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Sushama Murty* and R. Robert Russell**

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*Department of Economics, University of Warwick: s.murty@warwick.ac.uk

**Department of Economics, University of California, Riverside: robert.russell@ucr.edu

Abstract

We distinguish between intended production and residual generation and introduce the concept of by-production. We show that by-production provides the fundamental explanation for the positive correlation that is observed between intended production and residual generation. Most of the existing literature attributes the observed positive correlation to abatement options available to firms. We show that abatement options of firms add to the phenomenon of by-production in strengthening the observed positive correlation. The existing literature usually does not explicitly model abatement options of firms, but considers a reduced form of the technology, which satisfies standard disposability assumptions with respect to all inputs and intended outputs. We show that more than one implicit production relation is needed to capture all the technological trade-offs that are implied by by-production. From our model, we are able to derive a reduced form of the technology that is in the spirit of the one that is usually studied in the literature. However, we find that our reduced form technology violates standard disposability with respect to inputs and intended outputs that cause pollution. We derive implications from the phenomenon of by-production for the econometric and Data Envelopment Analysis (DEA) specifications of pollution-generating technologies. We derive a DEA specification of technologies that satisfy by-production. Such a specification can be used to study issues relating to measurement of efficiency, marginal abatement costs, productivity, *etc.*, of firms with technologies that generate pollution.

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1. Introduction.

Our reading of the environmental economics literature reveals three broad features of pollution that economists aim to capture. First, the generation of pollution/residuals seems to proceed hand-in-hand with the processes of consumption and production.[1] Second, the residuals so generated require the use of the assimilative capacity of the environment for their disposal. Third, the generation of the residuals and the consequent use of environmental resources for their disposal generate external effects on both consumers and producers and hence the need for policies to regulate the generation of pollution.[2]

In this paper, we confine ourselves to addressing the first feature alone. In particular, we focus on pollution generated by firms. We distinguish between outputs that firms intend to produce and outputs that unintentionally (incidentally) get generated by firms when they engage in the production of intended outputs. Pollution is such an unintended output. We are mainly concerned with studying the specification of technology sets that best captures the link between production of outputs intended by firms and the generation of pollution. Our work has a bearing on the literature that is concerned with measurement issues, such as measuring technical efficiency, marginal abatement cost, productivity, and growth when economic units also produce incidental outputs like pollution. Both Data Envelopment Analysis (DEA)[3] and econometric approaches are employed in this literature. Thus, we

[1] See, especially, Ayres and Kneese [1969] and Førsund [2009].

[2] See Murty [2010] for a general equilibrium study of the second feature in the light of the first feature. See, *e.g.,* Murty and Russell [2005] for analysis of the third feature.

[3] DEA is a mathematical programming approach to the construction of data-based technologies and the concomitant calculation of technological efficiency of individual firms (or other organizations). See Färe, Grosskopf, and Lovell [1994] for a thorough description.

are also interested in exploring DEA and econometric specifications for sound estimation/construction of pollution-generating technologies from data.

It is reasonable to say that, in the case of pollution generated by firms, there are some specific aspects about the process of transformation of inputs into intended outputs (*e.g.,* the use of certain inputs such as coal or the production of certain outputs such as varieties of cheese that have an offensive odor) that trigger additional reactions in nature and *inevitably* result in the generation of pollution as a by-product (abstracting from abatement activities). In this paper, we refer to these natural reactions, which occur alongside intended production by firms, as *by-production*[4] of pollution.

In the case of technologies exhibiting by-production, we observe an inevitability of a certain *minimal* amount of the incidental output (the by-product), given the quantities of certain inputs and/or certain intended outputs. Inefficiencies in production could generate more than this minimal amount of the unintended output. At the same time, in such technologies, we also observe the usual menu of *maximal* possible vectors of intended outputs, given an input vector. Such a menu generally reflects the negative tradeoffs in the production of intended outputs when inputs are held fixed, as production of each of these commodities is costly in terms of the inputs used. Inefficiencies in intended production may imply that less than this maximal amount may get produced. An increase in the amounts of the inputs used increases the menu of intended output vectors that are technologically feasible. At the same time, it increases the minimal amount of the unintended output that can be generated.[5]

The above underscores two crucial points to note about pollution-generating technologies:

(i) technologies of pollution-generating firms do not satisfy free disposability of by-products such as pollution (pollution cannot be disposed of below the minimal level described above if inputs and intended outputs are held fixed) and

[4] A word that is not in the dictionary but perhaps should be.

[5] *E.g.*, a greater amount of usage of coal increases the quantity of both smoke and electricity generated.

(ii) in such technologies there is a mutual interdependence between changes in inputs, intended outputs, and pollution—an interdependence that we will argue is more correlation than causation.

We show that a single implicit production relation is not rich enough to capture, simultaneously, all the trade-offs between commodities that are implied by the phenomenon of by-production. At the same time, a sound foundation must be identified for introducing multiple production relations to capture correctly the features of by-production. We feel that the resolution to the problem lies in early work of Frisch [1965] on production theory, in which he envisaged situations where the correct functional representation of a production technology may require more than one implicit functional relation between inputs and outputs. More recently, Førsund [2009] explores these ideas of Frisch.[6] In this paper, we build on the works of Frisch and Førsund to identify the production relations that can simultaneously capture all the aspects of by-production.

