum_i w_{it}^* \Delta (\ln q_{it}) = \Delta (\ln m_t) - \sum_i w_{it}^* \Delta (\ln p_{it})$$

which is true up to errors of the third order in the log-changes in income and prices. Defining

$$(2.19) \quad \Delta (\ln q_t) = \sum_i w_{it}^* \Delta \ln q_{it}$$

which can be interpreted as the log-change in the volume of consumption, we can replace the expression within the first pair of square brackets of (2.15) by $\Delta (\ln q_t)$ without essentially affecting the validity of (2.15). The resulting demand equation is:

$$(2.20) \quad w_{it}^* \Delta (\ln q_{it}) = \mu_i \Delta (\ln q_t) + \sum_j v_{ij} [\Delta (\ln p_{jt}) - \sum_k \mu_k \Delta (\ln p_{kt})]$$

The advantage of using (2.20) rather than (2.15) becomes clear when both sides of (2.20) are summed up over i :

$$(2.21) \quad \sum_i w_{it}^* \Delta (\ln q_{it}) \\ = \sum_i \mu_i \Delta (\ln q_t) + \sum_i \sum_j v_{ij} [\Delta (\ln p_{jt}) - \sum_k \mu_k \Delta (\ln p_{kt})]$$

In view of (2.9), (2.8) and (2.11) the second term on the right-hand side can be seen to be

$$(2.22) \quad \sum_i \sum_j v_{ij} [\Delta (\ln p_{jt}) - \sum_k \mu_k \Delta (\ln p_{kt})] \\ = \sum_j \phi \mu_j \Delta (\ln p_{jt}) - \sum_k \phi \mu_k \Delta (\ln p_{kt}) = 0$$

Since the marginal value shares, the μ_i , add up to one, the right-hand side of (2.21) is exactly equal to the left-hand side. In this way the demand equations fully reflect the allocation aspect of the consumer's decision, or in other words demand equations of type (2.20) possess the well-known adding-up property. Demand equations like (2.15) don't have this property.

To economize on notation we shall denote the operation of taking the first (backward) difference in the logarithms by D , i.e. for some variable x_t we have

$$Dx_t = \Delta (\ln x_t) = \ln x_t - \ln x_{t-1}$$

Using this notation for the log-changes and (2.3) demand equation (2.20) can be rewritten as follows

$$(2.23) \quad w_{it}^* Dq_{it} = \mu_i Dq_t + \sum_j v_{ij} D\bar{p}_{jt}'$$

where the μ_i and v_{ij} ($i, j = 1, \dots, n$) are constants to be estimated when not a priori specified to be zero [see (iv)].

It is not difficult to see that the finite-changes-version of (2.13), the equation for the log-changes in the marginal utility of income, is

$$(2.24) \quad D\lambda_t = \frac{1}{\phi} Dq_t - \sum_k \mu_k Dp_{kt}$$

so that by using (2.3) and (2.9) an alternative expression for (2.23) is:

$$(2.25) \quad w_{it}^* Dq_{it} = \sum_j v_{ij} Dp_{jt} + \phi \mu_1 D\lambda_t = \sum_j v_{ij} Du_{jt}$$

where

$$(2.26) \quad Du_{jt} = Dp_{jt} + D\lambda_t$$

stands for the log-changes in the marginal utility of commodity j . Equation (2.25) provides an explicit relation between the left-hand variable of our demand equations and the log-changes in the marginal utilities of all commodities.

3. THE STOCHASTIC COMPONENTS

Apart from observational errors and approximation errors demand equation (2.23) completely specifies changes in demand as the consequence of changes in disposable income and prices. However, income and prices are not the only determinants of demand. One can think of many other factors relevant for the consumer's decision. Some of these factors may cause changes in effective restrictions on consumer behavior other than the budget constraint. More important are those factors which lead to changes in the utility function. In this article we will concentrate on the latter type. Instead of studying the effects of each possible factor separately we will consider their combined effect as a random variable. This means that the utility function is assumed to be stochastic.

We will assume that the random components of utility are independent of income and prices. We therefore can as a start keep income and prices constant in our analysis of the effects of random utility on the quantities demanded. Secondly, we will assume, following Theil and Neudecker [18], that the randomness of the utility function is confined to the linear part of it. This means that in the quadratic approximation

$$(3.1) \quad u(q) = a'q + \frac{1}{2} q'Uq$$

where a is a n -element vector and U the $n \times n$ - matrix of second-order derivatives, the vector a is supposed to be random. In the vector of marginal utilities, u_q , which is approximated by

$$(3.2) \quad u_q = a + Uq \quad u_q = [u_i]$$

this random vector appears in an additive way. When maximizing his utility under the budget constraint, the consumer equates the vector of marginal utilities to λp , the marginal utility of income times the given vector of prices, by means of adjusting the quantities. Hence, in the condition

$$(3.3) \quad u_q = a + Uq = \lambda p$$

also the vector q and the scalar λ are stochastic, though the non-stochastic restriction $p'q = m$ on the vector q remains valid.

In the preceding section we have defined the price coefficients v_{ij} to be equal to $(\lambda/m) p_i p_j u^{ij}$. To avoid random coefficients we will replace in this definition λ by its expectation: $\xi(\lambda)$. Thus, using matrix notation we define

$$(3.4) \quad N = \frac{\xi(\lambda)}{m} P U^{-1} P$$

where $N = [v_{ij}]$ and P is a diagonal matrix with the elements of the vector p as diagonal elements. Since by assumption $p > 0$, also P^{-1} is defined. As is easy to see, we have

$$(3.5) \quad U = \frac{\xi(\lambda)}{m} P N^{-1} P$$

The existence of N^{-1} follows directly from the existence of P^{-1} and U^{-1} . Using (3.5) we rewrite (3.3) as follows:

$$(3.6) \quad a + \frac{\xi(\lambda)}{m} P N^{-1} P q = \lambda p$$

The vector of value shares $w = [w_i]$ is by definition equal to $(1/m) P q$. Premultiplying (3.6) by $[1/\xi(\lambda)] P^{-1}$, which is the inverse of the diagonal matrix of the expected marginal utilities, we obtain

$$(3.7) \quad \frac{1}{\xi(\lambda)} P^{-1} a + N^{-1} w = \frac{\lambda}{\xi(\lambda)} \iota$$

where ι is a n -element column vector with all elements equal to unity.

Consider a vector of random shocks (Δa) . As the consequence of (Δa) , there will be random shocks (Δw) and $\Delta \lambda$, related to each other by

$$(3.8) \quad s + N^{-1}(\Delta w) = \frac{\Delta \lambda}{\xi(\lambda)} \iota$$

where $s = [1/\xi(\lambda)]P^{-1}(\Delta a)$, i.e., the initial random shift in the marginal utilities divided by the expected marginal utilities. The vector (Δw) is subject to the restriction $t'(\Delta w) = 0$, since the budget constraint has to be satisfied before and after the random shock. Consequently, we have the following system

$$(3.9) \quad \begin{bmatrix} N^{-1} & t \\ t' & 0 \end{bmatrix} \begin{bmatrix} (\Delta w) \\ \Delta \lambda / \xi(\lambda) \end{bmatrix} = \begin{bmatrix} -s \\ 0 \end{bmatrix}$$

The solution of (3.9) for (Δw) and $\Delta \lambda / \xi(\lambda)$ can be obtained by premultiplying (3.9) by the inverse of the matrix on the left-hand side. This inverse is given by

$$(3.10) \quad \begin{bmatrix} N^{-1} & t \\ t' & 0 \end{bmatrix}^{-1} = \frac{1}{t' N t} \begin{bmatrix} t' N t \cdot N - (N t)(N t)' & N t \\ (N t)' & -1 \end{bmatrix}$$

Realizing that N is symmetric [see (2.8)] and that according to (2.9) and (2.11) we have

$$(3.11) \quad N t = \phi \mu \quad t' N t = \phi$$

where μ is a n -column vector with the μ_i for elements, we can also write:

$$(3.12) \quad \begin{bmatrix} N^{-1} & t \\ t' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} N - \phi \mu \mu' & \mu \\ \mu' & 1/\phi \end{bmatrix}$$

Hence:

$$(3.13) \quad \begin{bmatrix} (\Delta w) \\ \Delta \lambda / \xi(\lambda) \end{bmatrix} = \begin{bmatrix} N - \phi \mu \mu' & \mu \\ \mu' & 1/\phi \end{bmatrix} \begin{bmatrix} -s \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -(N - \phi \mu \mu') s \\ -\mu' s \end{bmatrix}$$

For an element of (Δw) we then have

$$(3.14) \quad \Delta w_i = - \sum_j (v_{ij} - \phi \mu_i \mu_j) s_j$$

as the expression for the change in the value share as the consequence of random shocks in the utility function. Recalling that until now income and prices have been kept constant this random change in the value share of the i^{th} commodity is in view of (2.17) equal to the value of the dependent variable of the demand equations.

Allowing for changes in income and prices over time does not complicate the picture as long as the random shocks are uncorrelated with these changes. The right-hand side of (3.14) then represents the random component of our demand equations. The completed demand equation is now:

$$(3.15) \quad w_{it}^* Dq_{it} = \mu_i Dq_t + \sum_j v_{ij} D\bar{p}_{jt} + v_{it}$$

where v_{it} is a disturbance term for which we can write

$$(3.16) \quad v_{it} = - \sum_j (v_{ij} - \phi \mu_i \mu_j) s_{jt}$$

The time subscript has now been attached to the variable s_j to indicate that it represents a random shift over time in the marginal utilities. Note that (3.16) and the price term in (3.15) are of similar structure. We can write for this price term by using (2.3) and (2.9)

$$\begin{aligned} \sum_j v_{ij} D\bar{p}_{jt} &= \sum_j v_{ij} Dp_{jt} - \sum_j v_{ij} \sum_k \mu_k Dp_{kt} \\ &= \sum_j v_{ij} Dp_{jt} - \phi \mu_i \sum_j \mu_j Dp_{jt} \\ &= \sum_j (v_{ij} - \phi \mu_i \mu_j) Dp_{jt} \end{aligned}$$

which shows that apart from sign the relative random shocks in the marginal utilities, the s_{jt} , have the same effect as the log-changes in the deflated prices. Also here we have a specific effect: $\sum_j v_{ij} s_{jt}$

and a more general effect: $-\phi\mu_i \sum \mu_j s_{jt}$, which according to (3.13) can also be written as $\phi\mu_i \Delta\lambda/\xi(\lambda)$, if income and prices are constant. The general effect is thus seen to be proportional to the random relative shift in the marginal utility of income

A direct consequence of (3.16) and of (2.6) and (2.9) is that

$$\begin{aligned}
 (3.17) \quad \sum_i v_{it} &= - \sum_i \sum_j (v_{ij} - \phi\mu_i \mu_j) s_{jt} \\
 &= - \sum_j \left(\sum_i v_{ij} - \phi\mu_j \right) s_{jt} \\
 &= - \sum_j (\phi\mu_j - \phi\mu_j) s_{jt} = 0
 \end{aligned}$$

for all values of s_{jt} ($j = 1, \dots, n$). Hence the sum of the disturbance terms of all equations is zero. This is consistent with the adding-up property of our complete set of demand equations. The sum over i of the left-hand side of the demand equations (3.15) has to be identically equal to the sum over i of the right-hand side. This is assured by (2.6), (2.19), (2.22) and (3.17). Another consequence of (3.17) is that the covariance matrix of the v_{it} for a given t is singular, since the v_{it} are linearly dependent.

