QUANTIFYING LONG RUN AGRICULTURAL RISKS AND EVALUATING FARMER RESPONSES TO RISK

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ON THE BEHAVIOR OF THE COMPETITIVE PRODUCER UNDER MULTIVARIATE RISKS

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1. Introduction

The effect of output price uncertainty on the level of production by the competitive producer is one of the most familiar results in all of the literature on uncertainty. There is a straightforward correspondence between a concave utility function defined on profits and a positive risk premium (Arrow (1965), Pratt (1964)), and between each of those conditions and a lower level of output than would be forthcoming from the firm facing the same mean price but with certainty (Sandmo (1971)). Following Sandmo's influential paper, other aspects of uncertainty and the competitive firm have been explored in numerous studies. Nearly all of these studies make use of the expected utility hypothesis, in which the producer makes decisions so as to maximize the expected value of a utility function defined only on the level of profits. The process by which profits produce utility, presumably through the consumption of goods after income is received, is left implicit.

There are many situations where the producer's objective function is multivariate, defined on several random arguments, so a generalization of the theory of the firm under uncertainty is necessary. Stiglitz (1969), Kihlstrom and Mirman (1974), Epstein (1975), and Karni (1979) have examined the effects of multivariate uncertainty on consumer behavior. In this paper, we apply their results to a generalization of the Sandmo (1971) model, to analyze the behavior of the firm under multivariate risk; this differs from the case where several *sources* of uncertainty affect a single random argument, profits. Our model can therefore be thought of as a combination of two existing models, Sandmo's (1971) model of the firm's behavior under price uncertainty and Epstein's (1975) model of consumer choices under uncertainty. We proceed by extending the Sandmo framework to incorporate multivariate uncertainty about the prices of consumer goods, or by endogenizing the joint distribution of prices and income in Epstein's model.

There are many examples where firms face multivariate risks. For instance, any enterprise engaged in both production and consumption, such as an agricultural household, will have a multivariate objective function. Such firms face uncertainty about both profits and the prices of goods consumed from profits. This is especially important in developing countries where risks may be large relative to total wealth and access to capital markets is limited.

Assuming that the only source of income is profit from production, and that the production choice is made *ex ante,* while consumption decisions are made solely *ex post,* we show below that the single–period objective function of the producer is the usual indirect utility function. It includes profits, as in the standard case, but also includes the vector of prices of goods consumed. When only the price of output is uncertain (univariate risk), the level of output is not affected by preferences for consumption goods (ordinal preferences), but it is affected by the degree of risk aversion, which is a cardinal property. However, in the presence of multivariate risk, where the price of output *and* the prices of consumption goods are uncertain, output decisions are affected by both, and the separability between consumption and

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production decisions breaks down.[2] Thus, cardinal properties of the function representing the ordinal preferences and the ordinal preferences themselves affect the level of output. This finding is analogous to the results of Kihlstrom and Mirman (1974) or Karni (1979) for the analysis of consumer behavior under multivariate risk—in situations involving multivariate risk, ordinal preferences play an important role in an agent's attitude toward risk.

The paper proceeds as follows. In section 2, we present the model and discuss the cases in which the objective function of the producer can be reduced to an indirect utility function defined on profits and prices. We then examine the conditions under which it will reduce to a function of profits alone. In section 3, we derive the conditions under which the output level is the same for both the multivariate risk model and the classic one and show that these conditions are very restrictive. Following that, in section 4, we establish the main results of the paper concerning production decisions under multivariate price risk. Both the short and long runs are considered, using a multivariate risk premium. We find that many of the traditional results concerning the level of output under uncertainty may no longer hold. The paper concludes with a summary of the results and suggestions for further research.

2. Modeling Producer Behavior Under Multivariate Uncertainty

We consider a one-period model of a firm engaged in both production and consumption. A single period model could be constructed to allow *ex ante* and *ex post* decisions with respect to both production and consumption. The issue of *ex post* flexibility with respect to production decisions has been considered in many studies (*e.g.* Turnovsky (1973), Epstein (1978), Wright (1984)) and is beyond the scope of the current paper. Thus, in the model below we consider *ex ante* choices of output and some of the consumption goods and *ex post* choices of other goods.

Hence, there are two types of goods; those which are precommitted before the realization of prices, and those which are chosen when prices are known. We denote the M goods of the first type by $z = (z_1, z_2, ..., z_M) \geq 0$ and the N goods of the second type by $x = (x_1, x_2, ..., x_N) \geq 0$. The prices of the goods in z and x are denoted by $q = (q_1, q_2, \ldots, q_M)$ and by $p = (p_1, p_2, \ldots, p_N)$, respectively. We assume that the producer's objective function is his utility function $U(z, x)$, defined over both types of goods. The output price is denoted by p_y and may be contained in p if the producer consumes a portion of his output. $U(z, x)$ is assumed to be continuous in z and x, non-decreasing, and quasi-concave for $x, z \geq 0$.

The producer is subject to the following constraints. Initial wealth is W_0 (assumed to be non-random) and the budget constraint is given by

$$W_0 + \pi = q'z + p'x \ ,$$

where

$$\pi = p_y y - C(y) - T,$$

T denotes fixed costs, $C(\cdot)$ is a variable cost function with $C' > 0$, $C'' \geq 0$, and output y is assumed to be non-stochastic.[3] Finally, we assume that *ex ante* knowledge of the producer

[2] A similar finding concerning non-separability was found by Roe and Graham-Tomasi (1986), in a dynamic model of an agricultural household facing uncertainty about yields.

[3] This assumption is not essential, but simplifies the analysis and allows comparison with Sandmo's model.

concerning q, p, and p_y can be summarized by a subjective probability distribution function $F(q, p, p_y)$ with finite moments.

The producer must choose output y and the level[4] of x and z so as to maximize the expected value of a Von Neumann–Morgenstern utility function:

$$\max_{y,\,z,\,x \geq 0} \int U(z, x(y, \omega))\, d\mu(\omega)$$

subject to

(i) $q'(\omega)z + p'(\omega)x(\omega) \leq W_0 + \pi(\omega, y)$

and

(ii) $\pi(\omega, y) = p_y(\omega) \cdot y - C(y) - T,$

where by ω we denote the state of the world and μ is the producer's subjective probability measure defined on ω. The maximization problem can be solved in two stages—maximization with respect to x for a given realization of prices and prior choices of z and y, and then maximization with respect to z and y. As Epstein (1975) argued, since consumption plans for x can be revised when prices are realized, the first maximization problem may be taken inside the integral to obtain a revised objective function, the variable indirect utility function $g(z, s, p)$, where s is the amount of total wealth available for consumption of x, and is defined by

$$s(\pi, z, q) = W_0 + \pi(p_y, y) - q'z.$$

The objective becomes

$$\max_{y,\,z \geq 0} E\, g(z, s(\pi, z, q), p) = \int_{p_y} \int_{q} \int_{p} g(z, s(\pi, z, q), p)\, dF(q, p, p_y).$$

Without additional restrictions the objective function $g(\cdot)$ does not reduce to the traditional objective function of firms $(u(\pi))$. Both include profit as an argument, but the former also includes the vector of consumption goods z and the random vectors p and q. The maximization problem of the producer is therefore a multivariate risk problem; the utility function depends on more than one random argument.

Two assumptions are required for $g(\cdot)$ to reduce to the univariate objective function $u(\pi)$. First, all consumption decisions must take place *ex post*, so z and q can be ignored. Second, the price vector p must be known when the production decision is made. If the first assumption holds, $g(\cdot)$ reduces to the ordinary indirect utility function $V(\pi, p)$. If, in addition, p is known to be fixed at some level $\hat{p}$, then the objective function is $V(\pi, \hat{p})$. This, of course, may be treated as a function of profit alone.

The second assumption is unrealistic, since it is hard to imagine that only the price of output is random, while the prices of all consumption goods are known in advance. For instance, p_y may be contained in p, which violates this assumption immediately. The first assumption is more plausible, since situations which involve *ex ante* consumption decisions are relatively rare. For this reason and for the sake of comparability with the conventional

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The main results hold under more general forms of stochastic profits.

[4] To avoid the possibility of bankruptcy, we assume that the choice of z is constrained by $Pr(q'z \leq W_0 + \pi) = 1$ (see also Epstein (1975) and Karni (1981)).

model, we adopt the first assumption in the rest of the analysis and consider the implications of relaxing the second one.

