RISK CONCEPTS REVISITED: A MATHEMATICAL APPROACH

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1. Introduction

An examination of the risk literature reveals a rather striking schism between theory and application. On the one hand, a relatively few highly theoretical works such as von Neumann & Morgenstern and Herstein & Milnor have set forth a mathematically rigorous framework on which to (at least attempt to) base risk analysis. On the other hand, while such works are routinely cited as constituting the logical foundation for applied studies, the requirements imposed by their axioms and logical constructs have not really been given serious attention in the less theoretical literature. As a result, some of the logical consequences of these theories have gone unrecognized, while other claimed consequences require additional assumptions. An example of a misconstrued consequence can be found in the common claim that risk aversion is equivalent to the concavity of the utility function of income. This assertion is now known to be false, though it can be rescued by an additional—if not entirely innocent—assumption of continuous risk preferences (Weiss, Tech. Bull.). As another example, it has only recently been recognized that the classical axioms for the expected utility hypothesis do not require that an individual's preferences between "certainties" determine his preferences between risky prospects. When this requirement is removed, different farmer production decisions become possible (Weiss, 1983).

In this paper, we shall present a way of looking at risk problems that adheres more rigorously to "first principles" than is customary. An advantage of this approach is that it forces us to confront questions that otherwise might be missed, and it requires us to remain aware of precisely what we are assuming when dealing with specific problems.

In Section 2, we begin by clarifying several fundamental risk concepts. We introduce the notion of a "lottery" as the formalization of the idea of a risky prospect. We describe "lottery spaces" as convex sets of lotteries and draw the connection between convexity and compound lotteries. We characterize "measurable utility functions" as utility functions that represent risk preference orderings and, in addition, have a linearity property that mimics the computation of the expected utility of a risky prospect. Finally, we relate how a risk preference ordering can give rise to what we call an "induced utility function"—the type of utility function defined on the number line that is used in virtually all applied studies.

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In Section 3, we show how the concepts previously defined may be applied to the analysis of a specific problem. Taking as a point of departure the reported sighting (Masson) of a *discontinuous* utility function of farmers' income, we address the question of how such a function could be used to describe an individual's behavior under risk. In the traditional case in which a utility function of income is *continuous*, risk preferences are derived from it by comparing the expected-utility integrals of alternative risky prospects. However, when the utility function is *discontinuous*, these integrals need not exist. Thus, even the justification for permitting a discontinuous function of income to be called a "utility function" is unclear. To resolve this dilemma, we establish a new method, not requiring integrals, for deriving risk preferences from *any* (bounded) numerical-valued function defined on the number line.

2. Some Risk Concepts Revisited

Lotteries

The theory of behavior under risk is concerned with the choices that individuals make when confronted with alternative risky prospects. These risky objects of choice are customarily called "lotteries." Intuitively, a lottery may be conceived of as a game of chance in which various prizes occur with preassigned probabilities. The prizes may be money amounts or even other lotteries (i.e., the opportunity to play other lotteries and receive *their* prizes). In the latter case, one speaks of a "compound" lottery.

The intuitive concept of lottery used in economics is governed by an important convention: two lotteries are considered "equivalent" if they have the same sets of ultimate prizes occurring under the same probability laws, regardless of the processes by which these prizes are achieved. In short, the internal compound structure of a lottery is ignored. In effect, the objects of choice are not individual lotteries as one intuitively conceives them, but rather *equivalence classes* of individual lotteries.

At first glance, it might appear that one could define a lottery (or, more precisely, the corresponding equivalence class) mathematically as simply a random variable whose possible values were the various ultimate prizes, these occurring according to the desired probability law. However, the calculation of an "overall" random variable to represent an empirical compound lottery in terms of its constituent sublotteries would be unacceptably complicated. Thus, random variables are not adequate as mathematical representations of lotteries. Rather, it turns out that *cumulative probability distribution functions* (*"c.d.f.'s"*) are more tractable representations. Since we shall be concerned here only with lotteries that ultimately (i.e., whatever their internal compound structure might be) yield income (i.e., numerical) prizes, we define a *lottery* to be a one-dimensional c.d.f.

