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Risk Modeling in Agriculture: Retrospective and Prospective

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DUALITY MODELS OF PRODUCTION UNDER RISK: A SUMMARY OF RESULTS FOR SEVERAL NONLINEAR MEAN-VARIANCE MODELS

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Introduction

In an earlier paper I presented a duality model of production and risk within the framework of a mean-variance utility function (Coyle 1992). However this model is extremely restrictive: the mean-variance utility function is assumed to be linear, prices are the only source of uncertainty (uncertainty regarding outputs or yields is ignored), and the model is static rather than dynamic. The assumption of a linear mean-variance utility function seems particularly objectionable. As a result, I suspect that this model is irrelevant to serious empirical research on production decisions under risk.

More recently I have been attempting to generalize this model by relaxing the most objectionable assumptions, as listed above. The purpose of this present paper is to summarize some of these results. The first section presents a static duality model assuming a nonlinear mean-variance utility function, where only prices are uncertain. The second section generalizes this nonlinear model to a dynamic setting. The third section presents a static duality model with a nonlinear mean-variance utility function and uncertain yields. These discussions are intended to provide an introduction and overview to three recent working papers (Coyle 1993, 1994a, 1994b, respectively). Proofs of propositions presented here, as well as further discussion, is to be found in these papers.

A Static Model with Price Uncertainty

In contrast to Coyle (1992), assume that the firm has a general nonlinear mean-variance utility function. There are two justifications for this assumption. First, Epstein (1985) has argued that decreasing absolute risk aversion (or systematically changing risk aversion) implies mean-variance utility. Since decreasing absolute risk aversion is commonly assumed, mean-variance utility is relatively unrestrictive. Second, if uncertainty is limited to one output price, then static nonlinear mean-variance and expected utility models are equivalent (Sinn; Meyer).

Suppose that the firm has a general nonlinear mean-variance utility function $U = U(W_0 + E\pi, V\pi)$, where $E\pi$ is expected profit in the current period, $V\pi$ is variance of profit, and W_0 is initial wealth. The certainty-equivalent version of this utility function is

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$$U = W_0 + E\pi - \alpha(W_0 + E\pi, V\pi)/2\ V\pi \tag{1}$$

i.e. the coefficient of absolute risk aversion α varies with $(W_0 + E\pi, V\pi)$. It is convenient to assume that

A.1: $d\alpha(.)/d(W_0 + E\pi) \neq 0 \quad \forall\ (W_0 + E\pi, V\pi)$

i.e. $\alpha(W_0 + E\pi, V\pi)$ is monotonically increasing or decreasing in $(W_0 + E\pi)$, and

A.2: $d\alpha(.)/d(W_0 + E\pi) \leq 0 \quad \forall\ (W_0 + E\pi, V\pi)$

i.e. absolute risk aversion (as measured by $\alpha \geq 0$) decreases as $(W_0 + E\pi)$ increases. This restriction of decreasing absolute risk aversion (e.g. Meyer; Sinn 1983, pp. 116-7) is a common assumption (a stylized fact) for risk preferences.

Profits are p y - w x, where (y,x) denotes a vector of output and input levels, and (p,w) is a vector of corresponding prices. It will be assumed that input prices w are nonstochastic and there is a single output, but these assumptions can easily be relaxed. The expected value and variance of profits conditional on (y,x) are

$$\begin{aligned} E\pi(y,x) &= Ep\ y - w\ x \\ V\pi(y,x) &= Vp\ y^2 \end{aligned} \tag{2}$$

where (Ep,Vp) are mean and variance of output price p.

General Model

The risk-averse competitive firm's utility maximization problem can be represented in terms of (1) as

$$\begin{aligned} U^*(Ep,w,Vp,W_0) = \max_{y\,\varepsilon\, s}\ & W_0 + Ep\ y - C(w,y) \\ & - \alpha(W_0 + Ep\ y - C(w,y),\ Vp\ y^2)/2\ Vp\ y^2 \\ & z \equiv (Ep,w,Vp,W_0)\varepsilon P \end{aligned} \tag{3}$$

where C(w,y) is the standard dual cost function

$$C(w,y) = \min_{x \,\varepsilon\, V(y)} \; w\,x \qquad (w,y)\varepsilon W\,. \tag{4}$$

A cost function $C^*(w,y)$ implicit in $U^*(Ep,w,Vp,W_0)$ can be defined as satisfying the following relation:

$$C^*(w,y) = \max_{z' \equiv (Ep,Vp,W_0)\,\varepsilon P} \; W_0 + Ep\,y - \alpha(W_0 + Ep\,y - C^*(w,y),\, Vp\,y^2)/2\; Vp\,y^2 - U^*(Ep,w,Vp,W_0) \qquad (w,y)\varepsilon W'. \tag{5}$$

Given knowledge of the utility function or equivalently $\alpha = \alpha(W_0 + E\pi, V\pi)$, and assumptions A.1-A.2 and a generalized convexity condition (below), the cost function C(w,y) can be recovered from the dual $U^*(Ep,w,Vp,W_0)$. Then the technology y = f(x) can be recovered from C(w,y) by standard arguments (Shephard; McFadden). This duality result can be stated as follows.