3. Equality of Output Levels Under Univariate and Multivariate Uncertainty

In section 2, we investigated the cases in which the multivariate objective function reduces to a univariate one, the traditional utility function defined on profits. In this section, we examine the following question: given that the objective functions of competitive producers will differ in the two environments, are there cases in which output is the same regardless of the probability distribution of prices? Since these cases turn out to be quite limited, the results will show that the additional uncertainty does affect behavior.

There are also empirical implications of multivariate risk. Suppose interest is in making inferences about a producer's degree of aversion to univariate income risk. We show below that if the prices of consumption goods are random, such inferences depend on knowing the entire probability distribution of prices *and* the nature of the producer's (ordinal) preferences for goods. Alternatively, knowing the conditions under which the output levels are the same in the multivariate and univariate models might facilitate correct inferences using only information about the marginal probability distribution of the output price and the observed level of production. We show below that only if the indirect utility function is separable in income and prices will the levels of output under univariate and multivariate uncertainty be the same.

The producer facing univariate income risk is assumed to maximize the expectation, over the distribution of p_y, of an indirect utility function defined over income[5] and fixed prices $\bar{p}$, taken to be the mean of the marginal distribution of p when it is random (the multivariate case). That makes the indirect utility function a function of income alone, since nothing else varies, and yields the conventional objective function.

Under multivariate risk, the producer's problem is[6]

$$\max_{y \geq 0} \int_{p_y} \int_{p} V(\pi, p)\, dF(p, p_y)$$

where $F(p, p_y)$ is the joint probability distribution function of all prices. Restrictions on the form of preferences are needed to establish equality between output under univariate and multivariate uncertainty. This is shown in proposition I for the case where $p_y \notin p$.

Proposition I: Denote the optimal output levels under multivariate and univariate risk as y^m and y^u, respectively. Then $y^m = y^u$ for all probability distributions of prices if and only if $V(\pi, p)$ can be written as $V^1(\pi) + V^2(p)$. That is, for output levels to be unaffected by the presence of the additional price uncertainty, the indirect utility function must be additively separable in π and p. The proof is presented in the Appendix.

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[5] To simplify the analysis below, we assume throughout the remainder of the paper that initial wealth is contained in the fixed costs term T and is therefore part of profits.

[6] The indirect utility function is assumed to be differentiable, and we assume the existence of unique interior solutions for the producer's problem, both in the univariate and multivariate models, as well as for the choice of consumption goods. While we have not introduced an explicit numeraire commodity, any of the deterministic prices in the model (input prices) can serve as numeraire.

The assumption of additive separability of $V(\cdot)$, required to support Proposition I, places extreme and implausible limitations on preferences. We show this in Corollary I.

Corollary I: The following statements are equivalent:

(i) $y^m = y^u$ for every probability distribution of prices.

(ii) $R = \eta_i = 1$ for each i, where R is the coefficient of relative risk aversion and η_i is the income elasticity of demand for good i.

(iii) The indirect utility function is of the form

$$V(\pi, p) = \log(\pi) - \log(G(p)),$$

where $G(p)$ is linearly homogeneous in p. The proof is presented in the Appendix.

Corollary I establishes that a very special form for the indirect utility function is required for the output level to be the same in the two environments for all risks. It rules out risk neutrality or risk-seeking behavior, as well as differences in income elasticities across goods. The case of risk aversion is also restrictive, since its extent is determined by $R = 1$. Each of these properties can be empirically tested, and is generally rejected.[7]

In Proposition I, we assumed that $p_y \notin p$. This does not hold if the producer consumes some of his own product. The conventional model of univariate uncertainty is no longer relevant. Instead, we compare levels of output under multivariate uncertainty (y^m) with the output level in a model where p_y alone is random ($y^{u'}$). In Proposition II, we establish the conditions for the two output levels to be the same. Even though $y^{u'}$ is the output level under output price risk, the same risk faced by the producer in Sandmo's model, we show in Proposition II that both cardinal and ordinal properties of preferences affect output.

Proposition II: Assume that $p_y \in p$, so the producer consumes some of his output and solves

$$\max_{y \geq 0} E_{p_y} V(\pi, p_y, \overline{p}_{i \neq y})$$

where $\overline{p}_{i \neq y}$ denotes the vector of goods prices excluding p_y. Then $y^m = y^{u'}$ if and only if the indirect utility function is of the form

$$V(\pi, p) = V^1(\pi, p_y) + V^2(p).$$

Thus, V is additively separable in income and all prices except p_y. The proof follows the approach used to prove Proposition I; the only difference is that $V_{\pi p_i} \equiv 0$ for all $i \neq y$. □

The form we obtain is slightly less restrictive, since all prices except for p_y are additively separable from income. As a result, fewer limitations are placed on the nature of preferences, and $R = \eta_i$ for each $i \neq y$, but is no longer required to be constant.

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[7] If the set of probability distributions under consideration is such that some prices are deterministic, then the restrictions on preferences are somewhat less restrictive. The income term in the indirect utility function must be separable only from the *random* prices, and income elasticities must equal R only for the goods whose prices are random. Interestingly, in a different context (measuring the benefits to consumers from price stabilization), Turnovsky, Shalit, and Schmitz (1980), found a similar functional form necessary for consumer surplus to be exact.

In the derivation of the special form of utility function necessary to support the equality of y^m and y^u (or $y^{u'}$ for the case of $p_y \in p$), we restricted the level of prices in p (other than p_y) in the univariate environment to be equal to the expected price level ($\bar{p}$) of the multivariate case. However, if the producer maximizes the expected value of the utility function in Proposition I, neither the price level (in the univariate case) nor its expected value (in the multivariate case) has any effect on the output level. We state this as Corollary II.

Corollary II: Denote the price level in the univariate model by $\hat{p}$ and the expected value of prices in the multivariate model by $\bar{p}$. Independence of the output level from both the level of prices ($\hat{p}$) in the univariate model and its expected value ($\bar{p}$) in the multivariate model is implied by $y^m = y^u$ for all probability distributions of prices.

Proof: Proposition I established that a necessary condition for $y^m = y^u$ for all price distributions is additive separability (in π and p) of the indirect utility function. It is easy to see that in this case both y^m and y^u are independent of $\bar{p}$ and $\hat{p}$. Thus, if the output level under univariate uncertainty equals the output level under multivariate uncertainty for all distributions of prices with mean $\bar{p} = \hat{p}$, it is also the same for any other price distribution with arbitrary mean. A similar result holds if $p_y \in p$, for the preferences in Proposition II. □

The forms above are the only utility functions for which $y^m = y^u$ or $y^m = y^{u'}$ for all price distributions. It may be the case, however, that there are distributions of interest for which the output levels are equal with less restrictive utility functions. A particular case which comes to mind is the case where p and p_y are independent, which, of course, requires that $p_y \notin p$. As already noted, this limits the number of applications.

One is tempted to assume that in the case of independence between p_y and p, y^u is equal to y^m. However, they are not in general equal. Under independence, the joint density of prices equals the product of the two marginal densities, so the producer solves

$$\max_{y \geq 0} \int_{p_y} \int_{p} V(\pi, p) g_1(p) g_2(p_y) \, dp \, dp_y \quad = \quad \max_{y \geq 0} \int_{p_y} E_p [V(\pi, p)] g_2(p_y) \, dp_y .$$

Only for an indirect utility function $V(\cdot)$ which is linear in p can the expectation over p in the first integral yield the function the producer would maximize in the case of univariate uncertainty, $V(\pi, E(p))$ or $V(\pi, \bar{p})$. However, even if $V(\cdot)$ is nonlinear in p, so that the producer facing multivariate uncertainty has a different objective, the choice of output could still be the same. This requires a special form of utility function, a special case of which is linear in prices.

Proposition III: If p and p_y are independent, then $y^u = y^m$ if and only if the indirect utility function takes the form

$$V(\pi, p) = a(p) + b(p) h(\pi).$$

The proof is provided in the Appendix.

Corollary III: Let p and p_y be independent. Then $y^m = y^u$ for all probability distributions of prices if and only if

$$V(\pi, p) = \begin{cases} \dfrac{1}{1-R}\left[\dfrac{\pi}{G(p)}\right]^{1-R} - H(p) & R \neq 1 \\ H(p)\log\left[\dfrac{\pi}{G(p)}\right] & R = 1 \end{cases}$$

where R is the coefficient of relative risk aversion and $G(p)$ and $H(p)$ are homogeneous of degrees 1 and 0, respectively, with G positive.