Bearing in mind the distinction between the empirical concept of lottery and our mathematical representation of it, consider an empirical compound lottery L that offers empirical lotteries L_1 and L_2 as prizes with probabilities p and 1 - p, respectively. Then, if the c.d.f.'s C_1 and C_2 are taken to represent L_1 and L_2, respectively, the c.d.f. $pC_1 + (1-p)C_2$ will represent L. (We stress that $pC_1 + (1-p)C_2$ is an algebraic combination not of numbers, but of functions. It is a function whose value at t is $pC_1(t) + (1-p)C_2(t)$.) *This simple relationship—the fact that compound empirical lotteries can be represented by convex combinations of c.d.f.'s—is central to the usefulness of c.d.f.'s as mathematical representations of empirical lotteries.*

For each r, the lottery F_r defined by

$$F_r(t) = \begin{cases} 0 \text{ if } t < r \\ 1 \text{ if } t \geq r \end{cases}$$

is called *degenerate*. A lottery, L, that can be expressed as a convex combination

$$L = \sum_{i=1}^{n} p_i F_{r_i}$$

of finitely many degenerate lotteries, F_{r_i} (where, of course, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$), is called *simple*. A lottery, L, is called *discrete* if it can be expressed as a convex combination

$$L = \sum_{i=1}^{\infty} p_i F_{r_i}$$

of a *sequence* of degenerate lotteries. Finally, a *continuous* lottery is one that is continuous as a function in the usual sense. In the second and third of these definitions, we do not require the F_{r_i}'s to be distinct, nor do we rule out the possibility that $p_i = 0$ for some i's. In particular, a degenerate lottery is necessarily simple, while a simple lottery is also discrete. Note that a simple lottery represents a risk situation in which the associated r_i's are the possible outcomes and occur with respective probabilities p_i. In particular, a degenerate lottery, F_r, represents the occurrence of the outcome r with probability 1.

Lottery Spaces

Since economic problems may involve diverse types of risk, economic agents may be confronted with different "choice sets" of lotteries in different circumstances. In risk theory, as in demand theory, an essential requirement is that a choice set be *convex* (see, for example, Herstein & Milnor). Recall that the set of all numerical-valued functions defined on the number line is a *vector space* under the usual operations of addition/subtraction of functions and multiplication of functions by numbers. A choice set of lotteries is *convex* if it is convex as a subset of this vector space. Recall also that a set is convex if and only if it contains all finite convex combinations of its elements. Thus, in view of the relationship—described earlier—between convex combinations of c.d.f.'s and compound empirical lotteries, the assumption of convexity for a choice set of lotteries amounts, intuitively, to the requirement that any compound lottery formed from lotteries in the choice set itself lie in the choice set. We see here the significance of the fact that empirical compound lotteries can be represented by convex combinations of c.d.f.'s. Henceforth, we shall call a convex set of lotteries a *lottery space*.

The following are examples of lottery spaces: the sets of all (1) lotteries; (2) simple lotteries; (3) discrete lotteries; (4) continuous lotteries; (5) lotteries that are c.d.f.'s of bounded random variables; and (6) lotteries with finite mean. We point out that (6) provides a natural setting within which to consider risk aversion.