Theorem 1. Suppose $U^*(.)$, C(.), $C^*(.)$ and $\alpha(.)$ are defined and continuous, and assume A.2. A. Then $C(.) \geq C^*(.)\ \forall\ (w,y)\ \varepsilon\ W$. B. Suppose that $\forall\ y^0\ \varepsilon S\ \exists\ z^0\ \varepsilon P$ such that $y^0\ \varepsilon y(z^0)$, i.e. any $y^0 \varepsilon S$ can be obtained as a solution to (3) for an appropriate choice of $z\varepsilon P$. Then $C(.) = C^*(.)\ \forall\ (w,y)\ \varepsilon\ W$.

The properties of the dual $U^*(Ep,w,Vp,W_0)$ for problem (3) are summarized as follows. The proof of this Theorem is obvious (perhaps in contrast to Theorem 1).

Theorem 2. Assume existence of the dual $U^*(Ep,w,Vp,W_0)$ for problem (3). Then

a) U^* is increasing in Ep, decreasing in w and increasing in W_0, given α decreasing in $(W_0 + E\pi)$ (A.2), and U^* is decreasing in Vp given $\alpha(.)>0$ and α increasing in Vp;

b) U^* is linear homogeneous in (Ep,w,Vp,W_0) if $\alpha(.)$ is homogeneous of degree zero in $(W_0 + E\pi, V\pi)$;

c) (assuming $U^*(Ep,w,Vp,W_0)$ and $\alpha(W_0 + E\pi,V\pi)$ are differentiable)

$$\begin{aligned}
&a)\quad y = dU^*(.)/dEp\ /\ [1 - d\alpha(.)/d(W_0 + E\pi)\ Vp\ y^2/2] \\
&b)\quad x^i = -\,dU^*(.)/dw^i\ /\ [1 - d\alpha(.)/d(W_0 + E\pi)\ Vp\ y^2/2] \qquad i=1,..,N \\
&c)\quad dU^*(.)/dVp = -\,y^2\,[\alpha/2 + d\alpha(.)/dVp\ Vp\ y^2/2] \\
&d)\quad dU^*(.)/dW_0 = 1 - d\alpha(.)/d(W_0 + E\pi)\ Vp\ y^2/2 \quad ;
\end{aligned} \tag{6}$$

d) (assuming $U^*(Ep,w,Vp,W_0)$ and $\alpha(W_0 + E\pi,V\pi)$ are twice differentiable)

$$U^*_{vv}(v) + [\alpha(W_0 + Ep\ y - w\ x,\ Vp\ y^2)\ Vp\ y^2/2]_{vv} \qquad (7)$$
symmetric positive semidefinite

where $v \equiv (Ep,w,Vp,W_0)$.

Theorem 2b implies that $U^*(.)$ is linear homogeneous in (Ep,w,Vp,W_0) if $\alpha(.)$ is homogeneous of degree zero in $(W_0 + E\pi, V\pi)$, i.e. the mean-variance utility function is homothetic. This is implied by constant relative risk aversion (Meyer; Sinn 1983, pp. 148-9, 153-7), which is a common assumption in the asset pricing literature. Nevertheless this assumption seems restrictive.

The envelope relations (6) can be used to specify output supply and factor demand equations. Substituting (d) into (a)-(b) of Theorem 2c,

$$a)\quad y = dU^*(.)/dEp\ /\ dU^*(.)/dW_0$$
$$b)\quad x^i = -dU^*(.)/dw^i\ /\ dU^*(.)/dW_0 \qquad i=1,..,N\ . \qquad (8)$$

In contrast to the homogeneity result, envelope relations similar to (8) have been noted in several studies.

An Alternative Model with Concave Mean-Variance Production Frontier

As in the previous model, suppose that the firm solves a maximization problem as in (3) over a general nonlinear mean-variance utility function or equivalently maximizes expected utility. However, in contrast to the previous model, suppose that the firm's production possibility frontier for mean and variance of profit is concave. Then the firm's nonlinear indifference curves in mean-variance space can be replaced by a family of parallel linear indifference curves, where the slope is defined by the equilibrium (local) coefficient of absolute risk aversion α^*, without changing the equilibrium production decision. Local concavity of the $E\pi$-$V\pi$ production frontier is likely given decreasing, constant, or "moderately" increasing returns to scale.

Given concavity of the production possibility frontier for mean and variance of profit, a solution (y^*,x^*) to the firm's maximization problem (3) conditional on parameters (Ep,w,Vp,W_0) also solves the following maximization problem:

$$U^{**}(Ep,w,Vp,\alpha^*) = \max_{(x,y)\varepsilon T}\ Ep\ y - w\ x - \alpha^*/2\ Vp\ y^2 \qquad z \equiv (Ep,w,Vp,W_0)\varepsilon P \qquad (9)$$

where α^* denotes the local coefficient of absolute risk aversion at solution to (3), i.e. $\alpha^* = \alpha(W_0 + Ep\ y^* - w\ x^*, Vp\ y^{*2})$. Since initial wealth W_0 affects production decisions in problem (3) only through its influence on the coefficient of risk aversion α, the objective function of (3) can be redefined as $Ep\,y - C(w,y) - \alpha(W_0 + Ep\,y - C(w,y), Vp\,y^2)/2\,Vp\,y^2$ without altering the solution to (3), and similarly the solutions to (3) and (9) are identical. Moreover, when the assumption of a concave mean-variance frontier is imposed, duality can be established without assuming decreasing absolute risk aversion.