In most of the existing literature, the standard building block employed in constructing pollution-generating technologies is the positive correlation between intended and unintended outputs that is usually observed in such technologies. This literature attributes this observed positive correlation to abatement activities by firms rather than directly to the phenomenon of by-production. Abatement activities of firms involve a diversion of their resources (inputs) to mitigate or clean up the pollution they produce.[7] The production of these abatement activities is hence costly, given fixed amounts of resources: the more resources are diverted to abatement activities, the less they are available for producing intended outputs. Hence,

[6] He employs a welfare maximization problem to show that the optimal government policies are counter-intuitive and meaningless when a single production relation is used to represent a pollution-generating technology.

[7] In this paper, we model abatement activities as outputs of the firm. Examples are end-of-pipe treatment plants (that treat and clean water to remove the pollutant) and production of outputs like scrubbers (which reduce sulphur emissions). We abstract from long-run abatement options of development, purchase, and installation of new technologies that generate less pollution. See *e.g.,* Barbera and McConnell [1998], where abatement activities include both a purchase of abatement capital and a diversion of some amounts of the usual inputs of a firm towards running of the abatement capital.

an increase in the level of abatement activities leads concomitantly to both lower residual generation and lower production of intended output.

In this literature, however, abatement activities are not explicitly modeled as another set of outputs produced by firms.[8] Rather, what is proposed is a "reduced form" of the technology in the space of inputs, by-products, and intended outputs. Special assumptions are made to ensure that such a technology exhibits a positive correlation between by-products and intended outputs, which is implicitly explained by abatement options open to firms. At the same time, it is also assumed that the technology satisfies the standard disposability assumptions with respect to *all* inputs and intended outputs. The approaches taken in the literature to model the positive correlation include: (a) a single equation functional formulation of the "reduced form" technology that treats pollution as a standard input,[9] (b) a single-equation distance function representation of the reduced form technology that treats pollution as an output and employs the assumptions of weak disposability and null-jointness with respect to intended and unintended outputs,[10] and (c) a non-parametric set-theoretic approach that also treats pollution as an output and employs weak disposability and null-jointness with respect to intended and unintended outputs.[11]

We propose a model of pollution-generating technologies that captures the salient features (i) and (ii) of the phenomenon of by-production identified above. Even without any reference to explicit abatement efforts by firms, the model generates a positive correlation

[8] For an exception, see Barbera and McConnell [1998].

[9] See, *e.g.,* Baumol and Oates [1988] and Cropper and Oates [1992]. (Unlike Cropper and Oates [1992], we abstract from the distinction between pollution and emissions.)

[10] See, *e.g.,* Pittman [1983], Färe, Grosskopf, Noh, and Yaisawarng [1993], Coggins and Swinton [1994], Hailu and Veeman [1999], Murty and Kumar [2002, 2003], and Murty, Kumar, and Paul [2006]. A technology satisfies weak disposability of intended and unintended outputs if the latter can be disposed of only in strict tandem with the disposition of the former, and it satisfies null jointness if zero pollution implies all intended output quantities are zero as well. See Section 4 for formal definitions of these concepts. The distance function (inaptly named since it does not satisfy the properties of a mathematical distance function) is a particular (homogeneous) representation of multiple-output technologies first formulated by Malmquist [1953] and Shephard [1953]. See Färe and Primont [1995] for a thorough treatment of this concept.

[11] See, *e.g.,* Färe, Grosskopf, and Pasurka [1986], Färe, Grosskopf, Lovell, and Pasurka [1989], Färe, Grosskopf, Noh, and Weber [2005], and Boyd and Clelland [1999]. See Zhou and Poh [2008] for a comprehensive survey of over a hundred papers employing this approach to the modeling of pollution-generating technologies.

between pollution generation and intended outputs. This is because the model recognizes and subsumes a residual generation mechanism that is set in motion when firms undertake production of intended goods. The residual generation mechanism is a relationship between pollution and commodities that cause pollution. If we assume that it is some inputs (*e.g.,* coal) that cause pollution, then an increase in these inputs causes an increase in pollution. At the same time, an increase in these inputs results also (under standard assumptions) in an increase in intended outputs (say electricity). Thus, the positive correlation between by-products and intended outputs exists, even in the absence of abatement activities.

We show that abatement options available to firms can also be explicitly factored into our model. When they are available, they form a part of both the production of intended output (as their production is also costly in terms of resources/inputs of the firm) and the residual generation mechanism (as they mitigate residual generation). From the full technology, we derive a reduced form technology that is in the spirit of those studied in the usual literature. We find, however, that while our reduced form technology satisfies standard disposability properties with respect to inputs and outputs that do *not* affect pollution generation, the disposability properties of this technology with respect to abatement and commodities (*e.g.,* coal) that cause pollution are ambiguous. In general, this reduced form technology does *not* satisfy free disposability in these inputs and outputs, contrary to common assumptions in the literature.