In view of the importance of normal distributions to the subject of risk, it is interesting to note that the set of all normal lotteries (i.e., normal c.d.f.'s) is *not* a lottery space. This fact is certainly known in other contexts (cf. Everitt & Hand) but does not seem to have been expressed clearly in the risk literature of economics. To establish it by means of a counterexample, let F be the N(0,1) c.d.f. and f the N(0,1) probability density function. Then, there is an x_0 such that $f(x_0) < 1/2(2\pi)^{1/2}$. Let G be the $N(2x_0,1)$ c.d.f. and g the $N(2x_0,1)$ probability density function. Now, if the c.d.f. $(1/2)F + (1/2)G$ were normal, then its derivative, $h = (1/2)f + (1/2)g$, would be a normal probability density function. However, this is impossible, since the inequalities

$$\begin{aligned} h(0) &= (1/2)f(0) + (1/2)g(0) \\ &> (1/2)f(0) \end{aligned}$$

$$= 1/2(2\pi)^{1/2},$$

$$h(x_0) = (1/2)f(x_0) + (1/2)g(x_0)$$

$$< [1/4(2\pi)^{1/2}] + [1/4(2\pi)^{1/2}]$$

$$= 1/2(2\pi)^{1/2},$$

and

$$h(2x_0) = (1/2)f(2x_0) + (1/2)g(2x_0)$$

$$> (1/2)g(2x_0)$$

$$= 1/2(2\pi)^{1/2}$$

show that $h(x_0)$ is smaller than both $h(0)$ and $h(2x_0)$, although x_0 lies between 0 and $2x_0$. This establishes that the set of all normal c.d.f.'s is not convex. Note that the "smallest" lottery space that contains all normal c.d.f.'s—by definition, the *convex hull* of the set of all normal c.d.f.'s—is the set of all convex combinations of finitely many normal c.d.f.'s.

Preferences and Measurable Utility

In risk theory, as in demand theory, the starting point is the economic agent's preference ordering over his choice set. By a *preference ordering*, $\succsim$, on a lottery space L we mean a complete, transitive, binary relation on L. If L is a lottery space and $\succsim$ a preference ordering on L, we call the pair $(L, \succsim)$ a *preference lottery space*. A *utility function*, U, for $(L, \succsim)$ is a numerical-valued function defined on L such that, for any L_1, L_2 in L, $U(L_1) \geq U(L_2)$ if and only if $L_1 \succsim L_2$.

If L is a lottery space and V a numerical-valued function on L, we call V *linear* if

$$V(tL_1 + (1-t)L_2) = tV(L_1) + (1-t)V(L_2)$$

whenever $0 \leq t \leq 1$ and L_1, L_2 are any lotteries in L. A linear utility function for a preference lottery space $(L, \succsim)$ is called *measurable*. The whole thrust of von Neumann & Morgenstern (in that work's discussion of utility) and Herstein & Milnor was to prove that, given plausible assumptions concerning $(L, \succsim)$, a measurable utility function would exist. The relation of measurable utility functions to "expected utility" will be explained once we have introduced the concept of "induced utility function."

The notion of "linearity" defined above can be sharpened to cover "infinite" convex combinations as follows (Grandmont): A set L of lotteries is called *σ-convex* if $\sum_{i=1}^{\infty} p_i L_i$ is in L whenever each $p_i \geq 0$, each L_i is in L, and $\sum_{i=1}^{\infty} p_i = 1$. A numerical-valued function V defined on a σ-convex set L is called *σ-linear* if

$$V\left(\sum_{i=1}^{\infty} p_i L_i\right) = \sum_{i=1}^{\infty} p_i V(L_i)$$

whenever each $p_i \geq 0$, each L_i is in L, and $\sum_{i=1}^{\infty} p_i = 1$. Clearly, a σ-convex set is convex, while a σ-linear function is linear. In our discussion of farmers' discontinuous utility functions of income, we shall need to consider σ-linear functions defined on the set of all discrete lotteries. Thus, we shall need:

Proposition: The set of all discrete lotteries is σ-convex.