The properties of the dual $U^{**}(Ep,w,Vp,\alpha^*)$ for (9) are summarized as follows. In establishing the homogeneity and envelope relations, the critical point is that W_0 is not explicitly an argument of problem (9) and assumption A.1 implies compensating changes in W_0, so (Ep,w,Vp) can vary independently of α^*. These homogeneity and envelope properties are analogous to the linear mean-variance case (Coyle 1992).

Theorem 3. Assume A.1, concavity of the profit mean-variance production frontier and existence of the dual $U^{**}(Ep,w,Vp,\alpha^*)$ for problem (9). Then

a) U^{**} is increasing in Ep, decreasing in w, decreasing in Vp ($\alpha^*>0$), and decreasing in α^*;

b) U^{**} is linear homogeneous in (Ep,w,Vp);

c) (assuming $U^{**}(Ep,w,Vp,\alpha^*)$ and $\alpha^*(Ep,w,Vp,W_0)$ differentiable)

$$\begin{aligned} &a)\quad y = dU^{**}(.)/dEp \\ &b)\quad x^i = -\,dU^{**}(.)/dw^i \qquad\qquad i=1,..,N \\ &c)\quad dU^{**}(.)/dVp = -\,\alpha^*/2\,y^2 \\ &d)\quad dU^{**}(.)/d\alpha = -\,Vp\,y^2/2 \end{aligned} \tag{10}$$

where derivatives of $U^{**}(.)$ are evaluated at $\alpha^* = \alpha^*(Ep,w,Vp,W_0)$;

d) (assuming $U^{**}(Ep,w,Vp,\alpha^*)$ and $\alpha^*(Ep,w,Vp,W_0)$ twice differentiable)

$$U^{**}_{vv}(Ep,w,Vp,\alpha^*(Ep,w,Vp,W_0)) + [\alpha^*(ep,w,Vp,W_0)\;Vp\;y^2/2]_{vv} \quad \textit{symmetric positive semidefinite} \tag{11}$$

where $v \equiv (Ep,w,Vp,W_0)$.

Theorem 3b indicates that the dual $U^{**}(Ep,w,Vp,\alpha^*)$ is linear homogeneous in (Ep,w,Vp). This result depends on the assumption of a concave mean-variance frontier; whereas homogeneity of the dual $U^*(Ep,w,Vp,W_0)$ in the previous section depended upon constant relative risk aversion. Whether or not either of these assumptions is appropriate is an empirical issue.

Since α^* is an argument of the dual $U^{**}(.)$ and is an unobserved variable, it is necessary to specify a functional form for risk preferences if (10a-b) is to be estimated. This can be either a structural equation $\alpha(W_0 + E\pi, V\pi)$ or a reduced form $\alpha^*(Ep,w,Vp,W_0)$. Theorems 2d and 3d indicate that a second order differential approximation to $\alpha(.)$ as well as $U^{**}(.)$ is adequate in terms of modeling the maximization hypothesis.

A Dynamic Model with Price Uncertainty

Static Expectations

The firm's risk preferences are modelled in terms of a general nonlinear mean-variance utility function $U = U(W_0 + EW, VW)$. W_0 is initial wealth, and W is present value of profits for the production plan. In forming its production plan, the firm assumes that output price p is a random variable distributed independently and with constant mean and variance over time: $p_t = \bar{p} + u_t$ where $Eu_t = 0$, $\text{cov } u = \sigma_u^2 I$, and in turn $Ep_t = \bar{p}$, $\text{cov } p = \sigma_u^2 I$. Then the mean and variance of the present value of wealth W from production are defined in terms of the mean and variance of output price p ($Ep \equiv \bar{p}$, $VP \equiv \sigma_u^2$) as

$$EW = \int_{t=0}^{\infty} [Ep\ y(t) - w\ x(t) - w^k\ I(t)]\ e^{-rt}\ dt$$
$$VW = \int_{t=0}^{\infty} [Vp\ y(t)^2]\ e^{-2rt}\ dt \tag{12}$$

where y(t) is the level of output for period t, x(t) is the corresponding vector of variable input levels, I(t) is gross investment in capital, w is the vector of variable input prices, w^k is the asset (purchase) price of capital, and r is the firm's discount rate. [I thank Wes Musser for pointing out an error in my initial specification of VW.] The certainty-equivalent version of the utility function is $W_0 + EW - \alpha(W_0 + EW,VW)/2\ VW$, but it is most convenient to define the utility function as net of W_0:

$$U = EW - \alpha(W_0 + EW,VW)/2\ VW\ . \tag{13}$$

As in Epstein's standard risk-neutral dynamic duality model (1981a), it is assumed here that the firm has static expectations for prices and technology over its infinite planning horizon. At any time t=0 the firm is assumed to make a plan for investment I and variable inputs x that solve the following optimal control problem:

$$J(Ep,w,w^{k},Vp,r,K_0,W_0) =$$

$$\max_{\{(x(t),\ I(t))\}} \quad EW - \alpha(W_0 + EW,VW)/2 \quad VW \tag{14}$$

$$s.\ t. \qquad EW \equiv \int_{t=0}^{\infty} [Ep\ f(x(t),K(t),I(t)) - w\ x(t) - w^{k}\ I(t)]\ e^{-rt}\ dt$$