In Section 2, we introduce our notation and show that a single implicit relation between outputs and inputs is not rich enough to capture, simultaneously, all the trade-offs between commodities that are implied by the phenomenon of by-production. In Section 3, we propose a model of a pollution-generating technology in which these inconsistencies in trade-offs are resolved, immaterial of whether or not abatement options are open to firms. We also derive a reduced form of our technology that is in the spirit of the one that is usually studied in the literature. In Section 4, we compare our (by-production) approach to modeling pollution-generating technologies with the standard approaches taken in the literature. In Section 5, we turn to DEA and econometric specifications of technologies that best capture the features of by-production that we have identified and provide some examples to explicate the differences between the DEA technologies derived using our specification and those derived using the assumptions of weak disposability and null-jointness. We conclude in Section 6.

2. Single equation representation of pollution-generating technologies.

2.1. The case without abatement output.

The vectors of input quantities (indexed by $i = 1, \ldots, n$), intended-output quantities (indexed by $j = 1, \ldots, m$), and incidental-output quantities (indexed by $k = 1, \ldots, m'$), are given, respectively, by $y \in \mathbf{R}^m_+$, $z \in \mathbf{R}^{m'}_+$, and $x \in \mathbf{R}^n_+$.

Suppose pollution is caused by the use of certain inputs like coal or because of the production of certain intended outputs like bleached paper. Suppose also that the firm does not participate in any abatement activity to reduce the pollution that it generates. A single equation formulation of such a pollution-generating technology, an extension of the standard functional representation of a multiple-output technology, is as follows:

$$T = \big\{ \langle x, y, z \rangle \in \mathbf{R}^{m+m'+n}_+ \big| \; f(y, z, x) \leq 0 \big\},$$

where f is differentiable, with derivatives with respect to inputs and intended outputs given by

$$\begin{aligned} &\text{(a)} \quad f_i(x, y, z) \leq 0, \; i = 1, \ldots, n, \\ &\text{(b)} \quad f_j(x, y, z) \geq 0, \; j = 1, \ldots, m, \end{aligned} \tag{2.1}$$

(where subscripts on f indicate partial differentiation with respect to the indicated variable).

The constraints (a) and (b) are standard differential restrictions to impose "free disposability" of, respectively, inputs and intended outputs:

$$\langle x, y, z \rangle \in T \;\wedge\; \bar{x} \geq x \Longrightarrow \langle \bar{x}, y, z \rangle \in T \tag{2.2}$$

and

$$\langle x, y, z \rangle \in T \ \wedge \ \bar{y} \leq y \Longrightarrow \langle x, \bar{y}, z \rangle \in T. \tag{2.3}$$

To capture the fact that pollution is an output of the production process whose disposal is not free, Murty [2010] introduces and formalizes an assumption that is the polar opposite of free output disposability with respect to the unintended outputs:

$$\langle x, y, z \rangle \in T \ \wedge \ \bar{z} \geq z \Longrightarrow \langle x, y, \bar{z} \rangle \in T. \tag{2.4}$$

Following Murty [2010], we refer to this property as "costly disposability" of residuals.[12] The differential restriction required to impose costly disposability on T is

$$f_k(x, y, z) \leq 0, \ \ k = 1, \ldots, m'. \tag{2.5}$$

Quantity vectors satisfying $f(x, y, z) = 0$ are points on the frontier of the technology; those satisfying $f(x, y, z) < 0$ are inefficient: more intended output could be produced with given quantities of inputs and pollution; less pollution could be generated with given intended output and input quantities; and smaller input quantities could be used to produce the given output quantities, given the pollution level.[13]

Assume, in this section and without loss of generality, that $m' = 1$. Suppose $f_k(\hat{x}, \hat{y}, \hat{z}) < 0$ for some $\langle \hat{x}, \hat{y}, \hat{z} \rangle$ satisfying $f(\hat{x}, \hat{y}, \hat{z}) = 0$. Then, from the implicit function theorem, there exist neighborhoods $U \subseteq \mathbf{R}_+^{m+n}$ and $V \subseteq \mathbf{R}_+$ around $\langle \hat{y}, \hat{x} \rangle \in \mathbf{R}_+^{m+n}$ and $\hat{z} \in \mathbf{R}_+$ and a function $\zeta : U \to V$ such that

$$\hat{z} = \zeta(\hat{y}, \hat{x}) \tag{2.6}$$

and

$$f\big(y, \zeta(y, x), x\big) = 0. \tag{2.7}$$

[12] Costly disposability implies the possibility of inefficiencies in the generation of pollution. If a given level of coal generates some level of smoke, then inefficiency in the use of coal may imply that this level of coal can also generate a greater amount of pollution. See also footnote 16 in Section 3.1.

[13] Of course, given the weak inequalities in the constraints (2.1) and (2.5), the set of efficient points is a subset of the frontier.