Proof: Suppose $L_1, L_2, L_3, \ldots$ is a sequence of discrete lotteries and $p_1, p_2, p_3, \ldots$ a sequence of nonnegative numbers such that $\sum_{i=1}^{\infty} p_i = 1$. We wish to prove that the lottery $\sum_{i=1}^{\infty} p_i L_i$ is discrete. Now, for each i, since L_i is discrete, there exist a sequence $t_{i,1}, t_{i,2}, t_{i,3}, \ldots$ of numbers and a sequence $q_{i,1}, q_{i,2}, q_{i,3}, \ldots$ of nonnegative numbers such that

$$\sum_{j=1}^{\infty} q_{i,j} = 1$$

and

$$L_i = \sum_{j=1}^{\infty} q_{i,j} F_{t_{i,j}}.$$

Thus,

$$(*) \qquad \sum_{i=1}^{\infty} p_i L_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i q_{i,j} F_{t_{i,j}}.$$

The right-hand side of (*) is written as a sum over all pairs (i,j) of positive integers. In order to express it as a convex combination of a *sequence* of degenerate lotteries, we use the fact from set theory that there exists a one-to-one correspondence, ϕ, between the set of all positive integers, and the set of all *ordered pairs* of positive integers. Intuitively, as n runs through all positive integers, $\phi(n)$ will run through all ordered pairs of positive integers. Thus, the right-hand side of (*) can be rewritten as

$$\sum_{n=1}^{\infty} p_{\phi(n)_1} q_{\phi(n)} F_{t_{\phi(n)}},$$

where $\phi(n)_1$ is the first coordinate of $\phi(n)$. Since each $p_{\phi(n)_1} q_{\phi(n)}$ is nonnegative and

$$\begin{aligned}\sum_{n=1}^{\infty} p_{\phi(n)_1} q_{\phi(n)} &= \sum_{i=1}^{\infty} p_i \sum_{j=1}^{\infty} q_{i,j} \\ &= \sum_{i=1}^{\infty} p_i \cdot 1 \\ &= 1,\end{aligned}$$

we conclude that $\sum_{i=1}^{\infty} p_i L_i$ is discrete.

Q.E.D.

It is important to understand that a linear function is really a measurable utility function in disguise. That is, although linear functions are defined in terms of lottery spaces and not *preference* lottery spaces, it is always possible, given a linear function U defined on a lottery space L, to construct a preference ordering—henceforth denoted $\succsim_U$—on L such that U is a (measurable) utility function for $(L, \succsim_U)$. In fact, $\succsim_U$ may be defined by the condition

$$L_1 \succsim_U L_2 \text{ if and only if } U(L_1) \geq U(L_2)$$

for each pair L_1, L_2 of lotteries in L. We shall make use of this observation when we discuss farmers' discontinuous utility functions in Section 3.

Induced Utility Functions

Utility functions defined on lottery spaces are the conceptually "fundamental" utility functions of risk theory. However, in most of the literature, these are ignored in favor of *another* type of "utility function:" a "utility function of [say] income." The latter type of function assigns numerical values not to lotteries, but to *numbers* (e.g., income levels). What, then, is its relationship to the more fundamental notion of utility?

Let U be a function defined on a lottery space L that contains all degenerate lotteries. If u is a numerical-valued function defined at each r by

$$u(r) = U(F_r),$$

we say u is *induced (on the number line) by U*. If, in particular, $(L, \succsim)$ is a preference lottery space for which U is a utility function, we call u the *utility function induced by U* and may interpret it as, for example, a "utility function of income." In this way, a utility function defined on a preference lottery space can give rise to a "utility function," u, defined on the number line. We may view u as akin to "U restricted to degenerate lotteries"--that is, by a slight abuse of language, "U restricted to certainties." Accordingly, u reflects only preferences over "certainties" (as embodied by U). It does not, in general, contain the "complete information" on preferences that U contains.

Conversely, given any *continuous* numerical-valued function, u, defined on the number line, we can define a preference lottery space and a utility function for it that induces u and thereby allows u to be interpreted as an induced utility function. To show this, define L_u to be the set of all lotteries L for which $\int_{-\infty}^{\infty} u(t)dL(t)$ is finite.