Proof: The form

$$V(\pi, p) = a(p) + b(p)\cdot h(\pi)$$

implies a coefficient of relative risk aversion of the form

$$R = -\frac{V_{\pi\pi}}{V_\pi}\cdot\pi = -\frac{b(p)h''(\pi)}{b(p)h'(\pi)}\cdot\pi = -\frac{h''(\pi)}{h'(\pi)}\cdot\pi.$$

This function does not depend on p. $V(\pi, p)$ is homogeneous of degree 0 in π and p, and hence, R itself must also be homogeneous of degree 0 in π and p (*e.g.* Deschamps (1973)). Since R is homogeneous of degree zero in π and p and does not depend on p, it must be a constant also with respect to π. Stiglitz (1969) and Hanoch (1977) found that R is constant if and only if the indirect utility function takes the above form. □

Although this form is more general than the ones needed to support Corollary I or Proposition II, it still restrictive in both its cardinal and ordinal properties. The main restrictions on behavior under risk are that R and the proportional risk premium are independent of the level of wealth. The corresponding measure of absolute risk aversion depends on wealth, but not on prices.

To see the restriction on ordinal preferences, note that the form we obtained for the case of $R \neq 1$ is a special case of the Gorman polar form

$$V(\pi, p) = \Phi[\pi/b(p)] + a(p).$$

An extensive discussion of the implications of these preferences is found in the demand literature (*e.g.* Deaton and Muellbauer (1980)). The form which corresponds to R=1 is the Bernoulli utility function (Bernoulli (1954)). Stiglitz (1969) showed that for R to be constant globally, the function $H(p)$ must be constant, so these forms reduce to special cases of homothetic preferences. Note that Corollary II is valid in this case as well, *i.e.* the level of output under the above forms of preferences does not depend on either the price level or its expected value.

The results established in these propositions make use not only of specific forms of preferences, but, in the last case, of the independence of p from p_y. For some cases, the latter assumption may be reasonable, and then one need only worry about testing the restrictions on preferences. If independence does not hold, we have shown that there is just one form for the indirect utility function for which production decisions are not affected by ordinal preferences. It places extreme limitations on both ordinal preferences and risk attitude, but it illustrates the key role played by the functional form of the indirect utility function.

4. Equilibrium Output Under Multivariate Uncertainty

The competitive producer facing multivariate risk was shown in the previous section to produce the same level of output as under univariate risk only under limited circumstances. When these do not hold, the usual results concerning risk aversion and the level of output (Sandmo (1971)) need not obtain. Indeed, even characterizing risk aversion, let alone the relationship between risk attitudes and the level of production, is complicated by the presence of multivariate risk. In this section, we examine in more detail the level of output in the presence of randomness in all prices.

4.1: A Multivariate Risk Premium for Income Risk

The univariate risk premium and the associated measures of risk aversion which were defined by Arrow (1965) and Pratt (1964) play an important role in the analysis of many situations in which firms face a univariate risk (*e.g.* Chavas and Pope (1981); Flacco (1983); Flacco and Larson (1987)). Based on Karni (1979), Finkelshtain and Chalfant (1988) developed a generalization of the univariate risk premium. They defined the income risk-premium as the amount the producer would pay to stabilize income with the prices of consumption goods random. It is given by $S(y, F)$, where

$$EV(\pi, p) = EV(\bar{\pi} - S, p).$$

The interpretation of S as an "income-risk" premium under multivariate risk is analogous to that of the regular Arrow-Pratt risk premium, which appears as a special case of S, when goods' prices are also fixed. This can be illustrated nicely for small risks.[8] As shown in the Appendix, a Taylor approximation of the above expression yields

$$S = -\frac{1}{2}\sigma^2_{p_y}\, y^2 \frac{V_{\pi\pi}}{V_\pi} - \sum_{i=1}^{N} \sigma_{p_i p_y}\, y\, \frac{V_{\pi p_i}}{V_\pi}$$

where $\sigma^2_{p_y}$ is the variance of p_y (so that $\sigma^2_{p_y} y^2$ is the variance of revenues, or profits with costs fixed) and $\sigma_{p_i p_y}$ is the covariance between p_y and the i^{th} price in p (so $\sigma_{p_i p_y}\, y$ is the covariance of income with the i^{th} price). The first term in this expression is the regular Arrow–Pratt risk premium, the amount that the producer is willing to pay to stabilize income when prices are fixed. The second term can be thought of as a monetary measure of the producer's aversion to the stochastic interaction between prices and income. If prices are fixed, the second term vanishes, and S reduces to the Arrow–Pratt (univariate) risk premium. However, S need not be of the same sign as the Arrow–Pratt risk premium when prices are random.

4.2: The Short Run: Optimal Behavior for a Single Firm

The necessary condition for maximization of the producer's objective function

$$E[V(\pi(y, p_y), p)]$$

is

$$E[V_\pi(p_y - C'(y))] = 0.$$

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[8] Following Karni (1979), we define small risks as those risks such that $Pr[(p_1, \ldots, p_n) \in b] = 1$ where b is an n–dimensional ball centered at $(\bar{p}_1, \ldots, \bar{p}_n)$, with radius ε which is arbitrarily close to zero.

The sufficient condition holds trivially for the case of an individual who is averse to univariate income risk, with a convex cost function, since $V_{\pi\pi} < 0$ and $C''(y) > 0$ guarantee that

$$E[V_{\pi\pi}(p_y - C'(y))^2 - V_{\pi}C''(y)] < 0.$$

We assume that it holds for other cases as well. The necessary condition can be rewritten as

$$E[V_{\pi} p_y] = E[V_{\pi}C'(y)]$$

or

$$E[V_{\pi}(p_y - \bar{p}_y)] = E[V_{\pi}(C'(y) - \bar{p}_y)].$$

The left-hand side is the covariance between the marginal utility of income and the output price. In the univariate case, when $V_{\pi\pi} < 0$, this covariance is clearly negative. This implies that the expected price of output exceeds marginal cost ($\bar{p}_y > C'(y)$). It is exactly this observation that leads, in the Sandmo case, to the conclusion that output is strictly less than the expected profit maximizing level.

In the multivariate case, the above expectations are taken with respect to the joint distribution of all prices, and V_{π} depends on the random vector p as well as π. As a result, we are unable to determine the sign of $Cov(V_{\pi}, p_y)$ without considerably more information about preferences and/or the distribution of prices, and the level of output is not necessarily below the expected profit-maximizing level. However, for distributions where p and p_y are independent, a less ambiguous result can be obtained.

Proposition IV: Denote the output under certainty by y^c. Then univariate income risk aversion ($V_{\pi\pi} < 0$) and independence of p and p_y are sufficient for $y^m < y^c$.

To prove Proposition IV, we need the following Lemma, which is analogous to a result involving the conditional variance (e.g. Mood, Graybill, and Boes (1974, p. 159)).

Lemma I: Let $x \in R^1$, $y \in R^N$ be random variables, and let $g: R^{N+1} \rightarrow R^1$ be an arbitrary function. Then

$$Cov[g(x,y), x] = E_y\{Cov[g(x,y), x] \mid y\} + Cov\{E[g(x,y) \mid y], E(x \mid y)\},$$

so that the unconditional covariance is equal to the expected value of the conditional one plus the covariance between the conditional expectations. Lemma I is proven in the Appendix. We now prove Proposition IV.

Proof: It was shown above that

$$(C'(y^m) - \bar{p}_y) = \frac{Cov(V_{\pi}, p_y)}{E[V_{\pi}(\pi, p_y)]}.$$

From Lemma I, it follows that this covariance can be expressed as

$$Cov(V_{\pi}, p_y) = E_p Cov[(V_{\pi}, p_y) \mid p] + Cov[E_{p_y \mid p}(V_{\pi} \mid p), E_{p_y \mid p}(p_y \mid p)].$$

The first term in this sum is the expectation, taken over all values of p, of the conditional covariance between $V_{\pi}(\pi, p)$ and p_y, for a given p. As long as p is fixed (at any point) and $V_{\pi\pi} < 0$, this term is negative. If it is always negative, so is its expectation over all p.