For the specification of the first- and second-order moments of the v_{it} we will make use of (3.16). Assuming that $\xi(s_{jt}) = 0$ for all i and t implies

$$(3.18) \quad \xi(v_{it}) = 0 \quad \text{all } i \text{ and } t$$

An alternative assumption for $\xi(s_{jt})$ would be: $\xi(s_{jt}) = \alpha_j$ independent of t . This means that we assume a trendlike shift in the linear part of the utility function. This shift gives rise to constant terms of the type $\sum (v_{ij} - \phi\mu_i \mu_j) \alpha_j$ in all demand equations. Another approach could be to make $\xi(s_{jt})$ dependent on some other variables like inventories. This would lead to the presence of these variables as explanatory variables in the demand equations. We will not explore these possibilities further in this article but maintain the original assumption: $\xi(s_{jt}) = 0$ and hence (3.18). Our demand equations have therefore no constant terms and no other explanatory variables than income and prices.

With respect to the second-order moments we will introduce the simplifying assumption of absence of correlation over time between the s_{jt} for all j and t , which implies that

$$(3.19) \quad E(v_{it}, v_{jt'}) = 0 \quad t \neq t', \quad \text{all } i \text{ and } j$$

For the contemporaneous second-order moments of the s_{jt} we will assume:

$$(3.20) \quad E(s_t s_t') = (\sigma^2/\phi) N^{-1}$$

where σ^2 is a positive scalar. Let $v_t = [v_{it}]$, i.e., the n -element vector of disturbances in period t so that according to (3.16):

$$(3.21) \quad v_t = - (N - \phi \mu \mu') s_t$$

We then have for the contemporaneous covariance matrix of v_t :

$$\begin{aligned} (3.22) \quad E(v_t v_t') &= (N - \phi \mu \mu') E(s_t s_t') (N - \phi \mu \mu') \\ &= (\sigma^2/\phi) [(N - \phi \mu \mu') N^{-1} (N - \phi \mu \mu')] \\ &= (\sigma^2/\phi) [(I - \phi \mu \mu' N^{-1}) (N - \phi \mu \mu')] \\ &= (\sigma^2/\phi) [N - \phi \mu \mu' - \phi \mu \mu' + \phi^2 \mu \mu' N^{-1} \mu \mu'] \end{aligned}$$

Because of (3.11) $\phi \mu' N^{-1} \mu = \mu' N^{-1} N \mu = \mu' \mu = 1$ since the sum over i of all μ_i equals one, according to (2.6). Thus the last term between square brackets on the last line of (3.22) becomes $+\phi \mu \mu'$ and we have

$$(3.23) \quad E(v_t v_t') = (\sigma^2/\phi) (N - \phi \mu \mu')$$

i.e., the matrix of the substitution effect of the prices doubles as the covariance matrix of the disturbance terms, apart from a scalar. This result is due to Theil [15].

The choice of (3.20) for the covariance matrix of s_t can be justified as follows. Assume again for the moment that income and prices are kept constant. The vector s has been defined before to be equal to $[1/E(\lambda)] P^{-1} (\Delta a)$, where P is a diagonal matrix with the (positive) elements of the price vector as diagonal elements. Hence,

$$E(ss') = \frac{1}{[E(\lambda)]^2} P^{-1} E[(\Delta a)(\Delta a)'] P^{-1}$$

from which follows that

$$(3.24) \quad E[(\Delta a)(\Delta a)'] = [E(\lambda)]^2 P E(ss') P = \frac{\sigma^2 [E(\lambda)]^2}{\phi} \cdot P N^{-1} P$$

where use has been made of (3.20). According to (3.4)

$$(3.25) \quad N^{-1} = \frac{m}{E(\lambda)} P^{-1} U P^{-1}$$

Therefore,

$$(3.26) \quad E[(\Delta a)(\Delta a)'] = \frac{\sigma^2 E(\lambda) m}{\phi} U$$

A sufficient condition for maximizing utility is that U , the matrix of second-order derivatives of the utility function, is negative definite. Since λ and m are always positive this condition implies that N and N^{-1} are also negative definite [see (3.4) or (3.5)]. This means that ϕ , defined as being equal to $-N$, is negative. Covariance matrix (3.26) is therefore a positive definite matrix, because on the right-hand side we have negative definite matrix multiplied by a negative scalar. Apart from this negative scalar the variances and covariances of the initial random shocks in the marginal utilities are equal to the corresponding elements of the matrix of second-order derivatives of the utility function. If the marginal utility of the i^{th} commodity is stable in the sense that a change in the quantity of that commodity has little effect on its marginal utility, i.e., $|\partial^2 u / \partial q_i^2|$ relatively small, then according to (3.26) it is also stable in the sense that the initial random shocks have a relatively small variance. If u_i does not change because of a change in q_j ($\partial^2 u / \partial q_i \partial q_j = 0$), i.e., if commodities i and j are want independent then also the initial shocks are uncorrelated according to (3.16). If the commodities i and j are closely related in the sense that more of j means a substantial decrease in the marginal utility of i - as one can expect to be for instance the case for veal and beef - then, according to (3.26) the initial random shocks in the marginal utilities of i and j have a relatively high positive correlation. Specification (3.26) therefore provides a plausible and useful link between the systematic and random variations in the marginal utilities of all n commodities.

Returning to the covariance matrix for the disturbances (3.23), it may be remarked that this covariance matrix is positive semidefinite with rank $n - 1$. The proof of this statement will not be given here, but it may be noted that postmultiplication of the matrix on the right-hand side of (3.23) by ι results in a zero-vector, because

$$(3.27) \quad (N - \phi \mu \mu') \iota = N \iota - \phi \mu (\mu' \iota) = \phi \mu - \phi \mu = 0$$

where use has been made of (3.11) and (2.6). The same result is obtained by premultiplication of (3.16) by ι' . This property is consistent with (3.17) where it was shown that the sum over i of all disturbance terms is identically equal to zero.

Furthermore, it is easily seen that under the condition of additivity of the utility function (3.23) implies rather simple formulations for the variance and covariances of v_{it} . In that case all $v_{ij} = 0$ for $i \neq j$, and $v_{ii} = \phi \mu_i$ according to (2.12), whence

$$(3.28) \quad \text{var } v_{it} = (\sigma^2/\phi)(v_{ii} - \phi \mu_i^2) = \sigma^2 \mu_i (1 - \mu_i)$$

$$(3.29) \quad \text{cov}(v_{it}, v_{jt}) = (\sigma^2/\phi)(v_{ij} - \phi \mu_i \mu_j) = -\sigma^2 \mu_i \mu_j \quad i \neq j$$

This corresponds with a multinomial distribution of the v_{it} with the marginal value shares as the probabilities.

Finally we will make use of the assumption that as far as prices and total available income are stochastic, they are independent of random fluctuations in the marginal utilities. Empirically, the response of prices and income to stochastic fluctuations in the marginal utilities and quantities demanded will be of very minor importance only. It may be possible to construct some examples for which this is not the case, but these will be of an exceptional nature. The assumption of independence implies zero covariance between the disturbance terms and the explanatory variables in (3.15)

It should be noted that because of the linear dependence of the disturbance terms and the restrictions on the coefficients our system of n equations consists actually of $n - 1$ independent equations. By deleting one equation no information is lost. To retain symmetry in the treatment of the different equations and to make clear that the consumer's decision problem is really an allocation problem, which is the cause of this dependence, we shall in this article continue to work with the complete n -equation system.

4. DEMAND EQUATIONS FOR COMPOSITE COMMODITIES

Since for several reasons large systems of demand equations are difficult to handle it is practical to reduce the number of equations by combining the equations of closely related commodities. If the utility function can be written as a sum of the utilities of groups of commodities such that each commodity belongs to one group only, a

natural way of combining commodities, and hence equations, is to construct composite commodities which correspond with these groups. In this section we shall show that under this condition of block additivity of the utility function rather simple demand equations for composite commodities result from linear aggregation of the demand equations for the relevant elementary commodities when suitable definitions of the aggregate variables are used.

As has been indicated in Section 2 (iv) block additivity or block independence means that U - and therefore U^{-1} and N - is block diagonal. This implies that the coefficient v_{ij} in the demand equation for the i^{th} elementary commodity is zero if j does not belong to the same group as i . Let us for the sake of convenience consider a group of only two commodities, commodities 1 and 2 say. Then $v_{ij} = 0$ for $i = 1, 2$ and $j > 2$. The demand equations for these commodities are

$$(4.1a) \quad w_{1t}^* Dq_{1t} = \mu_1 Dq_t + v_{11} D\bar{p}_{1t} + v_{12} D\bar{p}_{2t} + v_{1t}$$

$$(4.1b) \quad w_{2t}^* Dq_{2t} = \mu_2 Dq_t + v_{21} D\bar{p}_{1t} + v_{22} D\bar{p}_{2t} + v_{2t}$$

with $v_{12} = v_{21}$ and because of (2.9)

$$(4.2) \quad v_{11} + v_{12} = \phi\mu_1, \quad v_{21} + v_{22} = \phi\mu_2$$

The sum of (4.1a) and (4.1b) is given by

$$\begin{aligned} (4.3) \quad w_{1t}^* Dq_{1t} + w_{2t}^* Dq_{2t} \\ &= (\mu_1 + \mu_2) Dq_t + (v_{11} + v_{21}) D\bar{p}_{1t} + (v_{12} + v_{22}) D\bar{p}_{2t} + v_{1t} + v_{2t} \\ &= (\mu_1 + \mu_2) Dq_t + \phi\mu_1 D\bar{p}_{1t} + \phi\mu_2 D\bar{p}_{2t} + v_{1t} + v_{2t} \end{aligned}$$