Clearly, L_u is convex, hence a lottery space. Furthermore, L_u contains every degenerate lottery. Define a function, $\underline{U}$, on L_u by

$$U(L) = \int_{-\infty}^{\infty} u(t)dL(t)$$

for each L in $\mathcal{L}_u$. Then, U is a utility function for the preference lottery space $(\mathcal{L}_u, \succsim_U)$, where $\succsim_U$ was defined earlier. (In fact, U is measurable, since it is obviously linear.) Moreover, u is induced by U, since, for each degenerate lottery F_r,

$$U(F_r) = \int_{-\infty}^{\infty} u(t)dF_r(t)$$

$$= u(r).$$

Having defined the notion of "induced utility function," we are now able to clarify the connection—alluded to earlier—between measurable utility functions and "expected utility." Suppose that a measurable utility function U for a preference lottery space $(\mathcal{L}, \succsim)$ induces a utility function u on the number line. Since $\mathcal{L}$, by assumption, contains all degenerate lotteries, it must contain all simple lotteries. Let

$$L = \sum_{i=1}^{m} p_i F_{r_i}$$

be a simple lottery and X a random variable with L as its c.d.f. Then, by the linearity of U and the definition of u, we have

$$U(L) = \sum_{i=1}^{m} p_i U(F_{r_i})$$

$$= \sum_{i=1}^{m} p_i u(r_i)$$

$$= E(u(X)),$$

so that U(L) is the "expected utility" of L. Nevertheless, U(L) need *not* be—and was not claimed in von Neumann & Morgenstern and Herstein & Milnor to be—the expected utility of L (in the sense described) if L is *not* simple—if, say, L is continuous. In fact, although measurable utility functions *can* be used to rationalize expected utility maximization (that is, maximization of expected *induced* utility) over lottery spaces of *simple* lotteries, they are *also* capable of generating *other* decision criteria when the lotteries involved are *not* necessarily simple. For examples, see Weiss (1973, Tech. Bull.).

3. An Application: The Use of Discontinuous Utility to Model Farmers' Risk Preferences

We now illustrate how the preceding risk concepts can be applied to expose and resolve a potential methodological problem in agricultural risk modeling: We shall consider the use of discontinuous utility functions of income to model farmers' behavior under risk.

Roy, in the context of his "safety-first" theory (1952), suggested that the minimization of the chance of disaster could in some cases be interpreted as the maximization of the expected utility of a discontinuous utility function. While he referred only to a two-valued utility function and expressed doubt as to the practical significance of this conception, he apparently had no *methodological* objection to discontinuous utility, writing that there "would appear to be no valid objection to the discontinuity in the preference scale that the existence of a single disaster value implies." Later, Masson (1974) reported that data in O'Mara's study (1971) of peasant farming in Mexico provided actual evidence of utility functions with jump discontinuities. In the same spirit as Roy, Masson suggested that the jumps in utility might be associated with critical income levels.

Our purpose here is not to investigate the appropriateness of using discontinuous utility to model disaster avoidance; this subject might require an extensive discussion of its own. Rather, having adopted from the start the viewpoint that, within the classical paradigm, risk questions are basically concerned with—and should be reducible to—the study of preference orderings over lottery spaces, we wish to pose the following question, or set of questions, addressed by neither Masson nor Roy:

> *Given a discontinuous function defined on the number line and claimed to be a "utility function," as in Masson, how can one derive risk preferences from it? Need there even be* any *risk preference ordering corresponding to it? (If not, what can such a function have to do with behavior under risk?) Is there necessarily any utility function of lotteries that* induces *it? (If not, then by what right is it claimed to be a "utility function?") Could two utility functions representing* incompatible *preference orderings* both *induce it? (If so, then of what use would it be in applied research?*)

The problem I raise is one of "measurement without theory." I submit that we cannot justify interpreting a discontinuous numerical function as a "utility function" or ascribe any usefulness to it in representing behavior under risk—whatever its empirical origin—until the preceding questions have been satisfactorily answered.