The trade-off between each intended output and unintended output k (with inputs and all other outputs held fixed) implied by the implicit function theorem is

$$\frac{\partial \zeta(x,y)}{\partial y_i} = -\frac{f_j(x,y,z)}{f_k(x,y,z)} \geq 0, \quad j = 1, \dots, m. \tag{2.8}$$

The trade-off between each input and unintended output k (with intended outputs and all other inputs held fixed) is

$$\frac{\partial \zeta(\hat{y},\hat{x})}{\partial x_i} = -\frac{f_i(x,y,z)}{f_k(x,y,z)} \leq 0, \quad i = 1, \dots, n. \tag{2.9}$$

Noting that all these trade-offs are evaluated at points in the technology set that are technically efficient (that is, $f(y,x,z) = 0$), the foregoing formulation of a pollution-generating technology seems to be inconsistent with the phenomenon of by-production for the following reasons:

(a) The existence of the function ζ satisfying (2.8) as a strict inequality implies that there exists a rich menu of (technically efficient) $\langle y, z \rangle$ combinations, with varying levels of z, that are possible with *given levels of all inputs.* If pollution is generated by input usage, then this menu is contrary to phenomenon of by-production, since the phenomenon implies that at fixed levels of inputs (*e.g.,* coal), there will be only *one* technically efficient (minimal) level of pollution.[14]

(b) Furthermore, if pollution is generated by inputs such as coal, as is very often the case, the negative trade-offs between pollution generation and inputs (derived by holding the levels of intended outputs fixed), apparent in (2.9), are inconsistent with by-production, which implies that this trade-off should be positive.

How should one interpret the trade-offs observed under single equation modeling of pollution-generating technologies when one abstracts from abatement options? As discussed

[14] If pollution is caused by some intended outputs (*e.g.,* bad odor from some varieties of cheese produced by a dairy) and (2.9) holds as a strict inequality, then it implies that there exists a rich menu of (technically efficient) $\langle x, z \rangle$ combinations, with varying levels of z, that are possible with *given levels of all intended outputs.* Such a menu is inconsistent with by-production.

above, these trade-offs are not reflective of the phenomenon of by-production. Rather, the positive trade-offs observed in (2.8) between each intended output and pollution and the negative trade-offs observed in (2.9) between each input and pollution suggest that this approach treats pollution like any other input in production: first, increases in its level, holding all other inputs fixed, increases intended outputs and, second, pollution is a substitute for all other inputs in intended production—the same level of intended outputs can be produced by decreasing other inputs and increasing pollution. This also does not seem to be intuitively correct: it is not a correct description of the role pollution plays in intended production.[15]

2.2. The case with abatement output.

Consider the case where the technology of a pollution-generating firm is defined by a single restriction on all inputs and outputs, including the abatement output:

$$T = \left\{ \langle x, y, z, y^a \rangle \in \mathbf{R}^{n+m+m'+1} \middle| \, f(y, z, x, y^a) \leq 0 \right\}. \tag{2.10}$$

We assume that

$$f_a(x, y, z, y^a) \geq 0. \tag{2.11}$$

This restriction captures the fact that the abatement output is also freely disposable:

$$\langle x, y, y^a, z \rangle \in T \;\wedge\; \bar{y}^a \leq y^a \Longrightarrow \langle x, y, \bar{y}^a, z \rangle \in T, \tag{2.12}$$

so that producing it is costly in terms of input usage, implying a negative trade-off between it and the other intended outputs. In that case, the implicit function theorem can again be invoked to show that, the trade-off between the abatement output and pollution, evaluated in a local neighborhood of a (technically efficient) point $\langle \hat{y}, \hat{z}, \hat{x}, \hat{y}_a \rangle \in \mathbf{R}_+^{n+m+m'+1}$ such that $f(\hat{x}, \hat{y}, \hat{z}, \hat{y}_a) = 0$ and $f_k(\hat{x}, \hat{y}, \hat{z}, \hat{y}_a) < 0$, is

$$\frac{\partial \zeta(x, y, y_a)}{\partial y^a} = -\frac{f_a(x, y, z, y_a)}{f_k(x, y, z, y_a)} \geq 0 \tag{2.13}$$

whenever $f(x, y, z, y_a) = 0$, contradicting the fact that abatement output is produced by firms to mitigate, and not to enhance, pollution.

[15] See also footnote 24 in Section 4.

3. A by-production approach to modeling pollution.

The above analysis reveals that a single implicit production relation between outputs and inputs is not rich enough to capture, simultaneously, all the trade-offs between commodities that are implied by the phenomenon of by-production. We build on the works of Frisch [1995] and Førsund [2009] to show that the phenomenon of by-production can be captured by employing more than one production relation when specifying pollution-generating technologies. In particular, the by-product generating mechanism needs to be explicitly distinguished from the production relation that describes the production of intended commodities. We show that when this is done the inconsistencies among trade-offs elucidated in Section 2 get resolved.

3.1. A by-production approach: the case without abatement.

In this sub-section, we abstract from explicit abatement efforts. The production of the intended output sets a residual-generation mechanism in motion, leading to the generation of the by-product. To fix our ideas on the salient aspects of by-production and to simplify notation, we continue to assume, without loss of generality, that $m' = 1$ and that the pollution is generated by usage of a single input (such an input could be coal), say input $\imath$, and the production of one of the firm's intended outputs, say output $\jmath$. Denote the input and output quantity vectors purged of the quantity of input $\imath$ and quantity of the output $\jmath$ by x^1 and y^1, respectively. Specify the technology as

$$T = T_1 \cap T_2, \tag{3.1}$$

where

$$T_1 = \big\{ \langle y^1, y_\jmath, z, x^1, x_\imath \rangle \in \mathbf{R}^{m+n+1} \ \big| \ f(y^1, y_\jmath, x^1, x_\imath) \leq 0 \ \big\}, \tag{3.2}$$

$$T_2 = \big\{ \langle y^1, y_\jmath, z, x^1, x_\imath \rangle \in \mathbf{R}^{m+n+1} \ \big| \ z \geq g(y_\jmath, x_\imath) \big\}, \tag{3.3}$$

and f and g are continuously differentiable functions. The set T_1 is a standard technology set, reflecting the ways in which the inputs can be transformed into intended outputs. The standard free disposability properties (2.3) and (2.4) can be imposed on this set by assuming that f satisfies

$$\begin{aligned} &f_i(x,y) \le 0,\ i = 1,\dots,n,\ \text{and} \\ &f_j(x,y) \ge 0,\ j = 1,\dots,m. \end{aligned} \tag{3.4}$$