If p and p_y are independent, the second term will be zero. This holds because $E_{p_y|p}(p_y|p)$ does not depend on p and is equal to $\bar{p}_y$. Hence, the unconditional covariance is negative, $\bar{p}_y > C'(y^m)$, and $y^m < y^c$.[9] □

As a result of the independence of p and p_y, the covariance between V_π and p_y is always negative for producers who are risk averse in the univariate sense. Once more, then, expected price exceeds marginal cost at the optimal output. As in Sandmo, aversion to univariate income risk causes output to be less than under expected profit maximization. However, the *level* of output is not, in general, the same as under univariate risk, even with independence. As we showed in section 2, those levels coincide only when the indirect utility function takes one of the homothetic forms of Proposition III.

The result of Proposition IV relies on the independence of p and p_y. When p and p_y are not independent, the second term in the expression for the covariance is no longer zero. It could be positive, and may well exceed the first term in magnitude. When that occurs, it will be optimal for the producer facing multivariate risk to produce *more* than the level which maximizes expected profits. In Propositions V and VI, we show that this does not rely on special preferences and that, for any indirect utility function, except for the very restrictive forms of Propositions I and II, there is a price distribution that guarantees it.

Proposition V: Assuming that $p_y \notin p$,

(i) $y^m < y^c$ for every probability distribution of prices if and only if the indirect utility function is of the form

$$V(\pi, p) = \log(\pi) - \log(G(p)).$$

Thus, for any other indirect utility function, there is always some probability distribution that implies $y^m \geq y^c$.

(ii) There is no indirect utility function such that $y^m \geq y^c$ for all distributions of prices.

Proof: First we prove part (i). (Sufficient) Proposition I established that, for this form, $y^m = y^u$, but since this utility function is concave in π, $y^u < y^c$, as shown by Sandmo (1971), and the result follows for y^m.
(Necessary) To establish the necessary condition, we show in the Appendix that, unless $V_{\pi p_i} \equiv 0$ for each i, it is always possible to find a distribution of prices for which $y^m > y^c$. Part (ii) is proved similarly in the Appendix. □

The result that, for almost every indirect utility function, there is always some distribution of prices such that output under uncertainty will exceed output under certainty is surprising enough to justify some discussion. For illustration, let us assume that $Cov(p_y, p_i) > 0$ for each i. In such a case, the producer's income lottery turns into a multidimensional one in income and prices. The producer is facing a "package deal": a high income with high prices or low income with low prices. The fundamental reason for risk aversion is the decreasing marginal utility of income ($V_{\pi\pi} < 0$), which means that, in a fair lottery, the "high income" result is not enough to offset the "low income" outcome, because in terms of utility the "low income dollars" are worth more than the "high income dollars". If $V_{\pi p_i} > 0$, then the

[9] The cases where $V_{\pi\pi} > 0$ and $V_{\pi\pi} = 0$ can be shown, similarly, to result in $y^m > y^c$ and $y^m = y^c$.

increasing consumption prices that go along with the "high income" outcome of the lottery leads to an increase in the value (in utility terms) of the "high income dollars" and decreases the value (in utility terms) of the "low income dollars". The net result is an output level which appears to indicate less risk aversion. In an extreme case, the producer may even appear to behave as a risk seeker. We turn now to Proposition VI in order to examine the case where $p_y \in p$.

Proposition VI: Assuming that $p_y \in p$, $y^m \overset{<}{\underset{>}{=}} y^c$ for every probability distribution of prices if and only if

$$(i)\ V(\pi, p) = V^1(\pi, p_y) + V^2(p),$$

and

$$(ii)\ \eta_y \overset{>}{\underset{<}{=}} R\left[\frac{s_y - \beta}{s_y}\right]$$

where β is the share of expected revenue from production in total wealth, s_y is the expected budget share of the good x_y (the quantity of the output good being consumed by the producer), and η_y is the income elasticity of the demand for x_y. For any other utility function, y^m will exceed $y^{u'}$ for some distributions and will be less for other, regardless of condition (ii).

Proof: (Sufficient) Given the indirect utility function in (i), necessary condition for the producer's maximization problem is

$$E[V^1_\pi(\pi, p_y)(p_y - C'(y))] = 0.$$

The total derivative of V_π with respect to p_y is

$$\frac{dV_\pi}{dp_y} = V_{\pi\pi} y + V_{\pi p_y} = -\frac{V_\pi}{\bar{p}_y}[R\beta + s_y(\eta_y - R)].$$

Rearranging the above expression, we obtain

$$\frac{dV_\pi}{dp_y} \overset{<}{\underset{>}{=}} 0 \quad \textit{if and only if} \quad \eta_y \overset{>}{\underset{<}{=}} R\left[\frac{s_y - \beta}{s_y}\right]$$

Since

$$\frac{dV_\pi}{dp_y} \overset{>}{\underset{<}{=}} 0 \iff Cov(V_\pi, p_y) \overset{>}{\underset{<}{=}} 0 \iff \bar{p}_y \overset{<}{\underset{>}{=}} C'(y) \iff y^m \overset{>}{\underset{<}{=}} y^c,$$

the sufficient part is established.

(Necessary) The proof follows the same line as in the necessary part of Proposition V. □

Proposition II established that for the above indirect utility function, $y^m = y^{u'}$. Hence condition (ii) of Proposition VI also determines the relationship between the certainty level of output (y^c) and the level of output under uncertainty ($y^{u'}$) about the output price alone, when the producer consumes the good he produces. Since this is a typical case in many developing economies (*e.g.* Wright and Williams (1988)), it is interesting to use condition (ii) to derive some qualitative results regarding the relationship between these output levels. Even though $p_{i \neq y}$ is treated as fixed in Corollary IV, and the only random variable is p_y, the utility function has two random arguments. As a result, the problem is one of multivariate risk and preferences for goods affect the response to the uncertainty.

Corollary IV: Let only the output price be random and let $p_y \in p$.

(i) If the producer is a net supplier (buyer) of his output (*i.e.*, $x_y < (>)\, y$) and averse to univariate risk ($V_{\pi\pi}<0$), then a non-negative (non-positive) income elasticity of demand for x_y is sufficient for $y^{u'} < (>)\, y^c$. If $x_y = y$, then the proposition holds with the conditions on η_y changed to positive (negative), respectively. Thus, in the typical case where farmers are net suppliers, it is sufficient for x_y to be a normal good for Sandmo's result to hold.

(ii) $\dfrac{dV_\pi}{dp_y}$ increases (is constant) (decreases) with R if and only if $x_y \; > (=) (<) \; y$. Thus if the producer is a net buyer of the good being produced,[10] then the *larger is R* the larger is the change of marginal utility of income with a change in p_y.

Proof: From Proposition VI,

$$\frac{dV_\pi}{dp_y} \begin{array}{c} < \\ = \\ > \end{array} 0 \text{ if and only if } \eta_y \begin{array}{c} > \\ = \\ < \end{array} R \left[\frac{s_y - \beta}{s_y} \right]$$

but

$$\frac{s_y - \beta}{s_y} = \frac{x_y - y}{x_y}$$

and the Corollary follows. □

Propositions I to VI established the relationships of y^m, y^u, and $y^{u'}$ with y^c. To study the factors that affect output under multivariate uncertainty, it is convenient to express the necessary condition in terms of the risk premium S, defined above. As in studies which focus on univariate risk (e.g. Chavas and Pope (1981); Flacco and Larson (1985)), the producer's maximization problem can be reformulated in terms of the certainty equivalent. Using the income risk premium S,

$$\max_{y \geq 0} E[V(\pi, p)] = \max_{y \geq 0} \bar{\pi} - S.$$

The first-order condition for the producer can then be rewritten in terms of S:

$$\bar{p}_y - C'(y) = \partial S / \partial y.$$

Using a Taylor series approximation of $\partial S / \partial y$, at an optimal output level we obtain

$$\frac{\partial S}{\partial y} = -\sigma^2_{p_y} y \frac{V_{\pi\pi}}{V_\pi} - \sum_{i=1}^{N} \sigma_{p_i p_y} \frac{V_{\pi p_i}}{V_\pi}.$$

[10] Which requires of course that there are alternative sources of safe income to finance the purchase of $(x_y - y)$ and other consumption goods.