For *continuous* functions, satisfactory answers to these questions are virtually immediate. For, as we noted earlier, the integral formula

$$U(L) = \int_{-\infty}^{\infty} u(t)dL(t)$$

provides us with a rule for determining--for each numerical-valued continuous function u defined on the number line--a unique measurable utility function U that, in turn, induces u. Furthermore, this integral formula associates distinct u's with distinct U's, since $u_1 \neq u_2$ implies $u_1(t_0) \neq u_2(t_0)$ for some t_0, whence

$$\begin{aligned} U_1(F_{t_0}) &= \int_{-\infty}^{\infty} u_1(t)dF_{t_0} \\ &= u_1(t_0) \\ &\neq u_2(t_0) \\ &= \int_{-\infty}^{\infty} u_2(t)dF_{t_0} \\ &= U_2(F_{t_0}) \end{aligned}$$

and (thus) $U_1 \neq U_2$. In short, for continuous u's, the integral formula provides us with a *one-to-one correspondence* between the u's and their associated measurable utility functions. Since each of the latter determines a unique preference lottery space, a linkage between the u's and preference orderings on lottery spaces is established. (For completeness, we mention that the final link in this chain--the correspondence between measurable utility functions and preference lottery spaces--is *not* one-to-one. Rather, measurable utility functions U and V defined on the same lottery space will represent the *same* preference ordering if (and only if) $U = aV + b$ for some constants a and b with $a > 0$ (see Weiss (Tech. Bull.)). As a result, continuous functions u and v for which $u = av + b$ $(a > 0)$ will correspond to the same preference lottery space. Nevertheless, each u still corresponds to a unique preference lottery space.)

For *discontinuous* u's, however, the relationship to risk preference orderings is less apparent, for, in this case, the integral formula cannot, in general, be used. Consider, for example, an empirical study--such as O'Mara's--in which the subjects are asked to choose between various putative simple lotteries. These are necessarily discontinuous. (Discontinuous lotteries also arise when, for example, commodity price risk is mediated through the introduction of a support price, or the risk of some disaster is mediated through an insurance policy with a deductible.) Suppose that the estimated utility function, u, is discontinuous at an income level t_0, while the simple lottery L is also discontinuous at t_0. Then, the Stieltjes integral

$$\int_{-\infty}^{\infty} u(t)dL(t)$$

does not even exist, for its integrand and weighting function are discontinuous at the same point. (See Rudin for a careful review of the definition of the Stieltjes integral.) Yet, we would certainly require that our lottery space contain L. Thus, we need to find an alternative method of associating discontinuous functions with risk preference orderings.

It turns out that, if we restrict our attention to *bounded* functions (as should be appropriate for most empirical work), we can characterize the associated measurable utility functions more strongly. (For an alternative result without this restriction, see Weiss (1985).) We now present a result showing how any bounded function—whether continuous or discontinuous—can be uniquely associated with a preference lottery space. By way of preparation, recall that any σ-linear function determines a unique preference lottery space for which it is a measurable utility function. Recall also that L_d is the space of all discrete lotteries and (by our *Proposition*) is σ-convex, so that σ-linear functions may be defined on it. Then:

Theorem: The relation "U induces u" is a one-to-one correspondence between the set of all σ-linear functions defined on L_d and the set of all bounded numerical-valued functions defined on the number line.