$$VW \equiv \int_{t=0}^{\infty} [Vp\ f(x(t),K(t),I(t))^2]\ e^{-2rt}\ dt$$

$$\dot{K} = I - \delta\ K \qquad\qquad K(0) = K_0 .$$

Here the production function is specified as $y = f(x,K,I)$, i.e. output is a function of current period variable inputs x, capital stock K and gross investment I. Note that redefining the utility function in (14) as $W_0 + EW - \alpha(W_0 + EW,VW)/2\ VW$ rather than as (13) would not change the solution to (14). The utility function is defined as in (13)/(14) in order to make the relation between our results and standard risk-neutral dynamic duality envelope relations most obvious. Duality between J(.) and technology can be established assuming decreasing absolute risk aversion, in a manner somewhat similar to the static case.

Since (14) is an autonomous problem (e.g. expectations for prices and technology are static) over an infinite horizon, the current value of the objective function at solution does not depend explicitly on t_0. Thus (e.g. Arrow and Kurz; Epstein 1981a)

$$dJ(.)/dt = -\ r\ J(.)\ . \tag{15}$$

Then the Hamilton-Jacobi equation (at t=0) has the form

$$r\ J(Ep,w,w^{k},Vp,r,K_0,W_0)$$

$$= \max_{x,\ I}\ Ep\ f(x,K,I) - w\ x - w^{k}\ I - \hat{\alpha}(.)/2\ Vp\ f(x,K,I)^2 + dJ(.)/dK\ (I - \delta K) \tag{16}$$

where $\hat{\alpha}(.) \equiv \alpha(W_0 + EW^{*}|_{x_0,I_0}\ ,\ VW^{*}|_{x_0,I_0})$. $EW^{*}|_{x,I_0}$ and $VW^{*}|_{x_0,i_0}$ denote optimal mean and variance of wealth W conditional on the choice (x,I) at t=0.

Major properties of the dual J(.) are summarized in the following Theorem. Assuming a general intertemporal utility function that is nonseparable over time, the risk aversion function is defined below as $\alpha(.) \equiv \alpha(W_0 + \int_{t=0}^{\infty} [Ep\ f(x(t),K(t),I(t)) - w\ x(t) - w^{k}\ I(t)]\ e^{-rt}\ dt,\ \int_{t=0}^{\infty} Vp\ f(x(t),K(t),I(t))^2\ e^{-rt}\ dt)$. [See Arnade and Coyle for development of a dynamic duality model and an empirical application in the case of a linear mean-variance utility function].

Theorem 4. Assume that J(.) for problem (14) is defined and differentiable. Then

a) J is increasing in Ep and decreasing in (w,w^k) given decreasing absolute risk aversion (A.2), and decreasing in Vp if $\alpha(.)>0$ and $\alpha(.)$ nondecreasing in VW;

b) J is nondecreasing in K_0 given decreasing absolute risk aversion;

c) J is linear homogeneous in (Ep,w,w^k,Vp,W_0) if $\alpha(.)$ is homogeneous of degree zero in $(W_0 + EW,VW)$;

d) J is convex in (Ep,w,w^k,Vp,r,W_0) if $\alpha(.)$ is concave in (Ep,w,w^k,Vp,r,W_0);

e) $r\,J - dJ(.)/dK\,(I(0)^* - \delta\,K_0) + \alpha^*(.)/2\,Vp\,y(0)^{*2}$ is convex locally in $z \equiv (Ep,w,w^k,Vp,r,W_0)$, i.e.

$rJ_{zz} - J_{kzz}\,(I(0)^*-\delta K_0) + [\alpha^*(.)\,Vp]_{zz}\,y(0)^{*2}/2$ symmetric positive semidefinite;

f) i)
$$y(0)^* = r\,dJ(.)/dEp - d^2J(.)/dKdEp\,(I(0)^* - \delta\,K_0) + d\alpha(.)/d(W_0 + EW)\,Vp\,y(0)^{*2}/2\;\; dJ(.)/dEp \;/\; \{1 + dJ(.)/dW_0\}$$

ii)
$$x^i(0)^* = -r\,dJ(.)/dw^i + d^2J(.)/dKdw^i\,(I(0)^* - \delta\,K_0) - d\alpha(.)/d(W_0 + EW)\,Vp\,y(0)^{*2}/2\;\; dJ(.)/dw^i \;/\; \{1 + dJ(.)/dW_0\}$$
i=1,.,N

iii)
$$I(0)^* = -r\,dJ(.)/dw^k + d^2J(.)/dKdw^k\,(I(0)^* - \delta\,K_0) - d\alpha(.)/d(W_0 + EW)\,Vp\,y(0)^{*2}/2\;\; dJ(.)/dw^k \;/\; \{1 + dJ(.)/dW_0\}$$

iv)
$$r\,dJ(.)/dVp = \alpha^*(.)/2\,y(0)^{*2} + d^2J(.)/dKdVp\,(I(0)^* - \delta\,K_0) + d\alpha(.)/dVW\,Vp\,y(0)^{*2}\,dJ(.)/dVp \;/\; \{\alpha^*(.) + d\alpha(.)/dVW\,VW\}$$

v)
$$d\alpha(.)/d(W_0 + EW)\,Vp\,y(0)^{*2}/2 = d^2J(.)/dKdW_0\,(I(0)^* - \delta\,K_0) - r\,dJ(.)/dW_0$$

g)
$$J + r\,dJ(.)/dr = d^2J(.)/dKdr\,(I(0)^* - \delta K_0) + \{d\alpha(.)/d(W_0 + EW)\int_{t=0}^{\infty} E\pi(t)\,te^{-rt}\,dt + d\alpha(.)/dVW\int V\pi(t)\,te^{-2rt}\,dt\}\,Vp\,y(0)^{*2}.$$