This can be written in unitless or elasticity form as

$$\frac{\partial S}{\partial y} = \bar{p}_y \left[\gamma^2_{p_y} R\beta - \sum_i \gamma_{p_y p_i} s_i (R - \eta_i)\right] = \bar{p}_y \left[R(\gamma^2_{p_y}\beta - \sum_i \gamma_{p_y p_i} s_i) + \sum_i \gamma_{p_y p_i} s_i \eta_i\right]$$

where γ_{p_y} is the coefficient of variation (C.V.) of p_y, β is the share of the expected revenue in the total expected wealth, $\gamma_{p_y p_i}$ is the coefficient of covariation of p_y and p_i, and s_i is the expected share of the expenditure on good i in the total budget. Examination of these expression reveals that if $\sigma_{p_y p_i} > 0$, the output level decreases with the income elasticity of the demand for the i^{th} good,[11] the C.V. of p_y, and the share of expected revenue in total expected wealth, which is a measure of importance of the risky income. These results are reversed if $\sigma_{p_y p_i} < 0$. The coefficient of covariation between p_y and p_i and the share s_i have ambiguous effects, depending on which is bigger, R or η_i. Finally, the most surprising result is the effect of the index of relative risk aversion, which is ambiguous! If $\gamma^2_{p_y}\beta < \sum \gamma_{p_y p_i} s_i$, then y^m increases with R. In other words, *the larger the aversion to univariate income risk, the larger is the producer's output under multivariate uncertainty.* If the above relationship is reversed, then a larger R is associated with a smaller y^m.

A similar qualitative result holds without the assumption of "small risk" and a Taylor approximation, if prices follow the multivariate normal distribution. In that case, the generalized Stein/Rubinstein covariance formula (Wei and Lee (1988)) can be used to find that

$$\frac{Cov(V_\pi, p_y)}{V_\pi(\bar{\pi}, \bar{p})} = \frac{\partial S}{\partial y} = -\sigma^2_{p_y} y \frac{E[V_{\pi\pi}]}{V_\pi} - \sum_{i=1}^{N} \sigma_{p_i p_y} \frac{E[V_{\pi p_i}]}{V_\pi}.$$

There are even more complications due to multivariate risk. One involves possible changes in the cost of production. Suppose that there is a change in the cost of production, so that $C'(y)$ is altered. It is then possible that $\partial S/\partial y$ changes sign, and equilibrium output may actually move from one side of y^c to the other, without any change in risk attitudes or the probability distribution of prices. This observation is illustrated by Figure I, in which the short–run equilibrium of the firm is described. It is drawn under the assumption of small risk, using the Taylor approximation. Under certainty and cost structure mc^0, the firm produces $y_0{}^c$. The multivariate risk effect causes an increase in the production level to $y_0{}^m$. So, given marginal cost mc^0, $y^m > y^c$. A shift in marginal cost to mc^1 would yield production levels $y_1{}^c$ and $y_1{}^m$ under certainty and multivariate risk, respectively. Hence, after the shift in marginal cost, $y^m < y^c$. Thus, comparisons of the uncertain output to the certain one across producers or over time could reflect technological changes and not differences in risk attitudes.

A second complication involves the relationship between the risk premium S and its derivative with respect to output y. In the univariate case when the risk premium is positive, so is the marginal risk premium (*e.g.* Flacco and Larson (1987)), so output is always less than under certainty. In the multivariate case the sign of S is not necessarily equal to that of $\partial S/\partial y$, which can be seen from the two Taylor approximations. For example, it could be the case that S is negative, implying that the producer prefers income fluctuations with random prices, yet $\partial S/\partial y$ could be positive, and he would still produce less than under stabilization of

[11] This should be interpreted carefully since a greater income elasticity for one good means a smaller one for others. However, if prices of other goods are certain or uncorrelated with p_y, then the statement is correct.

all prices.

The implications from these results for determining preferences from behavior should be emphasized: comparisons of the level of observed output to the one which maximizes expected profits *do not,* in general, reveal anything about aversion to univariate income risk. Estimates of coefficients of risk aversion, similarly, are at least biased, and possibly not even of the correct sign, if the observed level of output is used in the framework of a univariate model to obtain them. Only for the special cases discussed in Section 3 can any conclusions about income risk preferences be revealed by observed output. However, even then, strong assumptions about the probability distribution of prices (for instance, independence of p_y and p) may be needed. The only preferences for which observed output can be used for estimation of the risk aversion functions for *any* distribution of prices are next to useless, since they imply that $R = 1$.

4.3 Long Run Output

To relate the attitude towards risk and the long run behavior of the producer, we define below aversion to multivariate income risk.

Definition 1: A producer is said to be averse to a specific multivariate income risk if and only if $S \geq 0$ where S is evaluated at the optimal level of y.

Finkelshtain and Chalfant (1988) showed that $S \geq 0$ for all risks if and only if the indirect utility function takes the form of Proposition I, and that there is no indirect utility function such that $S \leq 0$ for all risks. Thus, in general, aversion to multivariate income risk is a meaningful notion only if the discussion is restricted to a specific risk, since in some cases agents will prefer stabilized income and in others they will not. In Proposition VII, we establish the conditions for the industry to be in a long run equilibrium, with each producer just indifferent between the alternatives of producing a positive output level or quitting business.

Proposition VII: Expected profits are larger (smaller) than average cost if and only if the producer is averse to (seeks) multivariate income risk in the sense of Definition 1.

Proof: In long run, fixed costs (T) do not exist and the producer is indifferent between his two alternatives if and only if

$$E[V(\pi + W_0, p)] = E[V(\bar{\pi}+W_0-S, p)] = E[V(W_0, p)]$$

or $\bar{\pi} - S = 0$. In other words, the expected value of profits, $\bar{\pi}$, which equals $\bar{p}_y y - C(y)$, is just equal to the risk premium S.[12] □

Since in the long run, the necessary condition for a short run equilibrium holds as well, the complete long run condition is

$$\bar{p}_y = \frac{C(y)}{y} + \frac{S}{y} = \frac{\partial[C(y) + S(y)]}{\partial y}.$$

[12] In section 4.2, we assumed an interior solution. However, the global maximum could be not to produce at all, and then the local maximum identified in section 4.2 is meaningless. The condition for staying in business in the short run is identical to the long run condition with a positive level of fixed cost (T).

In the long run, therefore, the average risk premium drives a wedge between expected price average cost. This is analogous to results derived by Flacco and Larson (1987) for the univariate case. As is true in the univariate case, production does not take place where long-run average cost is at minimum. Moreover, since in general S could change its sign with the distribution of prices, so could the wedge between average cost and the expected price of output. It follows that risk aversion in the univariate sense ($V_{\pi\pi}$<0) is neither sufficient nor necessary for average cost to be less than expected price, yet aversion to multivariate income risk, in the sense of Definition 1, is both necessary and sufficient to ensure that expected price is larger than average cost. We turn now to the question of comparisons between different producers.

Definition 2: Producer i is said to be more averse to a specific multivariate income risk than producer j if and only if $S^i(y) > S^j(y)$ for all y.

Once more, the definition is restricted to a specific risk, since except for very limited cases discussed in Finkelshtain and Chalfant (1988), $S^i - S^j$ will vary in sign with the distribution of prices. In Proposition VIII, we establish the relationship between the notion of "more averse to income risk" defined above and the expected price which is required by each producer to enter the market.

Proposition VIII: Given identical cost functions and probability beliefs, a producer with greater aversion to income risk, in the sense of Definition 2, will enter the market at a higher expected price than a producer with smaller aversion to the specific income risk.

Proof: Let y^j and y^i be the long run optimal output levels of producer$j$ and i respectively. From the condition for entering the market, producer j is just indifferent between entering and not doing so if and only if

$$\bar{p}_y = \frac{C(y^j)}{y^j} + \frac{S^j(y^j)}{y^j},$$

where y^j is the long run optimal level of output for producer j. Since y^j is the optimal output level, for every other level of output, including y^i, and it must be true that

$$\bar{p}_y < \frac{C(y^i)}{y^i} + \frac{S^j(y^i)}{y^i}.$$

By assumption, $S^i(y^i) > S^j(y^i)$. Upon substituting $S^i(y^i)$ in the above inequality, we find that producer i will not enter the market unless the expected price is greater than $\bar{p}_y$. □

Flacco (1983) found similar results in the context of univariate uncertainty about the output price. In the univariate case, if technology, probability beliefs, and the Arrow-Pratt risk aversion measures are all identical, all firms will enter (exit) the market at the same expected price. However, under multivariate risk, even if the above conditions hold, producers will require different levels of expected price according to their ordinal preferences. Flacco (1983) notes that differences in risk attitudes between firms can explain the empirical observation that different producers will exit the market at different levels of expected price. Our result that the ordinal preferences also matter provides an additional explanation. It also shows once more how the existence of multivariate uncertainty makes it difficult to generalize about risk attitudes from observed behavior, when that behavior is interpreted in the context of models of univariate risk.