Let

$$(4.4) \quad W_{gt}^* = \sum_{i \in S_g} w_{it}^*$$

be the sum of the value shares for the commodities belonging to group S_g . Let furthermore

$$(4.5) \quad DQ_{gt} = \sum_{i \in S_g} \frac{w_{it}^*}{W_{gt}^*} Dq_{it}$$

DQ_{gt} can be interpreted as the log-change in the volume-index of consumption on S_g [Cfr. (2.19)]. The left-hand variable of (4.3) is then $W_{1t} DQ_{1t}$, assuming that commodities 1 and 2 form together S_1 . We also have

$$(4.6) \quad M_g = \sum_{i \in S_g} \mu_i$$

which can be interpreted as the marginal value share of S_g , while

$$(4.7) \quad V_g = \sum_{i \in S_g} v_{it}$$

Furthermore, we define the deflated aggregate price variable as follows

$$(4.8) \quad \begin{aligned} D\bar{P}'_{gt} &= \sum_{i \in S_g} \frac{\mu_i}{M_g} D\bar{p}'_{it} \\ &= \sum_{i \in S_g} \frac{\mu_i}{M_g} Dp_{it} - \sum_{k=1}^n \mu_k Dp_{kt} \end{aligned}$$

[cfr. (2.3)], which can also be written as

$$(4.9) \quad D\bar{P}'_{gt} = \sum_{i \in S_g} \frac{\mu_i}{M_g} Dp_{it} - \sum_h M_h \sum_{k \in S_h} \frac{\mu_k}{M_h} Dp_{kt}$$

By using these definitions we can rewrite (4.3) in the following way:

$$(4.10) \quad W_{1t}^* DQ_{1t} = M_1 Dq_t + \phi M_1 D\bar{P}'_{1t} + V_{1t}$$

or more generally

$$(4.11) \quad W_{gt}^* DW_{gt} = M_g Dq_t + \phi M_g D\bar{P}'_{gt} + V_{gt}$$

It may be noted that this demand equation for S_g is an analogue to the demand equation for an elementary commodity under additivity of the utility function, because in that case we have in an equation like (3.15) all $v_{ij} = 0$ for $i \neq j$ and $v_{ii} = \phi \mu_i$.

The analogue also extends to the moments of the disturbance terms V_{gt} . Obviously if for all i and t $\varepsilon v_{it} = 0$ we have

$$(4.12) \quad E V_{gt} = E \sum_{i \in S_g} v_{it} = \sum_{i \in S_g} E v_{it} = 0$$

Furthermore we have

$$(4.13) \quad \text{var } V_{gt} = \sum_{i \in S_g} \text{var } v_{it} + \sum_{\substack{i, j \in S_g \\ i \neq j}} \text{cov}(v_{it}, v_{jt})$$

which becomes on the basis of (3.23)

$$(4.14) \quad \text{var } V_{gt} = (\sigma^2/\phi) \left[\sum_{i \in S_g} (v_{ii} - \phi \mu_i^2) + \sum_{\substack{i, j \in S_g \\ i \neq j}} (v_{ij} - \phi \mu_i \mu_j) \right]$$

$$= (\sigma^2/\phi) \left[\sum_{i, j \in S_g} v_{ij} - \phi \sum_{i, j \in S_g} \mu_i \mu_j \right]$$

Since under the assumption of block-additivity all $v_{ik} = 0$ for $k \notin S_g$ it follows from (2.9) that

$$(4.15) \quad \sum_{j \in S_g} v_{ij} = \phi \mu_i$$

Making use of this property and of definition (4.6) we can write for (4.14)

$$(4.16) \quad \text{var } V_{gt} = (\sigma^2/\phi) \left[\sum_{i \in S_g} \phi \mu_i - \phi \sum_{i \in S_g} \mu_i M_g \right]$$

$$= \sigma^2 (M_g - M_g^2)$$

which can be compared with (3.8), the variance of v_{it} under complete additivity.

Let S_h be another composite different from S_g . We then have

$$(4.17) \quad \text{cov}(V_{gt}, V_{ht}) = (\sigma^2/\phi) \left[\sum_{i \in S_g} \sum_{k \in S_h} (v_{ik} - \phi \mu_i \mu_k) \right]$$

$$= -\sigma^2 \sum_{i \in S_g} \mu_i \sum_{k \in S_h} \mu_k = -\sigma^2 M_g M_h \quad g \neq h$$

all v_{ik} being zero because by definition $k \notin S_g$. It is easily seen that also (4.17) is a macro-analogue to (3.29), the covariance of disturbance terms for elementary commodities under complete additivity of the utility function.

It may be observed that the simplicity of (4.11) is somewhat misleading. To obtain (4.11) we have made use of the deflated aggregate price variable $\bar{DP}_{gt}$ which according to its definition (4.9) requires knowledge of the partial marginal value shares μ_i/M_g . It would defeat the purpose of aggregation if all these quantities had to be estimated from single demand equations. They can of course be obtained rather simply from budget studies. An alternative way would be to use the assumption that within each group the income elasticities of the elementary commodities are equal, i.e.:

$$(4.18) \quad \eta_{it} = \frac{\mu_i}{w_{it}^*} = E_{gt} \quad \text{for all } i \in S_g$$

Then

$$(4.19) \quad \frac{\mu_i}{M_g} = \frac{\eta_{it} w_{it}^*}{\sum_{i \in S_g} \eta_{it} w_{it}} = \frac{E_{gt} w_{it}^*}{\sum_{i \in S_g} E_{gt} w_{it}} = \frac{w_{it}^*}{w_{gt}^*}$$

so that the partial marginal value shares are equal to the partial average value shares. Assumption (4.18) is rather restrictive in the case when only a very few aggregates, each consisting of a large number of elementary commodities, are used. Therefore, in the empirical part of this article the following procedure has been used to obtain the aggregate price variables

$$(4.20) \quad DP_t' = \sum_{i \in S_g} \frac{\mu_i}{M_g} DP_{it}$$

for the four composites considered. First, (4.19) has been used to construct aggregate price variables for sixteen intermediate aggregates. In the second stage: the construction of the series of aggregates price variables for the four final aggregates out of these sixteen, use has been made of estimates of the partial value shares for the intermediate groups derived from budget studies. In this way it was possible to avoid too much detailed estimation without using assumption (4.18) throughout.

Because of the analogy between the macro-equation (4.11) and the micro-equation (3.15) under conditions of complete additivity we

will in the sequel use the micro-notation even if the dependent variable refers to a composite commodity rather than an elementary commodity. Hence,

$$(4.21) \quad w_{it}^* Dq_{it} = \mu_i Dq_t + v_{ii} \bar{Dp}_{it} + v_{it}$$

where $v_{ii} = \phi \mu_i$, can be a demand equation for an elementary commodity under complete additivity or a demand equation for a composite commodity under block additivity, where the groups correspond with the blocks.

5. LINEARIZING THE DEMAND EQUATIONS

Even if the assumption of additivity is used estimation of demand equations of the type (4.21) is not a trivial matter if all theoretical restrictions are being employed. This becomes clear when we use (2.3) to write

$$(5.1) \quad w_{it}^* Dq_{it} = \mu_i Dq_t + v_{ii} Dp_{it} - v_{ii} \sum_k \mu_k Dp_{kt} + v_{it}$$

with

$$(5.2) \quad v_{ii} = \phi \mu_i \quad \phi = \sum_j v_{jj}$$

In (5.1) the right-hand side is non-linear in the coefficients μ_k and v_{ii} , while (5.2) specifies non-linear relations between μ_i and all v_{ii} . Restriction (5.2) can be used to eliminate μ_i from (5.1). The result is

$$(5.3) \quad w_{it}^* Dq_{it} = \frac{v_{ii}}{\sum_k v_{jj}} Dq_t + v_{ii} Dp_{it} - \frac{v_{ii}}{\sum_k v_{jj}} \sum_k v_{kk} Dp_{kt} + v_{it}$$

which clearly shows that each equation is a non-linear expression in all v_{kk} .

To facilitate estimation of the v_{kk} we shall linearize (5.3) in the coefficients to be estimated. Use will be made of a set of given initial values for the v_{kk} , indicated by v_{kk}^0 , which are all negative. Corresponding with (5.2) we define

$$(5.4) \quad v_{ii}^0 = \phi^0 \mu_i^0 \quad \phi^0 = \sum_j v_{jj}^0$$

The coefficient of Dq_t in (5.3) is then linearized as follows:

$$\begin{aligned}
 (5.5) \quad \frac{v_{ii}}{\sum_j v_{jj}} &\approx \frac{v_{ii}^0}{\phi^0} + \frac{1}{\phi^0} (v_{ii} - v_{ii}^0) - \left(\frac{1}{\phi^0}\right)^2 v_{ii}^0 \sum_j (v_{jj} - v_{jj}^0) \\
 &= \mu_i^0 + \frac{v_{ii}}{\phi^0} - \mu_i^0 - \frac{\mu_i^0}{\phi^0} \sum_j v_{jj} + \mu_i^0 \\
 &= \mu_i^0 + \frac{v_{ii}}{\phi^0} - \frac{\mu_i^0}{\phi^0} \sum_j v_{jj}
 \end{aligned}$$

while we have for the coefficient of Dp_{kt} :

$$\begin{aligned}
 (5.6) \quad \frac{v_{ii}}{\sum_j v_{jj}} v_{kk} &\approx \frac{v_{ii}^0}{\phi^0} v_{kk}^0 \\
 &+ \frac{v_{kk}^0}{\phi^0} (v_{ii} - v_{ii}^0) + \frac{v_{ii}^0}{\phi^0} (v_{kk} - v_{kk}^0) - \left(\frac{1}{\phi^0}\right)^2 v_{ii}^0 v_{kk}^0 \sum_j (v_{jj} - v_{jj}^0) \\
 &= \mu_i^0 v_{kk}^0 + \mu_k^0 v_{ii} - \mu_k^0 v_{ii}^0 + \mu_i^0 v_{kk} - \mu_i^0 v_{kk}^0 - \mu_i^0 \mu_k^0 \sum_j v_{jj} + \mu_k^0 v_{ii}^0 \\
 &= \mu_k^0 v_{ii} + \mu_i^0 v_{kk} - \mu_i^0 \mu_k^0 \sum_j v_{jj}
 \end{aligned}$$

These linearizations are used to obtain the following linearized version of demand equation (5.3):

$$\begin{aligned}
 (5.7) \quad w_{it}^* Dq_{it} &= \mu_i^0 Dq_t + \frac{v_{ii}}{\phi^0} Dq_t - \frac{\mu_i^0}{\phi^0} \sum_j v_{jj} Dq_t + v_{ii} Dp_{it} - v_{ii} \sum_k \mu_k^0 Dp_{kt} \\
 &- \mu_i^0 \sum_k v_{kk} Dp_{kt} + \mu_i^0 \sum_j v_{jj} \sum_k \mu_k^0 Dp_{kt} + v_{it}
 \end{aligned}$$

The approximation error is of second and higher order in the differences between the **true** and initial values of v_{jj} .