Note, (3.2) imposes no constraint on z, that is, it is implicitly assumed that the by-product does not affect the production of intended outputs.[16]

The set T_2 reflects nature's residual-generation mechanism. T_2 satisfies costly disposability of pollution as defined in (2.4), with the function g defining the minimal level of pollution that gets generated for given levels of $x_\imath$ and $y_\jmath$.[17] The derivatives of g satisfy

$$\begin{aligned} &g_\imath(x_\imath, y_\jmath, z) \ge 0, \\ &g_\jmath(x_\imath, y_\jmath, z) \ge 0 \end{aligned} \tag{3.5}$$

The conditions in (3.5) capture the fact that the efficient (minimal) level of pollution rises with the increase in the usage of input $x_\imath$ or the production of the intended output $y_\jmath$. This means, however, that T_2 violates standard free disposability of input $x_\imath$. In fact it satisfies the polar opposite condition in these goods:[18]

$$\langle x^1, x_\imath, y^1, y_\jmath, z\rangle \in T_2 \ \wedge\ \bar{z} \ge z \ \wedge\ \bar{x}_\imath \le x_\imath \ \wedge\ \bar{y}_\jmath \le y_\jmath \Longrightarrow \langle x^1, \bar{x}_\imath, y^1, \bar{y}_\jmath, \bar{z}\rangle \in T_2. \tag{3.6}$$

It is easy to infer the disposability properties of T from the disposability properties of the intended production technology T_1 and the residual generation mechanism T_2

[16] This could be generalized, of course, allowing pollution to have an effect on intended production as well; *e.g.,* smoke could adversely affect the productivity of labour engaged in producing intended outputs. In that case, suitable adjustments can be made to the analysis below to take account of this generalization. These adjustments may impinge on the disposability assumptions that the pollution-generating technology satisfies and on the trade-offs among various commodities.

[17] Costly disposability, as defined in (2.4), could be considered to be too extreme. It implies that an infinite amount of z can be generated by given amounts of $x_\imath$ and $y_\jmath$. In general, there may also be an upper bound for the generation of z. Let the set $\mathcal{T}_2$ reflect the realistic bounds on the generation of the by-product z. Then $\mathcal{T}_2 \subseteq T_2$ and both sets have a common lower boundary defined by the function g (in fact, T_2 is a particular monotonic hull of $\mathcal{T}_2$). From the point of view of technical efficiency and the econometric and DEA approaches for constructing technologies of pollution-generating firms and using the constructs for measurement issues, it is only this lower bound that is important. Hence, we focus only on the set T_2.

[18] This assumption reflects the possibility of inefficiencies in the production of pollution: if given levels of coal and a pollution-generating intended output generate some amount of pollution, then inefficiencies in residual generation may imply that lower amounts of the coal input or the intended output can generate the same level of pollution if the firm operates more efficiently.

Theorem 1: *T satisfies free disposability with respect to all intended outputs and non-pollution causing inputs. However, it violates free-disposability with respect to the pollution causing input $x_\imath$. It satisfies costly disposability with respect to pollution z.*

The technology violates standard disposability conditions with respect to the pollution causing input $x_\imath$ because, while T_1 satisfies standard free-disposability conditions in $x_\imath$, T_2 satisfies the polar opposite conditions with respect to this input.

Quantity vectors in $\langle x, y, z\rangle \in T$ that satisfy $f(x, y) = 0$ and $z = g(x_\imath, y_\jmath)$ are the frontier points of T. If a quantity vector in $\langle x, y, z\rangle \in T$ is such that $f(x, y) < 0$, then it is technologically possible to increase the levels of the non-pollution causing intended outputs or decrease the levels of the non-pollution causing inputs without changing the production levels of the remaining goods. If a quantity vector in $\langle x, y, z\rangle \in T$ is such that $z > g(x_\imath, y_\jmath)$, then it is technologically possible to decrease the level of pollution without changing the production levels of all other goods.