6. Conclusions

This paper has examined the behavior of a competitive firm facing multivariate risk. We examined the multivariate risk which is present when there is uncertainty about the price of output and the prices of goods bought for consumption, a case which seems relevant for any self-employed individual. In this case, the usual indirect utility function of the consumer replaces the typical producer's objective function defined on profits alone. As a result, many of the familiar results from models of output price uncertainty may no longer hold. Only for very limited cases will the output under multivariate risk be the same as in the univariate case examined by Sandmo. While we considered neither the effects of multivariate risk on input choices, *etc.*, nor the reactions to other examples of multivariate risk, these cases also require generalizing the theory of producer behavior under univariate risk.

A multivariate risk premium was used to characterize producer behavior. Neither the Arrow-Pratt risk premium nor the multivariate one is sufficient to determine the relationship between the output levels under multivariate risk and certainty. As a result, inferences about a producer's aversion to univariate income uncertainty that are based on comparisons of the level of output to the expected profit maximizing level require information about both ordinal preferences *and* the probability distribution of all prices. Otherwise, such inferences will be biased by the effects of multivariate risk.

We showed that characterizing aversion to income risk under multivariate uncertainty or making comparisons between producers concerning levels of risk aversion must be limited to a *specific* price risk. An individual may be averse to income risk under one price distribution and prefer it with another. Similarly, one producer may have a larger risk premium than another under one probability distribution for prices, and a smaller one under a different distribution.

Finally, we showed that a producer with greater aversion to a specific multivariate income risk requires a larger expected price to enter the industry. Depending on both preferences and the price distribution, average cost may be greater than, less than, or equal to the expected price, even if the producer is averse to univariate risk.

APPENDIX

Proof of Proposition I: (Sufficient) Let $f(p, p_y)$ denote the joint probability density function of p_y and p, and let $f_i(\cdot)$ denote the corresponding marginal densities. If the restriction on preferences holds, the maximization problem becomes[13]

$$\max_{y\geq 0} \iint_{p_y p} [V^1(\pi) + V^2(p)] f(p, p_y)\, dp\, dp_y \;=\; \max_{y\geq 0} E_{p_y}[V^1(\pi)] + E_p[V^2(p)] \,.$$

The solution to the above maximization is identical to that of the univariate problem

$$\max_{y\geq 0} E_{p_y}[V^1(\pi)] \;+\; V^2(\bar{p}),$$

since the objective functions differ only by terms that are constant with respect to y.
(Necessary) We assume two particular distributions, G^1 and G^2. Let G^1 be given by

$$p_y = p_y{}^0 \text{ and } p = p^0 \quad \textit{with probability } ½$$

and

$$p_y = p_y{}^1 \text{ and } p = p^1 \quad \textit{with probability } ½,$$

where p^0 and p^1 differ only by the fact that a particular element of p takes on values $p_i{}^0$ or $p_i{}^1$, respectively. Let each 0 superscript denote a low value and each 1 superscript denote a high value. Also, assume that the two values for p_i are each $\Delta/2$ away from its mean value, given by the $i^{\underline{th}}$ element in $\bar{p}$. Let the second distribution be given by

$$p_y = p_y{}^0 \text{ and } p = p^1 \quad \textit{with probability } ½$$

and

$$p_y = p_y{}^1 \text{ and } p = p^0 \quad \textit{with probability } ½.$$

For both G^1 and G^2, all prices except p_i are assumed to remain fixed at the levels defined by $\bar{p}$. When p_y is low, profits equal $\pi^0 = \pi(p_y{}^0)$ and when it is high, profits are $\pi^1 = \pi(p_y{}^1)$. The necessary conditions for maximization of expected utility under G^1 and G^2 are

$$V_\pi(\pi^0, p^0)\,(p_y{}^0 - C'(y)) \;+\; V_\pi(\pi^1, p^1)\,(p_y{}^1 - C'(y)) \;=\; 0$$

and

$$V_\pi(\pi^0, p^1)(p_y{}^0 - C'(y)) \;+\; V_\pi(\pi^1, p^0)(p_y{}^1 - C'(y)) \;=\; 0,$$

respectively. Assuming that $V_\pi(\,\cdot\,) > 0$, it follows from these that

$$p_y{}^0 - C'(y) < 0 \qquad \text{and} \qquad p_y{}^1 - C'(y) > 0.$$

Subtracting the second condition from the first and rearranging, we find that

$$(p_y{}^0 \;-\; C'(y))[V_\pi(\pi^0, \bar{p}_i \;+\; \frac{\Delta}{2}, \bar{p}_{j\neq i}) \;-\; V_\pi(\pi^0, \bar{p}_i \;-\; \frac{\Delta}{2}, \bar{p}_{j\neq i})]$$

$$=\; (p_y{}^1 \;-\; C'(y))[V_\pi(\pi^1, \bar{p}_i \;+\; \frac{\Delta}{2}, \bar{p}_{j\neq i}) \;-\; V_\pi(\pi^1, \bar{p}_i \;-\; \frac{\Delta}{2}, \bar{p}_{j\neq i})]$$

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[13] The proof of this part is given for the case where the relevant distribution function is assumed to have proper densities. The proof of the discrete case would be similar. This comment is applicable to proofs of other propositions as well.

where $\overline{p}_{j \neq i}$ denotes the prices in $\overline{p}$ that are assumed constant. Dividing both sides of the equation by Δ and letting it approach 0, we obtain the following partial derivatives as the limits of both sides of the above equation:

$$V_{\pi p_i}(\pi^0, \overline{p})\,(p_y{}^0 - C'(y)) \;=\; V_{\pi p_i}(\pi^1, \overline{p})\,(p_y{}^1 - C'(y)).$$

Using the relationship between $p_y{}^0$, $p_y{}^1$, and marginal cost, this result implies that either the two cross derivatives have different signs, or that they are equal to zero for every value of π^0 and π^1. The former could not occur for choices of π^0 and π^1 arbitrarily close together, given that V_π is continuous. Thus, the latter alternative must hold, and $V_{\pi p_i}(\,\cdot\,) \equiv 0$. We can repeat the above argument for each $i = 1, \ldots, N$ and hence $V_{\pi p_i} \equiv 0$ for each i, which implies the separable form above. □

Proof of Corollary I: To show that (ii) follows from (i), note that by Roy's Identity the following expression can be derived (*e.g.* Newbery and Stiglitz (1981, p. 117)):

$$V_{\pi p_i} = \frac{S_i}{p_i} V_\pi (R - \eta_i)$$

where S_i is the share of the i^{th} good in total expenditure. Proposition I established that (i) implies that the left-hand side is zero for every i, and so, each η_i must equal R. The budget constraint implies that

$$\sum_{i=1}^{N} S_i \eta_i = 1,$$

and hence each η_i, and R, must equal one.

We show now that (iii) follows from (ii). By (ii), R equals 1. Hanoch (1977) characterized all indirect utility functions consistent with constant relative risk aversion. In his Corollary 3 (p. 424), he showed that, for the case of $R = 1$,

$$V(\pi, p) \;=\; H(p) \log \left[\frac{\pi}{G(p)}\right]$$

where G and H are homogeneous of degrees 1 and 0, respectively. V is increasing in π, and its income elasticities of demand are identical to 1 if and only if $H(p) = H > 0$. Hence,

$$V(\pi, p) \;=\; H\;[\log(\pi) \;-\; \log(G(p))]$$

which is equivalent (both in the ordinal and cardinal sense) to the required form. To see that (i) follows from (iii), note that the form in (iii) is additively separable in π and p. □

Proof of Proposition III: (Sufficient) Using the special form for V and the independence of p and p_y, the producer's maximization problem is

$$\max_{y \geq 0} \;\; E[a(p)] + E[b(p)]\; E[h(\pi(y, p_y))].$$

Monotonicity of V in π, for every p, implies that the sign of $b(p)$ is the same for every p, and hence, the maximization problem above is equivalent to the univariate problem

$$\max_{y \geq 0} \;\; [a(\overline{p})] + [b(\overline{p})]\; E[h(\pi(y, p_y))].$$

(Necessary) We establish this using a particular price distribution. Let p_y equal either $p_y{}^l$ or $p_y{}^h$, each occurring with probability 1/2. Profits then will be either $\pi^l \equiv \pi(y, p_y{}^l)$ or $\pi^h \equiv \pi(y, p_y{}^h)$. Similarly, let p equal either p^l or p^h, each occurring with probability 1/2, where *all* prices are now assumed to vary between low and high levels.