The dual J(.) for the general nonlinear mean-variance or expected utility model (14) is linear homogeneous in (Ep, w, w^k, Vp, W_0) if $\alpha(W_0 + EW, VW)$ is homogeneous of degree zero in ($W_0 + EW, VW$), i.e. the mean-variance utility function is homothetic. This is implied by constant relative risk aversion.

In the general case where utility is nonseparable over time, it is necessary to employ envelope relations for the dynamic problem (14) as well as envelope relations for the Hamilton-Jacobi equation (16) in order to derive the relations in (f). Otherwise results are specified in terms of unobservable cumulative discounted plans $\int y(t)\, e^{-rt}\, dt$, $\int x(t)\, e^{-rt}\, dt$ and $\int I(t)\, e^{-rt}\, dt$ as well as observable first period plans y(0),x(0),I(0). This contrasts with standard risk-neutral models of the competitive firm, where the empirical equations are derived solely from envelope relations for the Hamilton-Jacobi equation. Combining both envelope relations leads to Theorem 4f. The risk aversion function $\alpha(.)$ can be eliminated by substituting (v) into (i-iii) for f:

a) $$y(0)^* = r\, dJ(.)/dEp - d^2J(.)/dKdEp\, (I(0)^* - \delta K_0) + \{d^2J(.)/dKdW_0\, (I(0)^* - \delta K_0) - r\, dJ(.)/dW_0\}\, dJ(.)/dEp \,/\, \{1 + dJ(.)/dW_0\}$$

b) $$x^i(0)^* = -r\, dJ(.)/dw^i + d^2J(.)/dKdw^i\, (I(0) - \delta K_0) - \{d^2J(.)/dKdW_0\, (I(0) - \delta K_0) - r\, dJ(.)/dW_0\}\, dJ(.)/dw^i \,/\, \{1 + dJ(.)/dW_0\}\,. \qquad i=1,..,N \qquad (17)$$

c) $$I(0)^* = -r\, dJ(.)/dw^k + d^2J(.)/dKdw^k\, (I(0)^* - \delta K_0) - \{d^2J(.)/dKdW_0\, (I(0)^* - \delta K_0) - r\, dJ(.)/dW_0\}\, dJ(.)/dw^k \,/\, \{1 + dJ(.)/dW_0\}\,.$$

These equations specify output supply, variable input demand and investment demand in terms of the dual J(.). They do not require specification of the function $\alpha(W_0 + EW, VW)$ or of EW, VW, which are unobservable. Since (c) is linear in I, it can be solved for a reduced form investment equation. Then reduced form output supply and variable input demand equations can also be obtained from (a)-(b).

Nonstatic Expectations

By relaxing the assumption of risk neutrality, the above models presumably are an improvement over standard deterministic dynamic duality models. However, by retaining the assumption of static expectations, these models do not incorporate an essential aspect of dynamic decisions under risk. The producer's degree of uncertainty about output prices is generally less for the immediate future than for the distant future, i.e. price uncertainty generally increases over the planning horizon.

The critical aspect for a dynamic model under risk is that the output price variance Vp changes over the planning horizon, but for generality we also allow the output price mean Ep to vary over the plan. In forming its production plan, the firm assumes that output price p is a random variable distributed independently but with mean and variance changing over time: $p_t = \bar{p}(t) + u_t$ where $Eu_t = 0$, var $u = \sigma_u^2(t)$, $\text{cov}(u_s,u_t) = 0$ $(s \neq t)$, and in turn $Ep_t = \bar{p}(t)$, var $p_t = \sigma_u^2(t)$, $\text{cov}(p_s,p_t) = 0$ $(s \neq t)$. The mean and variance of output price over the planning horizon are designated simply as Ep(t),Vp(t). For simplicity we will assume that mean and variance of p are generated by stationary deterministic processes, so the equations of motion for Ep and Vp are

$$\dot{E}p(t) = \theta_1(Ep(t)) \qquad \dot{V}p(t) = \theta_2(Vp(t)) \; . \tag{18}$$

Static expectations continue to be assumed for technology.