By reordering the terms in (5.7) we can obtain the following expression:

$$\begin{aligned}
 (5.8) \quad w_{it}^* Dq_{it} - \mu_i^0 Dq_t &= v_{ii} \left(Dp_{it} + \frac{1}{\phi^0} Dq_t - \sum_k \mu_k^0 Dp_{kt} \right) \\
 &- \mu_i^0 \sum_j v_{jj} \left(Dp_{jt} + \frac{1}{\phi^0} Dq_t - \sum_k \mu_k^0 Dp_{kt} \right) + v_{it}
 \end{aligned}$$

In view of (2.24) we can consider

$$(5.9) \quad D\lambda_t^0 = \frac{1}{\phi^0} Dq_t - \sum_k \mu_k^0 Dp_{kt}$$

as the initial value of the log-change in the marginal utility of income and hence

$$(5.10) \quad Du_{it}^0 = Dp_{it} + D\lambda_t^0$$

as the initial value of the log-change in the marginal utility of commodity i . Using (5.10) we can reformulate (5.8) as follows:

$$(5.11) \quad w_{it}^* Dq_{it} - \mu_i^0 Dq_t = v_{ii} Du_{it}^0 - \mu_i^0 \sum_k v_{kk} Du_{kt}^0 + v_{it}$$

Comparing (5.11) with the original equation (5.1), it can be seen that apart from the substitution of μ_i by μ_i^0 (and hence of $v_{ii}\mu_k = \mu_i v_{kk}$ by $\mu_i^0 v_{kk}$) the most important change is the replacement of the log-changes in the prices by the log-changes in the initial values of the marginal utilities of the commodities. The linearization error in (5.11) is of the second and higher order in the differences between v_{jj} and v_{jj}^0 . In the sequel we will proceed under the - perhaps heroic - assumption that this linearization error is small enough to be neglected.

6. MATRIX FORMULATION OF THE SYSTEM OF LINEARIZED EQUATIONS

As can be seen from (5.11), the linearized version of our demand equations, each equation contains all unknown coefficients v_{kk} . To obtain a unique set of estimates of these coefficients all equations should be estimated simultaneously. Furthermore, because of the interdependence of the disturbance terms of the equations only simultaneous estimation of all equations will produce efficient estimates. For this purpose it will prove to be convenient to reformulate the set of n equation in terms of matrices and vectors. The following definitions of n -element vectors will be used:

$$(6.1) \quad y_t = \begin{bmatrix} w_{1t}^* Dq_{1t} - \mu_1^0 Dq_t \\ w_{2t}^* Dq_{2t} - \mu_2^0 Dq_t \\ \vdots \\ w_{nt}^* Dq_{nt} - \mu_n^0 Dq_t \end{bmatrix}, \quad z_t = \begin{bmatrix} Du_{1t}^0 \\ Du_{2t}^0 \\ \vdots \\ Du_{nt}^0 \end{bmatrix}, \quad \mu_0 = \begin{bmatrix} \mu_1^0 \\ \mu_2^0 \\ \vdots \\ \mu_n^0 \end{bmatrix}, \quad v = \begin{bmatrix} v_{11} \\ v_{22} \\ \vdots \\ v_{nn} \end{bmatrix}$$

We shall make use of the $n \times n$ diagonal matrices Z_t and M_0 , which have the elements of the vectors Z_t and μ_0 , respectively, for diagonal elements.

The system of n demand equations (5.11) can now be written as follows:

$$(6.2) \quad y_t = Z_t v - \mu_0 z_t' v + v_t = (I - \mu_0 z_t') z_t v + v_t$$

where I is the $n \times n$ identity matrix. A further simplification can be obtained by defining the $n \times n$ matrix

$$(6.3) \quad X_t = (I - \mu_0 z_t') Z_t = Z_t - \mu_0 z_t'$$

because then

$$(6.4) \quad y_t = X_t v + v_t$$

It is of some importance to realize that

$$(6.5) \quad z_t' y_t = \sum_i w_{it}^* Dq_{it} - \sum_i \mu_i^0 Dq_t = Dq_t - Dq_t = 0$$

$$(6.6) \quad \iota' X_t = \iota' (I - \mu_0 \iota') Z_t = (\iota' - \iota') Z_t = 0$$

The next step is to formulate the system for all $t = 1, \dots, T$, for which it is considered to be relevant. Let

$$(6.7) \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_T \end{bmatrix}, \quad Z = \begin{bmatrix} z_1^1 \\ z_2^1 \\ \vdots \\ z_T^1 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_T \end{bmatrix}$$

Vectors y and v have nT elements each, while matrix X is of the order of $nT \times n$ and Z , the matrix of initial values of the log-changes in the marginal utilities, is of the order of $T \times n$. The extension of (6.4) for all t is then:

$$(6.8) \quad y = Xv + v$$

We shall assume that X has rank n .² It is possible that the variables and quantities which are used to define X , i.e., Dq_t , Dp_{it} ($i = 1, \dots, n$; $t = 1, \dots, T$), ϕ^0 and μ_i^0 ($i = 1, \dots, n$) are themselves outcomes of a random process. We have already introduced the assumption that the disturbance terms are independent of Dq_t and Dp_{it} for all i and t . We shall also assume that the initial value for ϕ and μ_i are selected independently of the disturbance terms in the demand equations. On the basis of these assumptions Dq_t , Dp_{it} , ϕ^0 and μ_i^0 and hence the elements of the matrix X can be taken as fixed for the purpose of the estimation of the v_{it} .

According to (3.18) and (3.19) we have

$$(6.9) \quad E(v) = 0$$

$$(6.10) \quad E(vv') = \Sigma = I \otimes \Omega$$

² It can be shown (see [4]) that X has rank n , if the rank of Z is at least 2 and no column of Z consists of zero's only. It is easy to check whether the data satisfy these very weak conditions.

where I is the $T \times T$ identity matrix, $\otimes$ denotes the operation of a Kronecker product and Ω is the $n \times n$ covariance matrix of the contemporaneous disturbance terms. Because of (3.17) we have

$$(6.11) \quad t' \Omega = 0 \quad \Omega t = 0$$

In Section 3 we have derived a further specification for Ω . Under the assumption of additivity specification (3.23) amounts to

$$(6.12) \quad \Omega = \sigma^2 (M - \mu \mu')$$

[See also (3.28) and (3.29)], where M is a diagonal $n \times n$ matrix with the elements of the vector μ for diagonal elements. The matrix $\sigma^2 (M - \mu \mu')$ is positive semidefinite with rank $n - 1$. The proof of this is left to the reader.

7. THE ESTIMATOR

To obtain point estimates of the elements of the vector v the following estimator, first proposed by Theil [15], will be used:

$$(7.1) \quad \hat{v} = \left(\sum_t X_t' M_0^{-1} X_t \right)^{-1} \left(\sum_t X_t' M_0^{-1} y_t \right) = (X' H X)^{-1} X' H y$$

where H is a $nT \times nT$ diagonal matrix defined by

$$(7.2) \quad H = I \otimes M_0^{-1}$$

In (7.2) I denotes a $T \times T$ identity matrix. In this section various properties of $\hat{v}$ will be discussed. First, however, it may be noted that $\hat{v}$ is an ordinary least-squares estimator of v in the system (6.4) after both sides of this matrix equation have been premultiplied by $M_0^{-\frac{1}{2}}$, i.e., after both sides of each demand equation have been divided by the square root of the relevant μ_i^0 . We then have

$$(7.3) \quad M_0^{-\frac{1}{2}} y_t = M_0^{-\frac{1}{2}} X_t v + M_0^{-\frac{1}{2}} v_t$$

The double sum over i and t of the squared disturbance terms $v_{it} / \sqrt{\mu_i^0}$ can be written as

$$(7.4) \quad \sum_t v_t' M_0^{-\frac{1}{2}} M_0^{-\frac{1}{2}} v_t = \sum_t v_t' M_0^{-1} v_t = \sum_t (y_t - X_t v)' M_0^{-1} (y_t - X_t v)$$

Differentiating this expression with respect to v leads to

$$(7.5) \quad -2 \sum_t X_t' M_0^{-1} y_t + 2 \sum_t X_t' M_0^{-1} X_t v$$

Substitution of v in (7.5) by $\hat{v}$ gives a zero vector, which implies that $\hat{v}$ minimizes (7.4).

This interpretation of $\hat{v}$ suggests that our estimator is the result of straightforward application of the principle of weighted least-squares. This, however, is not the case. Estimator (7.1) has been chosen because it is a best linear unbiased estimator under certain conditions, which with some luck might come close to being satisfied. As will be shown in Appendix A these conditions are that $\Omega = \sigma^2 (M - \mu\mu')$ and that at the same time $\mu = \mu_0$. In the present context the last condition ($\mu = \mu_0$), in the form of a strict equality, is not very realistic. If μ would be known to be equal to a given μ_0 the estimation problem would be reduced to the estimation of ϕ , since under additivity $v = \phi\mu$. As can be seen from (4.21) the demand equation is then a linear expression in ϕ and an entirely different estimation procedure would be in order. Here, however, our prior information is more restricted. We don't know μ exactly, but μ_0 might be close to it. Intuitively it is clear that then $\hat{v}$ is not too much different from the true minimum variance estimator. Other properties than that of minimum variance do not depend on the assumption that $\mu = \mu_0$ and hence $\Omega = \sigma^2 (M_0 - \mu_0\mu_0')$, as will be shown in the sequel.

The matrix $X'HX$ in (7.1) is of full rank, since X is of rank n , as has been assumed in the previous section, while also the matrix H has full rank. Thus $(X'HX)^{-1}$ is defined and our estimator exists. Furthermore, substitution of y in (7.1) by (6.8) results in:

$$(7.6) \quad \hat{v} = (X'HX)^{-1} X'HXv + (X'HX)^{-1} X'Hv = v + (X'HX)^{-1} X'Hv$$

Because of the fixed nature of X and H and the zero expectation of v we then have:

$$(7.7) \quad E(\hat{v}) = v + (X'HX)^{-1} X'HE(v) = v$$

i.e.: $\hat{v}$ is an unbiased estimator of v .