To sign the trade-offs between z and a non-pollution-causing intended output y_j at a frontier point of T, we invoke the implicit function theorem. Let $\langle \hat{y}, \hat{z}, \hat{x}\rangle$ be a frontier point of T. Then

$$\begin{aligned} f(\hat{y}, \hat{x}) &= 0 \\ \hat{z} - g(\hat{y}_\jmath, \hat{x}_\imath) &= 0. \end{aligned} \tag{3.7}$$

Denote y^{-j} to be the vector obtained by purging the j^{th} element from vector y, where $j \neq \jmath$. Suppose that $f_j(\hat{y}, \hat{x}) \neq 0$ and $g_\imath(\hat{x}_\imath, \hat{y}_\jmath) \neq 0$. Then the matrix

$$\begin{bmatrix} f_j(\hat{y}, \hat{x}) & f_\imath(\hat{y}, \hat{x}) \\ 0 & -g_\imath(\hat{x}_\imath, \hat{y}_\jmath) \end{bmatrix} \tag{3.8}$$

has full row rank. By the implicit function theorem, there exists a neighborhood U around $\langle \hat{y}^{-j}, \hat{x}^1, \hat{z}\rangle$ in $\mathbf{R}_+^{m+n}$, a neighborhood V around $\langle \hat{y}_j, \hat{x}_\imath\rangle$ in $\mathbf{R}_+^2$, and continuously differentiable mappings $\psi^j : U \to \psi^j(U)$ and $h : U \to h(U)$ with images

$$\begin{aligned} y_j &= \psi^j(y^{-j}, x^1, z) \\ x_\imath &= h(y^{-j}, x^1, z) \end{aligned} \tag{3.9}$$

such that $\left\langle \psi^j(y^{-j}, x^1, z),\ h(y^{-j}, x^1, z) \right\rangle \in V$ and

$$\begin{aligned} f(\psi^j(y^{-j}, x^1, z), y^{-j}, x^1, h(y^{-j}, x^1, z)) &= 0 \\ z - g(h(y^{-j}, x^1, z)) &= 0. \end{aligned} \tag{3.10}$$

In that case, assuming that $g_\imath(x_\imath, y_\jmath) > 0$, the trade-off between y_j and z is[19]

$$\frac{\partial \psi^j(y^{-j}, x^1, z)}{\partial z} = -\frac{f_\imath(y, x)\ h_k(y^{-j}, x^1, z)}{f_j(y, x)} \geq 0. \tag{3.11}$$

How should one interpret this non-negative "trade-off" between y_j and z seen in (3.11)? Starting at $\langle \hat{y}, \hat{x}, \hat{z} \rangle \in T$, an increase in z is attributable, because of the by-production phenomenon inherent in T_2, to an increase in $x_\imath$ if $y_\jmath$ is held fixed at $\hat{y}_\jmath$ (as $h_\imath(\hat{z}, \hat{y}_\jmath) > 0$). Under the conventional assumptions on intended production in (3.4), the trade-off between the pollution-generating input $\imath$ and intended output j is

$$-\frac{f_\imath(y, x)}{f_j(y, x)}; \geq 0, \tag{3.12}$$

hence, the increase in $x_\imath$ implies an increase in y^j. The "trade-off" in (3.11), thus, reflects a non-negative *correlation* between the residual and an intended output via $x_\imath$, because a change in $x_\imath$ affects both y^j (non-negatively in intended production) and z (positively with respect to residual generation); this is not a trade-off in the usual economic sense.

To summarize, the non-negative "trade-off" between an intended and an unintended output in the reduced form model is explained by (a) the phenomenon of by-production, which relates the use of inputs such as $x_\imath$ to the by-product z, and (b) the non-negative marginal product of input $x_\imath$ in producing intended outputs like j.

[19] Note that the function h is the inverse of g: $h(y^{-j}, x^1, z) = g^{-1}(z, y_\jmath)$, so that, if $z = g(x_\imath, y_\jmath)$ and $g_\imath(x_\imath, y_\jmath) > 0$, then the derivative of h with respect to z is $h_k(z, y_\jmath) = \frac{1}{g_\imath(x_\imath, y_\jmath)} > 0$.

3.2. A by-production approach: incorporating abatement activities.

We again keep the analysis simple by sticking to a single abatement output (as well as a single unintended output). On the other hand, we make the model more general to allow the possibility of input substitutability in the generation of the by-product.[20] We do so by partitioning the vector of all n inputs into n_1 non-residual-generating inputs and n_2 residual-generating inputs. Denote the respective input quantity vectors by x^1 and x^2. Let y^a denote the level of the firm's abatement activities, which are also costly in terms of the input resources of the firm. Similarly to the previous section, we specify the technology as $T = T_1 \cap T_2$, where

$$\begin{aligned} T_1 &= \left\{ \langle y, y^a, z, x^1, x^2 \rangle \in \mathbf{R}^{m+n+2} \;\middle|\; f(y, y^a, x^1, x^2) \leq 0 \right\} \\ T_2 &= \left\{ \langle y, y^a, z, x^1, x^2 \rangle \in \mathbf{R}^{m+n+2} \;\middle|\; z \geq g(y^a, x^2) \right\}. \end{aligned} \tag{3.13}$$

T reflects both the transformation of inputs into intended outputs and abatement output (as indicated by the definition of T_1) and the use of the abatement output by the firm to control the by-production of the residual that results from use of pollution-generating inputs in producing intended outputs (as indicated by the definition of T_2 in (3.13)). We confine ourselves again to a local analysis and posit the following signs of the partial derivatives at a frontier point $\langle \hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2, \hat{z} \rangle$ of T:

$$\begin{aligned} f_j(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) &\geq 0, \;\; j = 1, \ldots m, \\ f_a(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) &> 0, \\ f_i(y, \hat{y}_a, \hat{x}^1, \hat{x}^2) &\leq 0, \;\; i = 1, \ldots n, \\ g_a(\hat{y}_a, \hat{x}^2) &< 0, \\ g_\imath(\hat{y}_a, \hat{x}^2) &\geq 0 \quad \text{for all } \imath = n_1 + 1, \ldots, n, \\ g_\imath(\hat{y}_a, \hat{x}^2) &> 0 \quad \text{for some } \imath = n_1 + 1, \ldots, n. \end{aligned} \tag{3.14}$$