The general autonomous infinite horizon problem is

$$J(Ep_0,w,w^k,Vp_0,r,K_0,W_0) =$$

$$\max_{\{(x(t),\, I(t))\}} \quad EW - \alpha(W_0 + EW,VW)/2 \quad VW$$

$$\text{s. t.} \qquad EW \equiv \int_{t=0}^{\infty} [Ep(t)\; y(t) - w\; x(t) - w^k\; I(t)]\; e^{-rt}\; dt \tag{19}$$

$$VW \equiv \int_{t=0}^{\infty} [Vp(t)\; y(t)^2]\; e^{-2rt}\; dt$$

$$\dot{K} = I - \delta\; K \qquad K(0) = K_0$$

$$\dot{E}P(t) = \theta_1(Ep(t)) \qquad Ep(0) = Ep_0$$

$$\dot{V}p(t) = \theta_2(Vp(t)) \qquad Vp(0) = Vp_0 \; .$$

Since the above maximization problem is an infinite horizon autonomous problem (assuming stationary expectations), dJ(.)/dt = - r J(.). Then the Hamilton-Jacobi equation for this model at t=0 can be expressed as

$$
\begin{aligned}
& r\, J(Ep_0,w,w^k,Vp_0,r,K_0,W_0) - dJ(Ep_0,w,w^k,Vp_0,r,K_0,W_0)/dEp_0\ \dot{E}p(t_0) \\
& \quad - dJ(Ep_0,w,w^k,Vp_0,r,K_0,W_0)/dVp_0\ \dot{V}p(t_0) \\
& = \max_{x,\,I}\ Ep\ f(x,K_0,I) - w\,x - w^k\,I - \hat{\alpha}(.)/2\ Vp_0\ f(x,K_0,I)^2 \\
& \quad + dJ(.)/dK_0\ (I - \delta\ K_0)
\end{aligned}
\tag{20}
$$

where $\hat{\alpha}(.)$ is defined as in (16).

Various properties of the corresponding J(.) are summarized in the following Theorem. Here closed form solutions to the stationary processes (18) are denoted as

$$
Ep(t) = \beta_1(Ep_0,t) \qquad Vp(t)=\beta_2(Vp_0,t)\ . \tag{21}
$$

Theorem 5. Assume that J(.) for problem (19) is defined and differentiable. Then

a) J is decreasing in (w,w^k,r), increasing in Ep_0 ($d\beta_1(.)/dEp_0 \geq 0\ \forall\ t$), and decreasing in Vp_0 ($\alpha\geq 0$ and $d\beta_2(.)/dVp_0 \geq 0\ \forall\ t$) given decreasing absolute risk aversion and $\alpha(.)$ nondecreasing in VW;

b) J is nondecreasing in K_0 given decreasing absolute risk aversion;

c) J(.) is linear homogeneous in (Ep_0,w,w^k,Vp_0,W_0) if $\alpha(.)$ is homogeneous of degree zero in $(W_0 + EW,VW)$ and (18) are linear, i.e.

$$
\dot{E}p(t) = c_1\ Ep(t) \qquad \dot{V}p(t) = c_2\ Vp(t)\ ;
$$

d) J(.) is convex in (Ep_0,w,w^k,Vp_0,r,W_0) if $\beta_1(Ep_0,t)$ is convex in Ep_0 and $\alpha^*(.)\ \beta_2(Vp_0,t)$ is concave in (Ep_0,w,w^k,Vp_0,r,W_0) $(\alpha>0)$;

e)
$$
r\ J(.) - dJ(.)/dK\ (I(0)^* - \delta K_0) - dJ(.)/dEp\ \theta_1(Ep_0) - dJ(.)/dVp\ \theta_2(Vp_0) + \alpha^*(.)/2\ Vp\ y(0)^{*2}
$$

is convex locally in (Ep_0,w,w^k,Vp_0,r,W_0);

f) i) $y(0)^* = r\, dJ(.)/dEp - d^2J(.)/dKdEp\,(I(0)^* - \delta K_0) - d^2J(.)/dEp^2\,\theta_1(Ep_0)$

$- dJ(.)/dEp\, d\theta_1(Ep_0)/dEp - d^2J(.)/dVpdEp\,\theta_2(Vp_0)$

$+ d\alpha(.)/d(W_0 + EW)\, Vp\, y(0)^{*2}/2\;\; dJ(.)/dEp\, / \{1 + dJ(.)/dW_0\}$

ii) $x^i(0)^* = -r\, dJ(.)/dw^i + d^2J(.)/dKdw^i\,(I(0)^* - \delta K_0)$

$+ d^2J(.)/dEpdw^i\,\theta_1(Ep_0) + d^2J(.)/dVpdw^i\,\theta_2(Vp_0)$

$- d\alpha(.)/d(W_0 + EW)\, Vp\, y(0)^{*2}/2\; dJ(.)/dw^i\, / \{1 + dJ(.)/dW_0\}$

$i=1,..,N$

iii) $I(0)^* = -r\, dJ(.)/dw^k + d^2J(.)/dKdw^k\,(I(0)^* - \delta K_0)$

$+ d^2J(.)/dEpdw^k\,\theta_1(Ep_0) + d^2J(.)/dVpdw^k\,\theta_2(Vp_0))$

$- d\alpha(.)/d(W_0 + EW)\, Vp\, y(0)^{*2}/2\;\; dJ(.)/dw^k\, / \{1 + dJ(.)/dW_0\}$

iv) $r\, dJ(.)/dVp = -\alpha^*(.)/2\, y(0)^{*2} + d^2J(.)/dKdVp\,(I(0)^* - \delta K_0)$

$+ d^2J(.)/dEpdVp\,\theta_1(Ep_0) + d^2J(.)/dVp^2\,\theta_2(Vp_0) + dJ(.)/dVp\, d\theta_2(Vp_0)/dVp$