From (7.6) it follows that the sampling error of $\hat{v}$ is given by:

$$(7.8) \quad \hat{v} - v = (X'HX)^{-1} X'Hv$$

Hence

$$(7.9) \quad (\hat{v} - v)(\hat{v} - v)' = (X'HX)^{-1}X'Hvv'HX(X'HX)^{-1}$$

The covariance matrix of $\hat{v}$ is now

$$\begin{aligned} (7.10) \quad V &= E[(\hat{v} - v)(\hat{v} - v)'] \\ &= (X'HX)^{-1}X'H E(vv') HX (H'HX)^{-1} \\ &= (X'HX)^{-1}X'H (I \otimes \Omega) HX (X'HX)^{-1} \\ &= \left(\sum_t X_t' M_0^{-1} X_t \right)^{-1} \left(\sum_t X_t' M^{-1} \Omega M^{-1} X_t \right) \left(\sum_t X_t' M_0^{-1} X_t \right)^{-1} \end{aligned}$$

where use has been made of (6.10). If the T sets of observations on which the matrix X is based are repeated k times, the covariance matrix of the estimator which uses all these observations is given by $(1/k)V$. This means that for $k \rightarrow \infty$ the variances and covariances of the point estimates converge to zero, i.e.: $\hat{v}$ is a consistent estimator.

As a byproduct of the estimation of v we have estimates for ϕ and μ defined as

$$(7.11) \quad \hat{\phi} = \iota' \hat{v} \quad \hat{\mu} = (1/\hat{\phi}) \hat{v}$$

The variances and covariances of these point estimates also are of interest. The sampling error of $\hat{\phi}$ is given by

$$(7.12) \quad \hat{\phi} - \phi = \iota' (\hat{v} - v)$$

The (linearized) sampling error of $\hat{\mu}$ is

$$\begin{aligned} (7.13) \quad \hat{\mu} - \mu &= (1/\hat{\phi}) \hat{v} - (1/\phi) v \\ &\approx (1/\phi)(\hat{v} - v) - (1/\phi)^2 (\hat{\phi} - \phi) v \\ &= (1/\phi)(\hat{v} - v) - (1/\phi) \mu \iota' (\hat{v} - v) \\ &= (1/\phi)(I - \mu \iota')(\hat{v} - v) \end{aligned}$$

We then can write

$$(7.14) \quad \begin{bmatrix} \hat{v} - v \\ \hat{\phi} - \phi \\ \hat{\mu} - \mu \end{bmatrix} = \begin{bmatrix} I \\ \iota' \\ \frac{1}{\phi} (I - \mu \iota') \end{bmatrix} (\hat{v} - v)$$

The covariance matrix of the vector on the left-hand side of (7.14) is seen to be

$$(7.15) \quad V \begin{bmatrix} \hat{v} \\ \hat{\phi} \\ \hat{\mu} \end{bmatrix} = \begin{bmatrix} I \\ \iota' \\ \frac{1}{\phi} (I - \mu \iota') \end{bmatrix} V [I, \iota', \frac{1}{\phi} (I - \mu \iota')]$$

$$= \begin{bmatrix} V & V\iota' & \frac{1}{\phi} V(I - \iota\mu') \\ \iota'V & \iota'V\iota' & \frac{1}{\phi} \iota'V(I - \iota\mu') \\ \frac{1}{\phi} (I - \mu\iota')V & \frac{1}{\phi} (I - \mu\iota')V\iota & \frac{1}{\phi^2} (I - \mu\iota')V(I - \iota\mu') \end{bmatrix}$$

where V is the covariance matrix of $\hat{v}$ as given in (7.10). In (7.15) all variances and covariance of the estimators of $\hat{v}$, ϕ and μ can be found. Since the variances and covariance of the elements of the vector $\hat{\mu}$ are based on the approximation (7.13), which implies linearization errors of the second and higher degree in $(\hat{\phi} - \phi)$, these variances and covariances are only asymptotically valid. Furthermore, it may be noted that these variances and covariances are dependent on the unknown parameters ϕ and μ . For practical purposes these unknowns will have to be replaced by their estimates. The same applies to the matrix V for which knowledge of Ω is required [see (7.10)]. Also Ω will have to be substituted by its estimate. This matter will be taken up in the next section.

The properties of $\hat{v}$ discussed in this section are all derived under the assumption of absence of linearization errors in the linearized demand equations. This condition will not be satisfied in reality. Because of linearization errors in the demand equations a bias can be expected, while also the property of consistency will be affected. The covariance matrix V should also be adjusted to take account of the presence of this type of error. In this connection the desirability of iterative application of (7.1), where the initial values are replaced by estimates in the previous round, could be discussed. If iterative application of (7.1) converges on the same estimate for v for all reasonable initial vectors v_0 , one obtains a unique estimate for v independent of the choice of the initial vector v_0 . This is obviously an advantage. However, replacement of a vector v_0 , independent of the disturbances, by an estimate, which is correlated with the disturbance terms, has the consequence that the adjusted matrices X and H are not fixed anymore. The sampling error of the final $\hat{v}$ is in that case non-linear in the disturbances and the moments of its distribution are difficult to formulate for small samples, which means that we have not a clear idea about the statistical properties of the final $\hat{v}$. Therefore in this article we will confine ourselves to applying (7.1) only once.

8. ESTIMATION OF THE COVARIANCE MATRIX OF $\hat{v}$

As has been pointed out in the previous section an estimate of Ω is needed to obtain an estimate of V , the covariance matrix of the estimator $\hat{v}$. Some alternative possibilities for estimating Ω present themselves.

(i) According to (6.12) we have $\Omega = \sigma^2(M - \mu\mu')$, in which σ^2 and μ are unknown. The true vector μ could be replaced by its estimate $\hat{\mu}$. However, if the disturbances v_{it} are normally distributed, also the elements of $\hat{v}$ and $\hat{\phi} = \iota'\hat{v}$ are normally distributed. Since the $\hat{\mu}_i = \hat{v}_{ii}/\hat{\phi}$ are ratios of normally distributed variables, negative values for the $\hat{\mu}_i$ have a positive probability even if the true μ_i are all positive. Negative values for $\hat{\mu}_i$ lead to negative estimates of the variances of the disturbance terms [see (3.28)], which is not attractive.

(ii) Negative estimates of these variances can be avoided by using μ_0 in stead of $\hat{\mu}$ to replace μ in (6.12), since μ_0 is a vector of positive elements ($\mu_0 = \phi^0 v_0$, with all elements of v_0 negative and also $\phi^0 = \iota'v_0 < 0$). This is also not attractive. The vector μ_0 is based on initial values only and independent of the number of observations. Hence $M_0 - \mu_0\mu_0'$ cannot even be considered a consistent estimator of $M - \mu\mu'$.

(iii) Another possibility is to estimate Ω by the matrix of sample moments

$$(8.1) \quad \frac{1}{T} \sum_t \hat{v}_t \hat{v}_t'$$

where

$$(8.2) \quad \hat{v}_t = y_t - X_t \hat{v}$$

This matrix is also positive semi-definite for small samples. Because of (6.5) and (6.6) $\iota'\hat{v}_t = 0$. This implies that the rows and the columns of (8.1) add up to zero which corresponds with properties (6.11) for Ω . However, it can be expected that the diagonal elements of (8.1) have a negative bias, for as we have seen in the beginning of the preceding section $\hat{v}$ minimizes $\sum_t v_t' M_0^{-1} v_t$ which is equal to the weighted sum of the diagonal elements of (8.1) where the weights are T/μ_i^0 .

(iv) To take account of this type of bias we estimate Ω by

$$(8.3) \quad \frac{1}{T'} \sum_t \hat{v}_t \hat{v}_t'$$

Here T' is determined to reduce as much as possible the bias resulting from the minimizing procedure. Also (8.3) is positive semi-definite, while its rows and columns add up to zero for the same reason as in the case of (8.1). Most of the rest of this section will be devoted to the determination of T' .

Our starting point will be the transformed system (7.3). The covariance matrix of disturbance terms of this system is

$$(8.4) \quad E[(M_0^{-\frac{1}{2}}v_t)(M_0^{-\frac{1}{2}}v_t)'] = M_0^{-\frac{1}{2}}E(v_tv_t')M_0^{-\frac{1}{2}} = M_0^{-\frac{1}{2}}\Omega M_0^{-\frac{1}{2}}$$

This matrix also appears in (7.10), the expression for V , because we can write

$$(8.5) \quad V = (\sum_t X_t' M_0^{-1} X_t)^{-1} [\sum_t X_t' M_0^{-\frac{1}{2}} (M_0^{-\frac{1}{2}} \Omega M_0^{-\frac{1}{2}}) M_0^{-\frac{1}{2}} X_t] (\sum_t X_t' M_0^{-1} X_t)$$

T' will be determined in such a way that the expectation of the trace of

$$(8.6) \quad M_0^{-\frac{1}{2}} \left(\frac{1}{T'} \sum_t \hat{v}_t \hat{v}_t' \right) M_0^{-\frac{1}{2}}$$

equals the trace of (8.4). In this manner we can correct for the minimization bias mentioned above.

The trace of (8.4) can also be written as:

$$(8.7) \quad \text{tr} (M_0^{-\frac{1}{2}} \Omega M_0^{-\frac{1}{2}}) = \text{tr} (M_0^{-1} \Omega)$$

while we have for the trace of (8.6)

$$(8.8) \quad \text{tr} [M_0^{-\frac{1}{2}} \left(\frac{1}{T'} \sum_t \hat{v}_t \hat{v}_t' \right) M_0^{-\frac{1}{2}}] = \frac{1}{T'} \text{tr} [M_0^{-1} \sum_t \hat{v}_t \hat{v}_t']$$

Then T' is determined by

$$(8.9) \quad T' = \frac{E \text{tr} [M_0^{-1} \sum_t \hat{v}_t \hat{v}_t']}{\text{tr} (M_0^{-1} \Omega)} = \frac{\text{tr} [M_0^{-1} \sum_t E(\hat{v}_t \hat{v}_t')]}{\text{tr} (M_0^{-1} \Omega)}$$

for which a further evaluation of the numerator is needed.