It is easy to see that (3.13) and (3.14) imply that T_1 satisfies standard free disposability conditions for inputs, abatement output, and intended outputs. In addition, there is a

[20] For example, substituting a cleaner variety of coal for a less pure variety or vice-versa.

negative (or at least non-positive) trade-off between standard outputs and the abatement output and a positive (or a non-negative) trade-off between each intended output and the inputs in intended production.

With respect to residual generation, (3.13) and (3.14) imply that T_2 satisfies costly disposability for z and a condition that is polar opposite to standard input and output free disposability for y^a and x^2:

$$\langle x^1, x^2, y, y^a, z\rangle \in T_2 \ \wedge \ \bar{z} \geq z \ \wedge \ \bar{x}^2 \leq x^2 \ \wedge \ \bar{y}^a \geq y^a \Longrightarrow \langle x^1, \bar{x}^2, y, \bar{y}^a, \bar{z}\rangle \in T_2. \quad (3.15)$$

We call (3.15) "costly disposability of pollution, abatement output, and inputs that generate pollution."[21] The trade-offs between z and each of the pollution-generating inputs $x_\imath^2$ implied by (3.14) are non-negative and that between z and abatement output y^a is negative. Thus, the sign of g_a captures the mitigating effect abatement has on residual generation and the sign of $g_\imath$ captures the increase in pollution attributable to the increase in inputs causing pollution.

It is easy to infer the disposability properties of T from the above characteristics of T_1 and T_2:

Theorem 2: *T satisfies free disposability with respect to all intended outputs and non-pollution causing inputs. However, it violates free disposability with respect to each of the pollution-causing inputs and the abatement output. It satisfies costly disposability with respect to pollution.*

[21] This assumption reflects the inefficiencies in the production of pollution: if given levels of coal and abatement activities generate some amount of pollution, then inefficiencies in the use of coal or abatement activities imply that a lower amount of the coal input or a higher level of abatement activities could generate the same level of pollution if the firm operates more efficiently.

Let the inequalities in (3.14) hold. We now sign the trade-off between z and an intended output y_j at a frontier point of T. As in the previous section, we do so by employing the implicit function theorem. Let $\langle \hat{y}, \hat{y}_a, \hat{z}, \hat{x}^1, \hat{x}^2 \rangle$ be a frontier point of T. Then

$$\begin{aligned} f(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) &= 0 \\ \hat{z} - g(\hat{x}^2, y^a) &= 0. \end{aligned} \tag{3.16}$$

Let $f_j(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) \neq 0$ and $g_a(\hat{x}^2, y^a) \neq 0$. Then the matrix

$$\begin{bmatrix} f_j(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) & f_a(\hat{y}, \hat{y}_a, \hat{x}^1, \hat{x}^2) \\ 0 & -g_a(\hat{x}^2, \hat{y}_a) \end{bmatrix} \tag{3.17}$$

is full-row ranked. The implicit function theorem implies that there exists a neighborhood U around $\langle \hat{y}^{-j}, \hat{x}, \hat{z} \rangle$ in $\mathbf{R}_+^{m+n}$, a neighborhood V around $\langle y_j, y^a \rangle$ in $\mathbf{R}_+^2$, and continuously differentiable mappings $\psi^j : U \to \psi^j(U)$ and $h : U \to h(U)$ with images

$$\begin{aligned} y_j &= \psi^j(y^{-j}, x, z) \\ y^a &= h(y^{-j}, x, z) = g^{-1}(z, x^2) \end{aligned} \tag{3.18}$$

such that $\left\langle \psi^j(y^{-j}, x, z),\ h(y^{-j}, x, z) \right\rangle \in V$ and

$$\begin{aligned} f(\psi^j(y^{-j}, x, z), y^{-j}, x, h(y^{-j}, x, z)) &= 0 \\ z - g(h(y^{-j}, x, z), x^2) &= 0. \end{aligned} \tag{3.19}$$

In that case, the trade-off between y_j and z is

$$\frac{\partial \psi^j(y^{-j}, x, z)}{\partial z} = -\frac{f_a\big(y, x, y^a\big)\ h_k(y^{-j}, x, z)}{f_j(y, x, z, y^a)} \geq 0. \tag{3.20}$$

As in the previous section, this non-negative trade-off between an intended output and pollution at a frontier point of T reflects a correlation between these commodities; in this case, this correlation is effected by abatement effort of the firm to mitigate by-production of pollution.[22] Precisely, holding the levels of all inputs (including pollution-causing inputs) fixed, an increase in z must have come about because of reductions in abatement efforts y^a by firms, and hence an increase in resources diverted towards production of other intended outputs y (assuming, of course, that firms are operating efficiently).

[22] Note that, as in the previous section, a (generally different) positive correlation between the intended and unintended outputs effected by an input that causes pollution could also be derived.