The necessary conditions for the multivariate and univariate maximization problems are:

$$\frac{p_y{}^l - C'(y^m)}{p_y{}^h - C'(y^m)} = -\frac{V_\pi(\pi^h, p^h) + V_\pi(\pi^h, p^l)}{V_\pi(\pi^l, p^h) + V_\pi(\pi^l, p^l)}$$

and

$$\frac{p_y{}^l - C'(y^u)}{p_y{}^h - C'(y^u)} = -\frac{V_\pi(\pi^h, \bar{p})}{V_\pi(\pi^l, \bar{p})}.$$

If the output levels are equal, then the left-hand sides of the two conditions are the same, so we can rearrange the two right-hand sides, to obtain

$$\frac{V_\pi(\pi^h, p^l) + V_\pi(\pi^h, p^h)}{V_\pi(\pi^h, mean\,(p^h, p^l))} = \frac{V_\pi(\pi^l, p^l) + V_\pi(\pi^l, p^h)}{V_\pi(\pi^l, mean\,(p^h, p^l))}.$$

Note that these ratios, which we designate $r(\pi, p^l, p^h)$, must not depend on π, since for any arbitrary choice of π^h and π^l, $r(\pi^h, p^l, p^h) = r(\pi^l, p^l, p^h)$. If r does not depend on π, while its components, the derivatives of V do so,[14] then the numerator and denominator must have a common factor, such that any term involving π cancels:

$$r(\pi, p^l, p^h) = \frac{\kappa(\pi, p^l, p^h)\,[\lambda(p^l) + \zeta(p^h)]}{\kappa(\pi, p^l, p^h)\,[\phi(p^l, p^h)]}$$

where

(*i*) $\kappa(\pi, p^l, p^h)\phi(p^l, p^h) = V_\pi(\pi, mean\,(p^h, p^l))$,

(*ii*) $\kappa(\pi, p^l, p^h)\lambda(p^l) = V_\pi(\pi, p^l)$,

and

(*iii*) $\kappa(\pi, p^l, p^h)\zeta(p^h) = V_\pi(\pi, p^h)$.

Condition (ii) implies that κ does not contain p^h, while condition (iii) implies that it does not contain p^l, and hence from (i) it follows that V_π is of the form $b(p) \cdot h'(\pi)$. Integrating with respect to π yields the form

$$V(\pi,p) = a(p) + b(p) \cdot h(\pi). \ \square$$

Derivation of S: S is defined by

$$E[V(\pi, p)] = E[V(\bar{\pi} - S, p)].$$

By a second-order Taylor expansion of the left hand side of the equation around the point

[14] It might be that $V(\pi, p)$ is linear in π, in which case V_π is independent of π, so r itself is independent of π. However, this is simply a special case of the form $a(p) + b(p) \cdot h(\pi)$, where $h(\pi) \equiv \pi$.

$(\bar{\pi}, \bar{p})$,

$$EV(\pi, p) = V(\bar{\pi}, \bar{p}) + \frac{1}{2}\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\sigma_{ij}V_{ij} + o(tr\Psi)$$

where Ψ is the covariance matrix of the arguments of V. A second order approximation of the right hand side yields

$$EV(\bar{\pi} - S, p) = V(\bar{\pi}, \bar{p}) - S_1 \cdot V_1(\bar{\pi}, \bar{p}) + \frac{1}{2}\sum_{i=2}^{N+1}\sum_{j=2}^{N+1}\sigma_{ij}V_{ij} + o(tr\Psi^1)$$

where Ψ^1 is the covariance matrix of the prices of consumption goods. We ignore terms that contain S^2 since those are of the same order as the remainders. Now, by the assumption that the risk is small,[15] we can ignore the reminder terms and by setting the two expressions equal we can solve for the required representation of S. □

Proof of Lemma I: The conditional covariance is defined by

$$Cov\{[g(x,y), X] \mid Y\} = E_{X \mid Y}[g(x,y) \cdot X \mid Y] - E_{X \mid Y}[g(x,y) \mid Y] \cdot E_{X \mid Y}(X \mid Y).$$

So

$$\begin{aligned}
&E_Y[Cov\{[g(x,y), X] \mid Y\}] \\
&= E_Y\{E_{X \mid Y}[g(x,y) \cdot X \mid Y]\} - E_Y\{E_{X \mid Y}[g(x,y) \mid Y] \cdot E_{X \mid Y}(X \mid Y)\} \\
&= E_{X,Y}\{g(x,y) \cdot X\} - E_{X,Y}g(x,y)E_X(X) - \\
&\qquad \Big[E\{E_{X \mid Y}[g(x,y) \mid Y] \cdot E_{X \mid Y}[X \mid Y]\} - E_{X,Y}[g(x,y)]E_X(X)\Big] \\
&= Cov[g(x,y), X] - Cov\{E_{X \mid Y}[g(x,y) \mid Y], E_{X \mid Y}(X \mid Y)\}
\end{aligned}$$

By rearranging, we get the required formula. □

Proof of the Necessary Part of Proposition V: Let the price of output be perfectly correlated with the vector of consumption prices. This assumption implies a linear relationship between p_y and p_i, for $i = 1, \ldots, N$:

$$p_i = a_i + b_i p_y.$$

We will show that y^m can exceed y^c by showing that $\frac{dV_\pi}{dp_y}$ and hence $Cov(V_\pi, p_y)$ can be positive. The change in V_π resulting from a change in the level of p_y is given by the total derivative

$$dV_\pi = \left\{V_{\pi\pi}\frac{\partial \pi}{\partial p_y} + \sum_{i=1}^{N} V_{\pi p_i}\frac{\partial p_i}{\partial p_y}\right\}dp_y .$$

From the definition of profits, $\frac{\partial \pi}{\partial p_y} = y$, and for this special case, $\frac{\partial p_i}{\partial p_y} = \frac{\sigma_{p_y p_i}}{\sigma^2_{p_y}}$. To see

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[15] See Karni's definition above.

this, note that $p_i = a_i + b_i p_y$ implies $\frac{\partial p_i}{\partial p_y} = b_i$. It follows that $\sigma^2_{p_i} = b_i^2 \sigma_{p_y}{}^2$. By the assumption of perfect correlation,

$$|\rho_{p_i p_y}| = |\sigma_{p_i p_y}/\sigma_{p_i}\sigma_{p_y}| = |\sigma_{p_i p_y}/b_i \sigma_{p_y}{}^2| = 1.$$

Since $sign(b_i) = sign(\sigma_{p_i p_y})$, the term $\sigma_{p_i p_y}/b_i \sigma_{p_y}{}^2$ is always positive and therefore $b_i = \sigma_{p_i p_y}/\sigma_{p_y}{}^2$. Substituting these expressions into dV_π, we obtain

$$\frac{dV_\pi}{dp_y} = V_{\pi\pi}\, y + \sum_{i=1}^{N} V_{\pi p_i} \frac{\sigma_{p_y p_i}}{\sigma^2_{p_y}} .$$

We assume now that $b_i = b$ for each i and therefore $\sigma_{p_y p_i} = \sigma_{p_y p}$ for each i, hence

$$\frac{dV_\pi}{dp_y} > 0 \iff \begin{cases} \dfrac{\sigma^2_{p_y}}{\sigma_{p_y p}} < -\dfrac{\sum_{i=1}^{N} V_{\pi p_i}}{V_{\pi\pi} y} & \text{if} \quad \sigma_{p_y p} > 0 \quad (b > 0) \\[2ex] \dfrac{\sigma^2_{p_y}}{\sigma_{p_y p}} > -\dfrac{\sum_{i=1}^{N} V_{\pi p_i}}{V_{\pi\pi} y} & \text{if} \quad \sigma_{p_y p} < 0 \quad (b < 0) \end{cases}$$

In both cases, we assume that $V_{\pi\pi}$ is negative. The proof for the cases where $V_{\pi\pi} \geq 0$ is similar. The right-hand side expressions in the above inequalities will have the same sign as the numerator, which depends on the both profits and prices and could be positive or negative.