On the basis of (8.2), (6.4) and (7.8) we have

$$(8.10) \quad \begin{aligned} \hat{v}_t &= y_t - X_t \hat{\beta} = X_t \nu + v_t - X_t \hat{\beta} = v_t - X_t (\hat{\beta} - \beta) \\ &= v_t - X_t (X' H X)^{-1} X' H v = v_t - X_t (X' H X)^{-1} \sum_q X_q' M_0^{-1} v_q \end{aligned}$$

where $q = 1, \dots, T$. Hence

$$(8.11) \quad \begin{aligned} \hat{v}_t \hat{v}_t' &= v_t v_t' - X_t (X' H X)^{-1} \left(\sum_q X_q' M_0^{-1} v_q \right) v_t' - v_t \left(\sum_q v_q' M_0^{-1} X_q \right) (X' H X)^{-1} X_t' \\ &\quad + X_t (X' H X)^{-1} \left(\sum_q X_q' M_0^{-1} v_q \right) \left(\sum_q v_q' M_0^{-1} X_q \right) (X' H X)^{-1} X_t' \end{aligned}$$

Since $E(v_t v_q') = 0$ for $q \neq t$ the expectation of (8.11) becomes

$$(8.12) \quad E(\hat{v}_t \hat{v}_t') = \Omega - X_t (X'HX)^{-1} X_t' M_0^{-1} \Omega - \Omega M_0 X_t (X'HX)^{-1} X_t' \\ + X_t (X'HX)^{-1} \left(\sum_q X_q' M_0^{-1} \Omega M_0^{-1} X_q \right) (X'HX)^{-1} X_t'$$

Consequently we have

$$(8.13) \quad \text{tr} [M_0^{-1} \sum_t E(\hat{v}_t \hat{v}_t')] = \\ = T \text{tr} (M_0^{-1} \Omega) - 2 \text{tr} [(X'HX)^{-1} (\sum_t X_t' M_0^{-1} \Omega M_0^{-1} X_t)] \\ + \text{tr} [(X'HX)^{-1} (\sum_q X_q' M_0^{-1} \Omega M_0^{-1} X_q) (X'HX)^{-1} (\sum_t X_t' M_0^{-1} X_t)] \\ = T \text{tr} (M_0^{-1} \Omega) - \text{tr} [(X'HX)^{-1} (\sum_t X_t' M_0^{-1} \Omega M_0^{-1} X_t)]$$

since $X'HX = \sum_t X_t' M_0^{-1} X_t$. Using (8.13) we can write for T'

$$(8.14) \quad T' = T - \frac{\text{tr} [(X'HX)^{-1} (\sum_t X_t' M_0^{-1} \Omega M_0^{-1} X_t)]}{\text{tr} (M_0^{-1} \Omega)}$$

Both traces in (8.14) are positive. Thus $T' < T$, as was to be expected.

For practical purposes we cannot use (8.14) since it is an expression in Ω , the unknown matrix. Replacing Ω by its estimate (8.3) results in

$$(8.15) \quad T'' = T - \frac{\text{tr} [(X'HX)^{-1} \sum_t X_t' M_0^{-1} (\sum_q \hat{v}_q \hat{v}_q') M_0^{-1} X_t]}{\text{tr} [M_0^{-1} (\sum_t \hat{v}_t \hat{v}_t')]} \\ = T - \frac{\text{tr} [\sum_t M_0^{-1} X_t (X'HX)^{-1} X_t' M_0^{-1}] (\sum_t \hat{v}_t \hat{v}_t')}{\text{tr} [M_0^{-1} (\sum_t \hat{v}_t \hat{v}_t')]}$$

which does not contain unknown quantities. However, substitution of T' by T'' in (8.3) implies that the trace of (8.3) is not an unbiased estimator of (8.7). The bias will be small, because the leading term in (8.15) is T , while the second term is of lower order and only in this second term T'' differs from T' . This difference will be of even lower order than the second term itself.

To get an idea about the order of magnitude of T' we can replace Ω in (8.14) by $\sigma^2(M_0 - \mu_0 \mu_0')$ which would be correct if (6.12) holds and at the same time $\mu = \mu_0$. Then we have for the numerator of the fraction in (8.14):

$$\begin{aligned}
 (8.16) \quad & \text{tr} [(X'HX)^{-1} \sum_t X_t' M_0^{-1} \sigma^2 (M_0 - \mu_0 \mu_0') M_0^{-1} X_t] \\
 &= \sigma^2 \text{tr} [(X'HX)^{-1} \sum_t X_t' (M_0^{-1} - \mu \mu') X_t] \\
 &= \sigma^2 \text{tr} [(X'HX)^{-1} \sum_t X_t' M_0^{-1} X_t] = \sigma^2 n
 \end{aligned}$$

because of (6.6). Furthermore the trace in the denominator of (8.14) becomes:

$$\begin{aligned}
 (8.17) \quad & \text{tr} [M_0^{-1} \sigma^2 [M_0 - \mu_0 \mu_0']] = \sigma^2 \text{tr} (I - \mu \mu') \\
 &= \sigma^2 (n - \mu' \mu) = \sigma^2 (n - 1)
 \end{aligned}$$

Thus $T' = T - n/(n - 1)$. The bias correction is somewhat larger than one. This correction can be interpreted as an adjustment for the loss of degrees of freedom, because in $n - 1$ independent equations we estimate n coefficients, which amounts to the estimation of $n/(n - 1)$ coefficient per independent equation.

Our final estimate of Ω in (7.10) is given by

$$(8.18) \quad \frac{1}{T'} \sum_t \hat{v}_t \hat{v}_t'$$

so that we have for the estimate of V :

$$(8.19) \quad V_1 = \frac{1}{T'} (X'HX)^{-1} \left[\sum_t X_t' M_0^{-1} \left(\sum_q \hat{v}_q \hat{v}_q' \right) M_0^{-1} X_t \right] (X'HX)^{-1}$$

while the corresponding estimate for the covariance matrix of all point estimates $(\hat{v}, \hat{\phi}, \hat{\mu})$ is given by

$$(8.20) \quad \tilde{V}_1 = \begin{bmatrix} I \\ \mu' \\ \frac{1}{\hat{\phi}}(I - \hat{\mu} \mu') \end{bmatrix} V_1 \begin{bmatrix} I, \mu, \frac{1}{\hat{\phi}}(I - \hat{\mu} \mu') \end{bmatrix}$$

A number of approximations have been used to obtain these estimators of the covariance matrices. Summing up these amount to:

(i) replacement of Ω in (7.10) by $(1/T') \sum_t \hat{v}_t \hat{v}_t'$, where T' corrects for the minimization bias;

(ii) substitution of T' by T'' because T' still is an expression in the unknown Ω ;

(iii) neglecting the linearization error (in 7.13), which means that (7.15) holds only asymptotically for the variances and covariances of the elements of μ ;

(iv) replacement of V , ϕ and μ in (7.15) by their estimates.

One might argue that since (8.18) as an estimate of Ω does not make use of the theoretical covariance specification (6.12), the whole argument of Section 3 leading up to this specification is superfluous. This conclusion is not correct. The theoretical covariance specification (6.12) plays an important rôle in the derivation of the minimum variance property of $\hat{v}$, while it also serves to obtain an idea about the order of magnitude of T' . Since its major disadvantage is that it is expressed in terms of the unknown elements of μ , it can be used more fully in situations, not considered here, where the μ -vector is given. Furthermore, we can by using (6.12) obtain in principle another estimate of μ from the numerical results for (8.18). This certainly will not be very simple and is therefore not undertaken here.

9. THE DATA

The estimators of the two preceding sections will be applied to a system of four demand equations describing the consumption of four major groups of commodities and services in the Netherlands in the periods 1923-1939 and 1950-1961. These four major groups are 1. Food, 2. Pleasure Goods (i.e. drinks, tobacco, confectionery, sweets and so on), 3. Durables and 4. Other Commodities and Services, to be called Remainder. The data for these aggregates have been derived from [3] which contains series on annual expenditure and prices for 99 commodities for the period 1922-1939 and for 108 commodities for the period 1949-1961. For the purpose of this article these commodities are considered to be elementary commodities. These 99 and 108 commodities are in [3] grouped to form sixteen broad categories of consumption. Out of these sixteen composite commodities the four major groups have been constructed.³

The log-changes for expenditure and for prices of the elementary commodities make it possible to compute log-changes for the quantities. By means of (4.5) the log-changes in the volume indices of consumption of the 4 groups have been computed. To take into account the change in the size of the population the log-changes in the mid-year population have been subtracted from the log-changes in the volume indices. The results, the log-changes in the volume indices of consumption per capita, are next multiplied by the appropriate average value shares to obtain the values for the dependent variables of the demand equations. These values are presented in Table 9 of Appendix B, where also the time series for the log-change in the volume index of total consumption per capita is given.

³ The same data have been used for Theil and Mnookin's study [17] and Theil's monograph [16]. Note however, that the two-stage procedure in constructing the log-changes in the price indices was not used for [16] and [17]. Therefore the observations on the macro price variables given in Table 10 are different from those given in [16].

As has been indicated in Section 4 the construction of the log-changes in the price indices of the four groups has been performed in two stages. First, the log-changes in the price indices of the sixteen intermediate groups have been computed by taking the weighted sum of the log-changes in the price-indices of the elementary commodities belonging to the same group. The weights in this first stage are the partial average value shares. Secondly, the log-changes in the price indices of the four major groups have been determined by taking the weighted sum of the log-changes in the price indices of the intermediate groups belonging to the same major group. The weights are partial marginal value shares which were derived from Barten and Turnovsky [5]. The results are presented in Table 10 of Appendix B.

The two tables of Appendix B summarize all the empirical material used for the estimation of the model. These tables also contain data for some other years which have not been used in the estimation, because they refer either to obviously abnormal situations or to preliminary data (the last two years: 1962 and 1963). These observations can serve as testing ground for the numerically specified model.

10. NUMERICAL RESULTS: POINT ESTIMATES⁴

To apply the estimator (7.1) we do not only need the data described in the preceding section but also initial values for the v_{ii} . Since $\mu_i = v_{ii}/\phi$, where ϕ is the sum of all v_{ii} , it is sufficient to determine initial values for the μ_i and ϕ . For our purpose we have used $\phi^0 = -0,5$. This value corresponds with the value suggested by Frisch [6] for ϕ in the case of consumers in the median income bracket. To obtain initial values for the μ_i use has been made of an empirical study by Houthakker [9], where estimates of income elasticities for 11 European countries, the U.S.A. and Canada, taken together, are given. On the basis of these estimates and the appropriate value shares the following initial marginal value shares have been computed: 1. Food 0.21, 2. Pleasure Goods 0.08, 3. Durables 0.36 and 4. Remainder 0.35. Application of (7.1) and (7.11) produced the estimates given in Table 1.

In this table the estimated standard errors (the square roots of the diagonal elements of the estimated covariance matrix given in Table 7) are given in parentheses. All results for the v_{ii} are negative. This means that the assumption of a negative definite U matrix is not contradicted by these estimates. [As can be seen from (3.5) there is a straightforward relation between the diagonal elements

⁴ Many thanks are due to Mr. P.A.M. Zwolsman who has directed most of the computations leading to the results given in this and the next two sections.

TABLE 1. POINT ESTIMATES

	$\hat{v}_{ii}$	$\hat{\mu}_i$
1. Foods	-0.0660 (0.0243)	0.2061 (0.0488)
2. Pleasure Goods	-0.0307 (0.0080)	0.0959 (0.0179)
3. Durables	-0.1619 (0.0342)	0.5058 (0.0680)
4. Remainder	-0.0615 (0.0382)	0.1922 (0.0752)
ϕ	-0.3201 (0.0909)	

of N , the v_{ii} , and the diagonal elements of U in the case of additivity where both U and N are diagonal. Negative diagonal elements of N imply negative diagonal elements of U .] With the exception of $\hat{v}_{44}$ all estimates are more than twice the corresponding estimated standard error. The estimate for ϕ , viz. -0,32, differs considerably from the value suggested by Frisch. The significance of this difference is doubtful in view of the large value for the estimated standard error. Our result would mean that an increase of real income by one percent is associated with a decrease in the marginal utility of income by slightly more than three percent ($1/\hat{\phi} = 3.125$). Among the $\hat{\mu}_i$ the estimated marginal value share for Durables dominates. Its difference with its initial value is sizable as is the case for the estimated marginal value share for Remainder.