$+ d\alpha(.)/dVW\;\; Vp\;\; y(0)^{*2}\;\; dJ(.)/dVp\, / \{\alpha^*(.) + d\alpha(.)/dVW\; VW\}$

v) $d\alpha(.)/d(W_0 + EW)\, Vp\, y(0)^{*2}/2 = d^2J(.)/dKdW_0\,(I(0)^* - \delta K_0) - r\, dJ(.)/dW_0$

$+ d^2J(.)/dEpdW_0\,\theta_1(Ep_0) + d^2J(.)/dVpdW_0\,\theta_2(Vp_0)$

g) $J + r\, dJ(.)/dr = d^2J(.)/dKdr\,(I(0)^* - \delta K_0) + d^2J(.)/dEpdr\,\theta_1(Ep_0)$

$+ d^2J(.)/dVpdr\,\theta_2(Vp_0) + \{d\alpha(.)/d(W_0 + EW)\int_{t=0}^{\infty} E\pi(t)\, te^{-rt}\, dt$

$+ d\alpha(.)/dVW\int_{t=0}^{\infty} V\pi(t)\, te^{-2rt}\, dt\}\; Vp\;\; y(0)^{*2}.$

The properties of the dual J(.) for this model depend on the properties of equations of motion for Ep and Vp (19) as well as the utility function or $\alpha(.)$. For example, Theorem 5c indicates that J(.) is linear homogeneous in $(Ep_0, w, w^k, Vp_0, W_0)$ only if Ep(t) and Vp(t) are linear homogeneous in Ep_0,Vp_0 as well as $\alpha(W_0 + EW,VW)$ is homogeneous of degree zero in $(W_0 + EW,VW)$.

Theorem 5e indicates that the dynamic maximization hypothesis places restrictions on the second derivatives of dJ(.)/dEp and dJ(.)/dVp with respect to (Ep_0,w,w^k,Vp_0) when Ep and Vp vary over the planning horizon. This is in addition to restrictions on second derivatives of dJ(.)/dK as in standard models with static expectations.

Theorem 5f, where utility is nonseparable over time, is derived from envelope relations for the dynamic problem (19) as well as envelope relations for the Hamilton-Jacobi equation (20). The analysis is similar to Proposition 3f, where expectations are static and utility is nonseparable over time. Derivatives of $\alpha(.)$ are eliminated by substituting (v) into (i-iii) of Theorem 5f. This leads to a system of reduced form equations for investment demand, output supply and variable input demand.

A Static Model with Yield Uncertainty

The firm's risk preferences are modelled in terms of a general nonlinear mean-variance utility function $U = U(W_0 + E\pi,V\pi)$ as in (1). Assume for simplicity a scalar production function $y = f(x,e)$ where e is a stochastic (weather) variable with moments denoted by the vector q. Denote the output mean and variance functions as $Ey = Ey(x,q_1)$ and $Vy = Vy(x,q_2)$, where q_1 and q_2 are vectors of moments (or transformations of moments) influencing mean and variance of output (Ey,Vy), respectively. The following restrictions on the stochastic aspect of technology simplify the specification of the duality model:

T.1: $dEy(x,q_1)/dq_2 = 0$
T.2: $Vy = \beta_1(x)\,\beta_2(q_2)$.

T.1 states that moments influencing Vy do not influence Ey, and T.2 states that $Vy(x,q_2)$ is multiplicative in functions of x and q_2. These assumptions are often satisfied in empirical models. Also assume for simplicity that output price p is known with certainty.

The static dual indirect utility function is

$$U^*(p,w,q,W_0) = \max_{x\geq 0}\; W_0 + p\,Ey(x,q_1) - w\,x - \alpha(W_0 + p\,Ey(x,q_1) - w\,x,\, p^2\,Vy(x,q_2))/2\;p^2\,Vy(x,q_2)\,. \tag{22}$$

Duality between the dual $U^*(.)$ and technology $y = f(x,e)$ can be established assuming decreasing absolute risk aversion, in a manner somewhat analogous to Theorem 1.

Properties of the dual $U^*(.)$ are summarized as follows.

Theorem 6. Assume that $U^*(.)$ for (22) is defined and differentiable. Then

(a) U^* is decreasing in w and increasing in W_0 given A.2;

(b) $U^*(\lambda p, \lambda w, q_1, \lambda q_2, \lambda W_0) = \lambda\, U(p,w,q,W_0)$ if

(i) $\alpha(\lambda W_0 + \lambda E\pi, \lambda V\pi) = \alpha(W_0 + E\pi, V\pi)$ (constant relative risk aversion)

(ii) $Vy(x,\lambda q_2) = \lambda^{-1}\, Vy(x,q_2)$

(iii) $Ey(x,q_1)$ is independent of q_2 (T.1);

(c) i) $Ey = \{dU^*(.)/dp + (\alpha + d\alpha(.)/dV\pi\; V\pi)\, p\, Vy\} \;/\; dU^*(.)/dW_0$

ii) if $dEy(x,q_1)/dq_2^i = 0$ ($q_2^i \;\varepsilon\; q_2$) (T.1):

$$Ey = \{dU^*(.)/dp - dU^*(.)/dq_2^i\; (Vy \;/\; dVy(.)/dq_2^i\;)\; 2/p\} \;/\; dU^*(.)/dW_0$$

iii) $x^i = -\, dU^*(.)/dw^i \;/\; dU^*(.)/dW_0 \qquad i = 1,.,N$

iv) $$dU^*(.)/dq_2^i = -\,(\alpha + d\alpha(.)/dV\pi\;\; V\pi)\; p^2\, dVy(.)/dq_2^i$$
$$+\,(1 - d\alpha(.)/d(W_0 + E\pi)\;/2)\; p\; dEy(.)/dq_2^i \qquad q_2^i \;\varepsilon\; q_2$$

(d) $[U^*(.) - p\, Ey(.) + \alpha(.)/2\; p^2\, Vy(.)]_{vv}$ symmetric positive semidefinite where $v \equiv (p,w,q,W_0)$.