To show that the producer could produce more than the certainty level, we must show that there is always some price distribution such that the above inequalities hold. This means that we must find a price distribution such that b satisfies one of the inequalities above. A necessary condition for this is that b and the numerator have the same sign. We show below that the support of the distribution can be chosen so that the sign of the numerator is either always positive or always negative, independent of the value of b. Thus, the sign of b could always be chosen to be of the same sign as this numerator.

Assuming that the term $\sum_{i=1}^{N} V_{\pi p_i}(\pi, p)$ is continuous, there is some neighborhood (δ) of $(\bar{\pi}, \bar{p})$ in which the sign of $\sum_{i=1}^{N} V_{\pi p_i}$ is the same as at $(\bar{\pi}, \bar{p})$, regardless of the value of b. Thus, once we find a distribution of prices for which $(\pi, p) \in \delta$ holds with probability one, we can choose b to have the same sign as $\sum_{i=1}^{N} V_{\pi p_i}(\bar{\pi}, \bar{p})$ and the right magnitude to satisfy the required inequality.

The problem with finding such a distribution is that the distribution of prices does not determine the distribution of π and p by itself. Rather, every price distribution induces a distribution for the random variables π and p, through the producer's optimization problem. Hence, we need to show that the distribution of prices can be chosen to guarantee that $(\pi, p) \in \delta$ with probability one, independent of the producer's choices.

Let B be an $N+1$—dimensional closed ball, centered at $(\bar{p}_y, \bar{p}_1, \cdots, \bar{p}_N)$. We show that the support of the probability distribution of prices can always be restricted so that $Pr[(p_y, p_1, \ldots, p_N) \in B] = 1$, and that, for every $(p_y, p) \in B$, it is also true that $(\pi, p) \in \delta$. To do so, we show that for a given B, the boundaries of output y, and therefore π, depend solely on B; hence, by choice of B we can ensure that $(\pi, p) \in \delta$.

Let p_y and p be bounded, where $p_y^{\max} \equiv \max_B(p_y)$ and $p_y^{\min} \equiv \min_B(p_y)$. If the producer is rational, the corresponding values of profit are also bounded, independent of the producer's choices. To see that, note that the maximum output level which would be chosen by the producer is the one which for $p_y^{\max} = C'(y)$, which we denote by $\hat{y}$. As a result, the maximum profits associated with B are $\pi(\hat{y}, p_y^{\max})$. The minimum profits are the minimum between $\pi(\hat{y}, p_y^{\min})$ and $\pi(0, p_y) = -T$. Thus, the choice of B, over which the producer has no control, implies that profits are bounded for any (rationally chosen) level of output, which ensures that a B can be found so $(\pi, p) \in \delta$.

The above steps justify the claim that there is always some price distribution such that the term $\sum_{i=1}^{N} V_{\pi p_i}$ does not change sign as prices change anywhere in B, regardless of the producer's choices *and* regardless of the value of b. Without loss of generality, we can assume now that $g(p_y, p)$ was chosen so that $\sum_{i=1}^{N} V_{\pi p_i} > 0$. We then need only choose $\sigma^2_{p_y}$ and $\sigma_{p_y p} > 0$, so that

$$\frac{\sigma^2_{p_y}}{\sigma_{p_y p}} < \max_{(\pi, p) \in \delta} \left[\frac{\sum_{i=1}^{N} V_{\pi p_i}}{-V_{\pi\pi} y} \right]$$

Similarly, if $\sum_{i=1}^{N} V_{\pi p_i} < 0$, we choose $\sigma_{p_y p} < 0$.[16]

Thus, we have established that we can find a distribution of prices such that $\frac{dV_\pi}{dp_y} > 0$ with probability one. Part (i) is now established, since

$$\frac{dV_\pi}{dp_y} > 0 \iff Cov(V_\pi, p_y) > 0 \iff \bar{p}_y < C'(y) \iff y^m > y^c.$$

To prove (ii), note that the only preferences for which $\frac{dV_\pi}{dp_y}$ has the same sign for every risk are those for which $V_{\pi p_i} \equiv 0$ for each i. It was already established in part (i) that, for these preferences, $y^m < y^c$, so the proof is completed. □

[16] Note that the choice of the required B might restrict the choices of $\sigma^2_{p_y}$ and $\sigma_{p_y p}$. However, we still have the freedom to choose an arbitrary value for $\left| \frac{\sigma^2_{p_y}}{\sigma_{p_y p}} \right|$ by choice of b.

REFERENCES

Arrow, Kenneth J. *Aspects of the Theory of Risk-Bearing*. Helsinki, (1965). Reprinted in *Essays in the Theory of Risk Bearing*. Amsterdam: North-Holland, 1970.

Bernoulli, D. "Exposition of a New Theory on the Measurement of Risk" *Econometrica* 22(1954):23-36.

Chavas, J. P., and R. D. Pope "A Welfare Measure of Production Activities under Risk Aversion" *Southern Economic Journal* 48(1981):187-95.

Deaton, Angus, and J. Muellbauer. *Economics and Consumer Behavior*. Cambridge: Cambridge University Press, 1980.

Deschamps, Robert. "Risk Aversion and Demand Functions" *Econometrica* 41(1973):455-65.

Epstein, Larry. "A Disaggregated Analysis of Consumer Choice Under Uncertainty" *Econometrica* 43(1975):877-92.

______________. "Production Flexibility and the Behavior of the Competitive Firm Under Price Uncertainty" *Review of Economic Studies* 45(1978):251-61.

Finkelshtain, Israel and James A. Chalfant. "A Risk Premium for Stabilization of Income Under Multivariate Risk" Working Paper No. 496, Department of Agricultural and Resource Economics, University of California, Berkeley, 1988.

Flacco, Paul R. "Output, Entry, and Competitive Production Under Price Uncertainty" *Southern Economic Journal* 50(1983):565-71.

______________, and Douglas M. Larson. "Marginal and Average Risk Premiums in Long Run Competitive Equilibrium under Price Uncertainty" Paper presented at the American Agricultural Economics Association Meetings, August, 1987, Michigan State University, Lansing, MI.

Hanoch, Giora. "Risk Aversion and Consumer Preferences" *Econometrica* 45(1977):413-25.

Karni, Edi. "On Multivariate Risk Aversion" *Econometrica* 47(1979):1391-1401.

_________ "Risk Aversion And Saving Behavior" *International Economic Review* 23(1982):35-42.

Kihlstrom R. E., and L. J. Mirman "Risk Aversion with Many Commodities" *Journal of Economic Theory* 8(1974):361-88.

Mood, A. M., F. A. Graybill, and D. C. Boes. *Introduction to the Theory of Statistics*. New York: McGraw-Hill, 1974.

Newbery, David M. G., and Joseph E. Stiglitz. *The Theory of Commodity Price Stabilization: A Study in the Economics of Risk*. Oxford: Clarendon Press, 1981.

Pratt, J. W. "Risk Aversion in the Small and in the Large" *Econometrica* 32(1964):122-36.

Roe, Terry, and T. Graham–Tomasi. "Yield Risk in a Dynamic Model of the Agricultural Household" In *Agricultural Household Models: Extensions, Applications, and Policy*. Baltimore: Johns Hopkins University Press, (1986).

Sandmo, Angmar. "On the Theory of the Competitive Firm under Price Uncertainty" *American Economic Review* 61(1971):65-73.

Stiglitz, Joseph E. "Behavior Towards Risk with Many Commodities" *Econometrica* 37(1969):660-667.

Turnovsky, Stephen J. "Production Flexibility, Price Uncertainty and Behavior of the Competitive Firm" *International Economic Review* 14(1973):395-413.

__________________, H. Shalit, and A. Schmitz. "Consumer's Surplus, Price Instability, and Consumer Welfare" *Econometrica* 48(1980):135-52.

Wei, K. C. John, and Cheng F. Lee. "The Generalized Stein/Rubinstein Covariance Formula and its Application to Estimate Real Systematic Risk" *Management Science* 34(1988):1266-1270.

Wright, Brian D. "The Effects of Price Uncertainty on the Factor Choices of the Competitive Firm" *Southern Economic Journal* 51(1984):443-55.

______________, and Jeffrey C. Williams. "Measurement of Consumer Gains from Market Stabilization" *American Journal of Agricultural Economics* August(1988):616-27.

FIGURE 1 : Short run equilibrium under multivariate risk