A better idea of what these point estimates mean can be obtained from the values of elasticities which they imply. In the approach of this article elasticities are not constants but vary from year to year. We shall give the implied values for the elasticities for four years, viz. 1928, 1938, 1951 and 1961. The income elasticities are computed by dividing the $\hat{\mu}_i$ by w_{it}^* (see Section 2 (i)). The results are given in Table 2.

TABLE 2. INCOME ELASTICITIES FOR SELECTED YEARS

	1928	1938	1951	1961
1. Food	0.66	0.70	0.71	0.77
2. Pleasure Goods	1.06	1.11	0.94	0.95
3. Durables	2.14	2.46	1.83	1.87
4. Remainder	0.54	0.46	0.58	0.53

Variations in the values of the estimated income elasticities of Table 2 over the years are entirely due to variations in the w_{it}^* .

Obviously Durables are luxuries, according to the usual meaning of this term in this connection. The income-elasticities for Food are well below 1, thus confirming another time the well-known law of Engel about the decreasing share of expenditure on food. Pleasure Goods seem to belong in the category of semi-luxuries. The income elasticities for Remainder are strikingly low. This group encloses next to fuel, public utilities, rent, also services. As a whole this group seems to be more "necessary" than food. Only a more detailed analysis of the income elasticities of the components of this group can indicate to which specific type of expenditure the low values of the income elasticities are due.

As can be seen from (2.4) the direct price elasticities are given by

$$(10.1) \quad \frac{\partial(\ln q_i)}{\partial(\ln p_i)} = \frac{v_{ii} - v_{ii}\mu_i - \mu_i w_i}{w_i}$$

In the case of additive utilities, where $v_{ii} = \phi\mu_i$ (10.1) can be re-written as

$$(10.2) \quad \frac{\partial(\ln q_i)}{\partial(\ln p_i)} = \frac{\mu_i}{w_i} [\phi(1 - \mu_i) - w_i]$$

from which the relation between the direct price elasticity and the income elasticity (μ_i/w_i) is clear. For commodities where μ_i and w_i are small, as is the case for Pleasure Goods, the direct price elasticity is almost equal to $\phi\mu_i/w_i$, which is the ratio of the income elasticity of commodity i to the income elasticity of the marginal utility of income ($= 1/\phi$). In those cases the specific substitution effect of price changes dominates the two other components viz., the general substitution effect and the income effect, which are represented in (10.2) by $-\phi\mu_i^2/w_i$ and $-\mu_i$, respectively.

TABLE 3. DIRECT PRICE ELASTICITIES FOR SELECTED YEARS

	1928	1938	1951	1961
1. Food	-0.37	-0.38	-0.39	-0.40
2. Pleasure Goods	-0.40	-0.42	-0.37	-0.37
3. Durables	-0.84	-0.89	-0.80	-0.80
4. Remainder	-0.33	-0.31	-0.34	-0.33

Using the estimates for ϕ and μ_i and the observed values for the value shares in (10.2) direct price elasticities have been computed. Table 3 presents the results. All direct price elasticities have values between 0 and -1. The four groups are all demand inelastic. This one would expect to be the case for these highly aggregated composite commodities with few if any close substitutes. Here also Durables dominates, while the direct price elasticities for the other groups are almost equal.

If we add to the values given in Table 3 the point estimates for the μ_i we obtain estimates of the component of the direct price elasticities which corresponds with the total substitution effect, since the component corresponding with the income effect is $-\mu_i$. With the exception of Pleasure Goods, the $\hat{\mu}_i$ of Table 1 are slightly more than half the values for the direct price elasticities, which means that income effect and substitution effect for the other three groups are almost equally strong. The low value of $\hat{\mu}_2$ for Pleasure Goods accounts for the domination of the substitution effect for that group. In general, one can expect a relatively strong substitution effect for commodities with a small value share and a not too low income elasticity.

11. THE ESTIMATED COVARIANCE MATRIX OF THE STOCHASTIC COMPONENTS

The contemporaneous covariance matrix of the disturbance terms is estimated by

$$(11.1) \quad \frac{1}{T''} \sum_t \hat{v}_t \hat{v}_t'$$

where T'' is defined by (8.15). In our case T , the total number of sets of observations used for estimation is 29, while

$$(11.2) \quad \frac{\text{tr} \left[\sum_t M_0^{-1} X_t (X' H X)^{-1} X_t' M_0^{-1} \right] \left(\sum_t \hat{v}_t \hat{v}_t' \right)}{\text{tr} \left[M_0^{-1} \left(\sum_t v_t v_t' \right) \right]} = \frac{0.0139475}{0.0101424} = 1.3752$$

Thus

$$(11.3) \quad T'' = 29 - 1.3752 = 27.6248$$

This value can be compared with $29 - 1.3333 = 27.6667$, which is the value of T' in the case that $\Omega = \sigma^2 (M_0 - \mu_0 \mu_0')$ as has been shown in Section 8.

The numerical results for (11.1) are given in Table 4.

TABLE 4. THE ESTIMATED COVARIANCE MATRIX OF DISTURBANCES^a

	1. Food	2. Pleasure Goods	3. Durables	4. Remainder
1. Food	3239	232	-1983	-1488
2. Pleasure Goods	232	391	-471	-152
3. Durables	-1983	-471	-3325	-870
4. Remainder	-1488	-152	-870	2510

^a All entries are to be multiplied by 10^{-8}

With the exception of the covariance between Food and Pleasure Goods all covariances are negative. This is more or less to be expected since our estimate satisfies the property of Ω , the true covariance matrix, that its rows and columns add up to zero. Since the variances are positive, at least one covariance in each row and column has to be negative. To get a better picture of the relative importance of the covariances Table 5 presents the correlation coefficients of the disturbance terms, obtained by dividing the covariances of Table 4 by the square root of the product of the corresponding variances.

TABLE 5. ESTIMATED CORRELATION COEFFICIENTS OF DISTURBANCES

	1. Food	2. Pleasure Goods	3. Durables	4. Remainder
1. Food	1	0.207	-0.604	-0.522
2. Pleasure Goods	0.207	1	-0.413	-0.153
3. Durables	-0.604	-0.413	1	-0.301
4. Remainder	-0.522	-0.153	-0.301	1

Out of the six correlation coefficients three are more than 0.4 in absolute value, which indicates that it pays not to neglect the covariances of the disturbance terms of the demand equations for different commodities. It should be noted that the positive sign of the covariance between Food and Pleasure Goods is in contradiction with the theoretical covariance specification given in (3.29). However, its small size - also relatively - raises some doubts with respect to the significance of this result.

The variances of Table 4 give an idea of the importance of the stochastic or residual component of consumer behavior, i.e. the component which cannot be explained by changes in income and prices. The absolute size of these variances is less informative than the ratio of these variances to the sample second moments of the dependent variables of the demand equations. The following quantities

$$(11.4) \quad 1 - R_i^2 = \frac{\sum_t \hat{v}_{it}^2}{\sum_t (w_{it}^* D_{q_{it}})} \quad i = 1, \dots, n$$

have been computed once for the whole period, once for the subperiod before World War II (indicated by Period I) and once for the years from 1950 through 1961 (indicated by Period II). The share of the residual component as measured by (11.4) is 1 if income and price changes cannot describe $w_{it}^*D_{q_{it}}$ at all. It is zero if all residuals are zero. R_i^2 is the sample coefficient of determination where it is to be noted that the $w_{it}^*D_{q_{it}}$ are not deviations from their sample mean, since our demand equations don't contain an intercept. The results for (11.4) are given in Table 6.

TABLE 6. RESIDUAL COMPONENT SHARES

	Total Period	Period I	Period II
1. Food	0.427	0.437	0.413
2. Pleasure Goods	0.286	0.320	0.253
3. Durables	0.118	0.204	0.081
4. Remainder	0.296	0.477	0.126

According to these residual shares random fluctuations are of most relative importance for Food, while they play a remarkably minor rôle in the case of Durables. The shares for Period I are all smaller than the corresponding shares for Period II. This might be caused by the better quality of the data for Period II, but it should also be realized that in Period II the values for the dependent variables are larger on the average than in Period I (Cfr. Table 9 in Appendix B). With the exception of Food, changes in income and prices are of paramount significance in the explanation of changes in demand. It is a subject for further research why Food differs in this respect from the other groups.

12. MISCELLANEOUS RESULTS

The use of the estimated covariance matrix of the disturbances as given in Table 4 makes it possible to compute the estimated covariance matrix of the point estimates $\tilde{V}_1$ [see (8.20)]. This matrix is presented in Table 7. The estimated standard errors given in Table 1 are the square roots of the diagonal elements of this matrix. As has been explained in Section 8 the estimates of the variances and covariances given in Table 7 are the result of a number of approximations especially in the case of the variances and covariances of the $\hat{\mu}_i$. The small variances of $\hat{v}_{22}$ and $\hat{\mu}_2$ attract attention. These low values are due to the rather low results for the variances and covariances of the disturbances in the demand equation for Pleasure Goods (Cfr. Table 4). All covariances between the $\hat{v}_{ii}$ are positive. This results in a rather large

variance for $\hat{\phi}$, being the sum of all $\hat{v}_{ii}$. Furthermore, it is easy to verify that the sums of the last four columns and of the last four rows in Table 7 are zero vectors. This stems from the property that the sum of all $\hat{\mu}_i$ equals one.