Proof: It was implicitly shown in Grandmont that any σ-linear function is bounded. Thus, the σ-linear functions of concern here must induce utility functions that are bounded. To establish the one-to-one correspondence asserted by the *Theorem*, it remains to show that (1) different σ-linear functions on L_d must induce different numerical functions and (2) every bounded numerical function is induced by *some* σ-linear function on L_d. (1) amounts to the assertion that the relation "U induces u" associates each bounded numerical function with *at most* one σ-linear function on L_d, while (2) amounts to the assertion that this relation associates each bounded numerical function with *at least* one σ-linear function on L_d. (2) implies that every bounded numerical function is associated with *some* risk preference ordering on L_d; (1) implies that that risk preference ordering is *unique*.

To prove (1), suppose σ-linear functions U_1 and U_2 on L_d induce the same bounded function, u, on the number line. Then, for any discrete lottery

$$L = \sum_{i=1}^{\infty} p_i F_{r_i},$$

we have

$$\begin{aligned} U_1(L) &= \sum_{i=1}^{\infty} p_i U_1(F_{r_i}) \\ &= \sum_{i=1}^{\infty} p_i u(r_i) \\ &= \sum_{i=1}^{\infty} p_i U_2(F_{r_i}) \\ &= U_2(L), \end{aligned}$$

so that $U_1 = U_2$. This proves (1).

To prove (2), let u be an arbitrary bounded numerical function. We shall construct a σ-linear function, denoted u*, on L_d that induces u. In fact, for any discrete lottery

$$(**) \qquad L = \sum_{i=1}^{\infty} p_i F_{s_i},$$

put

$$u^*(L) = \sum_{i=1}^{\infty} p_i u(s_i).$$

Note that, for u*(L) to be well-defined, it is required that u* assign the same value to L no matter how L may be represented as a convex combination of degenerate lotteries. That is, suppose L can also be represented as

$$(***) \qquad L = \sum_{j=1}^{\infty} q_j F_{t_j},$$

where each $q_j \geq 0$ and $\sum_{j=1}^{\infty} q_j = 1$. Then, it is required that

$$\sum_{i=1}^{\infty} p_i u(s_i) = \sum_{j=1}^{\infty} q_j u(t_j).$$

However, this condition is satisfied. To prove this, suppose that (**) and (***) both hold. Without loss of generality, we may assume that each p_i and q_j is positive. Let S be the set of distinct terms appearing among $s_1, s_2, s_3, \ldots$ and T the set of distinct terms appearing among $t_1, t_2, t_3, \ldots$. For each s in S, let S_s be the set of subscripts, i, for which $s_i = s$. Similarly, for each t in T, let T_t be the set of subscripts, j, for which $t_j = t$. Then, it follows from (**) and (***) (by rearranging terms) that

$$\sum_{s \in S} \left[\sum_{i \in S_s} p_i\right] F_s = \sum_{t \in T} \left[\sum_{j \in T_t} q_j\right] F_t$$

(where, for example, the notation "$s \in S$" indicates that the sum is to be taken over all elements s of the set S. Note that these "sums" really are infinite series, in general.). Our next step is to prove that S = T. We shall accomplish this by showing that S and T are precisely the sets of points of discontinuity of the left and right sides, respectively. It will be enough to prove this for S.

Toward this end, for each s in S, put

$$P_s = \sum_{i \in S_s} p_i.$$

Note that each P_s is positive. Put

$$F = \sum_{s \in S} P_s F_s,$$

and consider any number r. For any numbers r_1, r_2 satisfying $r_1 < r < r_2$, we have

$$F(r_2) - F(r_1) = \sum_{s \in S} P_s[F_s(r_2) - F_s(r_1)].$$

Thus

$$F(r+) - F(r-) = \sum_{s \in S} P_s[F_s(r+) - F_s(r-)]$$

(where the "+" and "-" denote right- and left-hand limits). However, for each s in S, $F_s(r+) - F_s(r-)$ is either 1 or 0, according to whether r is, or is not, equal to s. Thus, $F(r+) - F(r-)$ is positive if and only if r is in S. However, F is discontinuous at r if and only if $F(r+) - F(r-)$ is positive. It follows that S must be the set of points of discontinuity of F. Arguing similarly for T, we conclude that S = T.