The (expected) output supply equation is simplified under restrictions T on the stochastic component of technology. T.2 implies $dVy(.)/dq_2^i = \beta_1(x)\; d\beta_2(q_2)/dq_2^i$, so $Vy \;/\; dVy(.)/dq_2^i = \beta_2(q_2) \;/\; d\beta_2(q_2)/dq_2^i \equiv \beta^i(q_2)$. Then (c-ii,iii) imply

$$Ey = \{dU^*(.)/dp - dU^*(.)/dq_2^i\; \beta^i(q_2)\; 2/p\} \;/\; dU^*(.)/dW_0$$
$$x^i = -\, dU^*(.)/dw^i \;/\; dU^*(.)/dW_0 \qquad i = 1,.,N\,. \tag{23}$$

The duality model simplifies further in the case of a Just-Pope production function $y = a(x) + b(x)^{1/2}\, e$, where $Ey = a(x) + b(x)^{1/2}\, Ee$ and $Vy = b(x)\, Ve$. Define $q_1 \equiv Ee$ and $q_2 \equiv 1/Ve$. Then Theorem 6 simplifies to the following.

Theorem 7. Assume that $U^*(.)$ for (22) is defined and differentiable, and the technology is Just-Pope: $y = a(x) + b(x)^{1/2}\, e$, $q_1 \equiv Ee$ and $q_2 \equiv 1/Ve$. Then

(a) U^* is decreasing in w and increasing in W_0 given A.2;

(b) $U^*(\lambda p, \lambda w, q_1, \lambda q_2, \lambda W_0) = \lambda U(p,w,q,W_0)$ if $\alpha(\lambda W_0 + \lambda E\pi, \lambda V\pi) = \alpha(W_0 + E\pi, V\pi)$ (constant relative risk aversion)

(c) i) $Ey = \{dU^*(.)/dp + (\alpha + d\alpha(.)/dV\pi\ V\pi)\ p\ Vy\} / dU^*(.)/dW_0$

ii) $Ey = \{dU^*(.)/dp + dU^*(.)/dq_2\ q_2\ 2/p\} / dU^*(.)/dW_0$

iii) $x^i = - dU^*(.)/dw^i / dU^*(.)/dW_0 \qquad i = 1,.,N$

iv) $dU^*(.)/dq_2 = (\alpha + d\alpha(.)/dV\pi\ V\pi)\ p^2\ Vy/q_2$

(d) $[U^*(.) - p\ Ey(.) + \alpha(.)/2\ p^2\ Vy(.)]_{vv}$ symmetric positive semidefinite where $v \equiv (p,w,q,W_0)$.

The simplest case arises when the utility function is linear mean-variance and the technology is Just-Pope. Then $U^*(\lambda p, \lambda w, q_1, \lambda q_2, \lambda W_0) = \lambda U^*(p,w,q,W_0)$ and

$$\begin{aligned} i) \quad Ey &= dU^*(.)/dp + \alpha\ p\ Vy \\ &= dU^*(.)/dp + dU^*(.)/dq_2\ q_2\ 2/p \\ ii) \quad x^i &= - dU^*(.)dw^i \qquad i = 1,.,N \qquad (24) \\ iii) \quad Vy &= 2/\alpha\ q_2/p^2\ dU^*(.)/dq_2\ . \end{aligned}$$

(i)-(ii) provide equations for Ey and x that are often linear in coefficients. In addition, (iii) provides an equation for Vy that is nonlinear in coefficients (since α must be estimated).

The above models assume a single (stochastic) output and no uncertainty regarding prices. However the analysis can be extended to more general cases of multiple outputs and uncertainty regarding both yields and prices.

Conclusion

This paper has summarized results for several working papers. These duality models of production under risk aversion and uncertainty allow for a nonlinear mean-variance utility function, dynamics, and yield as well as price uncertainty. The models generally appear to be tractable for econometric research, although they are more complex than an earlier static duality model with a linear mean-variance utility function and price uncertainty.

There are obvious limitations to these papers. For example, only mean and variance of distributions are considered here whereas distributions are unlikely to be symmetric. On the other hand, arguments relating mean-variance to decreasing absolute risk aversion (Epstein 1985) and expected utility (Meyer; Sinn) apparently do not require symmetry. Second, dynamics and yield uncertainty are not considered jointly. In particular the sequential nature of production decisions and uncertainty in agriculture is not considered.

It is hoped that these models or further extensions of these models will eventually be useful in empirical research. However econometric applications of decision models under risk presumably require a more detailed specification of agents' subjective probability distributions than in the case of risk neutrality, and this may continue to limit the usefulness of such models.

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