TABLE 7. ESTIMATED COVARIANCE MATRIX OF POINT ESTIMATES^a

	$\hat{v}_{11}$	$\hat{v}_{22}$	$\hat{v}_{33}$	$\hat{v}_{44}$	$\hat{\phi}$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	$\hat{\mu}_4$
$\hat{v}_{11}$	592	126	448	530	1696	-757	113	1281	-638
$\hat{v}_{22}$	126	64	162	205	557	-36	-32	373	-305
$\hat{v}_{33}$	448	162	1211	999	2820	417	338	674	-1428
$\hat{v}_{44}$	530	205	999	1462	3196	402	318	1929	-2649
$\hat{\phi}$	1696	557	2820	3196	8270	26	737	4258	-5021
$\hat{\mu}_1$	-757	-36	417	402	26	2381	120	-1261	-1241
$\hat{\mu}_2$	113	-32	338	318	737	120	321	110	-550
$\hat{\mu}_3$	1281	373	674	1929	4258	-1261	110	4623	-3472
$\hat{\mu}_4$	-638	-305	-1428	-2649	-5021	-1241	-550	-3472	5263

^a All entries are to be multiplied by 10^{-6}

Tables 9 and 10 of Appendix B give next to the 29 sets of observations used for estimation four other sets of values for the dependent variables and the log-changes in income and prices. For various reasons these values have not been used for estimation. They can serve now to give the explanatory power of the model in its numerically specified form a primitive test. In Table 8 one finds the residuals $\tilde{v}_{it}$ which remain if from the dependent variables the left-hand sides of (2.1) are subtracted, where use is made of the point estimates described earlier. On the last line of Table 8 the square roots of the diagonal elements of Table 4, i.e. the estimated standard deviation of the disturbances in the estimation period, are given to make comparison easy. The

TABLE 8. EXTRAPOLATION RESIDUALS^a

Years	1.Food	2.Pleasure Goods	3.Durables	4.Remainder
1922	117	-55	35	-98
1949	90	-6	138	-222
1962	4	-2	-31	28
1963	-16	-2	17	1
Standard deviation estimation period	57	20	58	50

^a All entries are to be multiplied by 10^{-4}

extrapolation is less a success for the 1922 and 1949 sets than for the last two. The former yearly transitions are more or less abnormal. For instance, rationing was almost completely abolished in the course of 1948 and 1949. The "normal" observations 1962 and 1963 are well predicted on the basis of our model, at least compared with the standard deviations of the residuals in the estimation period. Stochastic fluctuations seem to have been not very important for those years. In this connection it is good to realize that our model is not in the first place a prediction model. The rôle played by the stochastic components is essential and is not necessarily to be minimized in our approach, though it must be granted that the presence of these components reduces the possibilities of precision in forecasting. Further work on the explanation of the "random" fluctuations in the preferences is needed to make our system of demand equations more suitable for prediction.

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A P P E N D I X A

MINIMUM VARIANCE PROPERTIES OF (7.1)

In this Appendix it will be shown that (7.1) is a best linear unbiased estimator of ν if

$$(A.1) \quad \Omega = \sigma^2(M_0 - \mu_0\mu_0')$$

Consider for that purpose the $n \times Tn$ non-stochastic matrix A for which holds:

$$(A.2) \quad AX = I$$

The estimator Ay is a linear unbiased estimator of ν . It is obviously linear in y . Its sampling error is given by

$$(A.3) \quad Ay - \nu = AX\nu + Av - \nu = Av$$

where use is made of (6.8) and (A.2). Because of $E(v) = 0$ also $AE(v) = 0$ and hence

$$(A.4) \quad E(Ay) = \nu$$

This establishes the absence of bias. The covariance matrix of Ay is given by

$$(A.5) \quad \begin{aligned} E[(Ay - \nu)(Ay - \nu)'] &= E[(Av)(Av)'] = AE(vv')A' \\ &= A \Sigma A' = A(I \otimes \Omega)A' \end{aligned}$$

where (6.10) has been used.

Next, consider another $n \times Tn$ non-stochastic matrix G defined as

$$(A.6) \quad G = A - (X'HX)^{-1}X'H$$

Because of (A.2) we have

$$(A.7) \quad GX = AX - (X'HX)^{-1}X'HX = I - I = 0$$

Using (A.6) we can write

$$(A.8) \quad Ay = (X'HX)^{-1}X'Hy + Gy = \hat{\nu} + Gy$$

G_y is the difference between (7.1) and A_y , which represents the whole class of linear unbiased estimators of v . Again using (A.6) we can write for the covariance matrix of A_y :

$$(A.9) \quad E[(A_y - v)(A_y - v)'] = (X'HX)^{-1}X'H(I \otimes \Omega)HX(X'HX)^{-1} \\ + G(I \otimes \Omega)HX(X'HX)^{-1} + (X'HX)^{-1}X'H(I \otimes \Omega)G' + G(I \otimes \Omega)G'$$

The first term on the right-hand side of (A.9) is the covariance matrix (7.10) of $\hat{v}$. Replacing Ω by $\sigma^2(M_0 - \mu_0\mu_0')$ in this term gives

$$(A.10) \quad (X'HX)^{-1}X'H(I \otimes \Omega)HX(X'HX)^{-1} = \sigma^2(X'HX)^{-1}$$

since

$$\begin{aligned} X'H(I \otimes \Omega)HX &= \sum_t X_t' M_0^{-1} \Omega M_0^{-1} X_t \\ &= \sigma^2 \sum_t X_t' M_0^{-1} (M_0 - \mu_0 \mu_0') M_0^{-1} X_t \\ &= \sigma^2 \sum_t X_t' (M_0^{-1} - \mu \mu') X_t \\ &= \sigma^2 \sum_t X_t' M_0^{-1} X_t = \sigma^2 X'HX \end{aligned}$$

where use has been made of the property that $\mu'X_t = 0$ [see (6.6)]. To prove that $\hat{v}$ is a minimum variance estimator if (A.1) holds we will show that the other terms on the right-hand side of (A.9) either vanish or are nonnegative definite.

First we will discuss the second term. Since the third term is the transpose of the second term our conclusions will also apply to the third matrix. If we make use of (A.1) we can write

$$(A.11) \quad G(I \otimes \Omega)HX = G(I \otimes \Omega)(I \otimes M_0^{-1})X = \sum_t G_t \Omega M_0^{-1} X_t \\ = \sigma^2 \sum_t G_t (M_0 - \mu_0 \mu_0') M_0^{-1} X_t = \sigma^2 \sum_t G_t (I - \mu_0 \mu') X_t$$

where the $n \times n$ matrix G_t is implicitly defined by $G = [G_1, \dots, G_T]$. Because of (A.7) $GX = \sum_t G_t X_t = 0$, while also $\mu'X_t = 0$. Thus, the expression on the left-hand side of (A.11) is a zero matrix, which means that the second matrix in (A.9) is a zero matrix too. This result applies also to the third matrix.

The fourth matrix in (A.9), which can also be written as

$$(A.12) \quad G(I \otimes \Omega)G' = G \Sigma G' = \sum_t G_t \Omega G_t'$$

is a positive semi-definite matrix, since Ω is a positive semi-definite matrix. This matrix vanishes if $G = 0$ or if $G \Sigma = 0$ with $G \neq 0$. The

latter case implies $Gv \equiv 0$, because Ω is the covariance matrix of v . Thus, since $GX = 0$

$$(A.13) \quad Gy = GXv + Gv = 0$$

Hence if $G = 0$ or if $G\Sigma = 0$ with $G \neq 0$

$$(A.14) \quad Ay = [X'HX]^{-1}X'H + G]y = (X'HX)^{-1}X'Hy = \hat{v}$$

The covariance matrix of every linear unbiased estimator $Ay \neq \hat{v}$ consists of the sum of the covariance matrix of $\hat{v}$ and the positive semi-definite matrix $G\Sigma G'$. This means that the variance of every linear combination of the elements of $\hat{v}$ is at most equal to the variance of the corresponding combination of the elements of any alternative linear unbiased estimator $Ay \neq \hat{v}$. Therefore $\hat{v}$ is a best linear unbiased estimator.

It is to be noted that the assumption (A.1) is only used to arrive at the conclusion that (A.11) is a zero matrix. In more general terms the necessary condition for the minimum variance property of $\hat{v}$ is that Ω is specified in such a way that for $\Omega_1 = 0$ and $GX = 0$

$$G(I \otimes \Omega)HX + X'H(I \otimes \Omega)G'$$

is a positive semi-definite matrix.

APPENDIX B.

TABLES OF DATA

TABLE 9. OBSERVATIONS ON THE DEPENDENT VARIABLES ($w_{it}^* Dq_{it}$)
AND THE LOG-CHANGES IN THE VOLUME INDEX OF CONSUMPTION PER CAPITA (Dq_t)^a

Years	$w_{it}^* Dq_{it}$ for				Dq_t
	1. Food	2. Pleasure Goods	3. Durables	4. Remainder	
1922	312	-2	425	35	769
1923	7	-32	-95	-61	-180
1924	-36	20	-11	-31	-59
1925	-184	-14	-38	125	-110
1926	150	81	107	-16	322
1927	79	-2	129	21	227
1928	63	34	58	121	276
1929	-36	19	61	96	140
1930	66	18	147	157	389
1931	90	-24	-91	36	11
1932	66	-60	-14	-95	-104
1933	-105	-22	54	-4	-76
1934	-75	-34	-173	-19	-302
1935	6	2	-52	-65	-109
1936	-40	13	217	99	289
1937	-18	10	-52	-46	-107
1938	8	27	-152	36	-81
1939	130	40	225	124	519
1949	176	21	368	-111	453
1950	59	7	50	4	119
1951	53	-19	-285	-37	-287
1952	26	9	-67	-54	-85
1953	153	49	139	147	488
1954	130	38	309	60	536
1955	35	27	306	190	558
1956	88	80	331	151	650
1957	-68	30	-2	-14	-53
1958	68	-15	-86	-38	-71
1959	43	38	117	93	291
1960	91	41	277	142	551
1961	112	60	197	51	420
1962	65	36	158	112	371
1963	89	48	270	123	520

^a All entries are to be multiplied by 10^{-4}

TABLE 10. OBSERVATIONS ON THE PRICE VARIABLES (Dp_{it})^a

Years	Dp_{it} for			
	1. Food	2. Pleasure Goods	3. Durables	4. Remainder
1922	-1711	-610	-1387	-344
1923	-539	-131	-908	-151
1924	94	-49	-34	-16
1925	375	-105	43	-61
1926	-841	-794	-724	-168
1927	-403	-33	-110	-47
1928	157	-87	4	21
1929	-18	-384	-41	-69
1930	-709	-178	-751	-193
1931	-1195	-291	-664	-329
1932	-1575	-637	-1202	-468
1933	-428	-478	-756	-368
1934	16	-143	-283	-183
1935	-448	-578	-365	-257
1936	-188	-311	-879	-404
1937	767	137	750	318
1938	709	21	389	18
1939	-271	39	543	79
1949	497	857	229	397
1950	1332	361	875	530
1951	677	921	1327	855
1952	300	174	-763	318
1953	-107	-41	-165	-9
1954	475	246	78	539
1955	145	46	-17	356
1956	623	-81	-50	272
1957	508	683	88	584
1958	-276	384	-64	489
1959	206	-13	1	137
1960	-83	-28	157	348
1961	213	53	86	232
1962	440	87	88	305
1963	273	133	119	324

^a All entries are to be multiplied by 10^{-4}