Thus, putting

$$Q_t = \sum_{j \in T_t} q_j$$

for each t in T, we have

$$\sum_{s \in S} (P_s - Q_s)F_s = 0.$$

Consider any number s_0. Reasoning as before, we obtain

$$\sum_{s \in S} (P_s - Q_s)[F_s(s_0+) - F_s(s_0-)] = 0,$$

which reduces to

$$(P_{s_0} - Q_{s_0})[F_{s_0}(s_0+) - F_{s_0}(s_0-)] = 0,$$

whence $P_{s_0} = Q_{s_0}$. Since, by rearrangement of terms,

$$\sum_{i=1}^{\infty} p_i u(s_i) = \sum_{s \in S} \left[\sum_{i \in S_s} p_i\right] u(s)$$

$$= \sum_{s \in S} P_s u(s)$$

and

$$\sum_{j=1}^{\infty} q_j u(t_j) = \sum_{t\in T}\left[\sum_{j\in T_t} q_j\right]u(t)$$

$$= \sum_{s\in S} Q_s u(s),$$

it follows that

$$\sum_{i=1}^{\infty} p_i u(s_i) = \sum_{j=1}^{\infty} q_j u(t_j),$$

as was to be proved.

Thus, the function u* is well-defined. Furthermore, it is σ-linear. To see this, let $L_1, L_2, L_3, \ldots$ be a sequence of discrete lotteries and $p_1, p_2, p_3, \ldots$ a sequence of nonnegative numbers such that $\sum_{i=1}^{\infty} p_i = 1$. We wish to show that

$$u^*\left(\sum_{i=1}^{\infty} p_i L_i\right) = \sum_{i=1}^{\infty} p_i u^*(L_i).$$

However, in order to use the definition of u* to calculate the left-hand side, we need to rewrite $\sum_{i=1}^{\infty} p_i L_i$ as a convex combination of a sequence of degenerate lotteries. For this, we follow the notation and reasoning used in the proof of the *Proposition* (Sec.2), writing

$$L_i = \sum_{j=1}^{\infty} q_{i,j} F_{t_{i,j}}$$

for each i and employing the one-to-one correspondence, ϕ, between the set of all positive integers and the set of all ordered pairs of positive integers. We obtain

$$u^*\left(\sum_{i=1}^{\infty} p_i L_i\right) = u^*\left(\sum_{n=1}^{\infty} p_{\phi(n)_1} q_{\phi(n)} F_{t_{\phi(n)}}\right)$$

$$= \sum_{n=1}^{\infty} p_{\phi(n)_1} q_{\phi(n)} u(t_{\phi(n)})$$

$$= \sum_{i=1}^{\infty} p_i \sum_{j=1}^{\infty} q_{i,j} u(t_{i,j})$$

$$= \sum_{i=1}^{\infty} p_i u^*(L_i),$$

as desired. Thus, u^* is σ-linear. Finally, u^* induces u, since $u^*(P_t) = u(t)$ for each t by the very definition of u^*. This proves (2) and completes the proof of the *Theorem*.

Q.E.D.

In establishing a one-to-one correspondence between bounded numerical functions and measurable utility functions, the *Theorem* assumed that preference comparisons were to be made only between discrete lotteries. It is natural to ask whether a similar correspondence based on the relation "U induces u" holds when *continuous* lotteries (in conjunction with discrete lotteries) are allowed in the lottery space. It is shown in Weiss (Tech. Bull.) that such a correspondence does *not* hold. In fact, in the situation described, measurable utility functions representing *incompatible* risk preference orderings can induce the *same* bounded numerical function. It is plausible, however, that a *different* type of correspondence—based on the canonical decomposition of a lottery into its discrete and continuous parts (Weiss, Tech. Bull.)—could be effective for an expanded lottery space. We leave that investigation for another time.

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