From our analysis above, we can derive the reduced-form functional representation of the technology T. By substituting out abatement efforts from the function f in (3.13), we can rewrite T equivalently as

$$T = \{\langle y, y^a, z, x^1, x^2\rangle \in \mathbf{R}^{m+n+2} \mid \tilde{f}(x, y, z) \leq 0 \ \wedge \ \ y^a \geq h(y^{-j}, x, z)\}, \tag{3.21}$$

where

$$\tilde{f}(x, y, z) := f(y, h(y^{-j}, x, z), x). \tag{3.22}$$

Using (3.21), we can define a reduced-form technology in the space of intended and unintended outputs and inputs as

$$\tilde{T} := \{\langle y, z, x^1, x^2\rangle \in \mathbf{R}_+^{m+n+1} \mid \tilde{f}(y, z, x) \leq 0\}. \tag{3.23}$$

It is easy to check that, in the neighborhood of a frontier point $\langle y, y^a, z, x\rangle$ of T, the trade-off between an intended and an unintended output, $-\tilde{f}_j(x, y, z)/\tilde{f}_k(x, y, z)$, is given by (3.20) and hence is non-negative.

4. A comparison of conventional formulation of a pollution-generating technology and the by-production approach.

In the conventional literature, the standard building block for constructing pollution-generating technologies is the positive correlation that is usually observed in such technologies between intended and unintended outputs. Broadly speaking, there are two approaches in this literature: (a) a single-equation formulation of the technology and (b) a set-theoretic DEA formulation.[23] Both approaches attribute the positive correlation between unintended and intended outputs solely to "resource costly" abatement options available to firms. Further, what is modeled is a technology—quite in the spirit of $\tilde{T}$ in (3.23) above—in the space of intended and unintended outputs and inputs that exhibits a positive correlation between intended and unintended outputs but satisfies *all* of the standard free disposability assumptions with respect to intended outputs and inputs. The technology is modeled

[23] For references, see footnotes 9, 10 and 11 in Section 1.

only in reduced form because, although this literature attributes the positive correlation to abatement options available to firms, abatement activities are not explicitly modeled.

These approaches either treat pollution as a standard input (specifying a function decidedly in the spirit of $\tilde{f}$ in Section 3.2, with derivatives satisfying sign restrictions (2.1) and (2.5)) or they treat pollution as an output (with novel disposability conditions). In the latter case, the reduced-form technology is represented by either a parametric distance function or a set-theoretic DEA fconstruction. Here, too, all intended outputs and inputs are assumed to satisfy standard disposability conditions, but two key assumptions are made regarding the unintended outputs. The first,

$$\langle x, y, z \rangle \in \tilde{T} \;\wedge\; \lambda \leq 0 \Longrightarrow \langle x, \lambda y, \lambda z \rangle \in \tilde{T}, \tag{4.1}$$

is called "weak disposability", a concept originally attributable to Shephard [1953, 1974]. The second,

$$\langle x, y, z \rangle \in \tilde{T} \;\wedge\; z = 0 \Longrightarrow y = 0, \tag{4.2}$$

is called "null jointness". Weak disposability and null-jointness imply that, (a) while pollution is not freely disposable, it is possible to jointly and proportionately decrease pollution and the intended outputs and (b) production of *any* positive level of intended output always results in positive amounts of the residual being generated. This literature is predicated on the belief that these two assumptions can capture the fact that, starting at any efficient point of the technology, it is not possible to decrease pollution without decreasing the production of the intended outputs, and hence that, together, they model the reduced-form positive correlation between pollution and other intended outputs.

As argued in Section 2, the treatment of a by-product as any other productive input is contrary to the intuition we have about the role by-products such as pollution play in intended production.[24] It will be seen in the next section that the weak-disposability restriction on pollution-generating technologies does not preclude a *negative* correlation between intended and unintended outputs (a fact already noted in the literature cited in footnote 11 above). Moreover, it is possible to rationalize situations with abatement activities where no pollution is generated until a high-enough (positive) level of intended output is produced, a violation of null-jointness.

By explicitly modeling abatement activities as a part of both the process of intended production and the process of residual generation, the analysis in Section 3.2 above provides a theoretical foundation for the positive correlation between intended outputs and residuals that is assumed at the outset in the conventional literature. Moreover, the analysis in Section 3 demonstrates the existence of a much more fundamental cause of the positive correlation than the existence of resource-costly abatement options, namely, the use of certain inputs like coal or the production of certain intended outputs that generate pollution. The fact that pollution is caused by such inputs or outputs implies that a positive correlation between intended and unintended outputs exists even in the absence of abatement options. In other words, the models of technology in the conventional literatures can be interpreted as ones where the production relation that characterizes the residual generation mechanism (or the set T_2) involves only the abatement activities of the firm and the unintended outputs. In the model of by-production developed in Section 3, however, the residual-generating mechanism is primarily a relation in nature between inputs and intended outputs that cause pollution, with the level of pollution generated being conditioned also by the abatement activities of the firm.[25]

[24] In the literature, the treatment of pollution as any other productive input is often justified by considering the amount of pollution generated as a proxy for the amount of the assimilative capacity of environmental resources such as air and water used to absorb the pollution generated. However, a clear distinction needs to be made between these environmental resources, which definitely serve as inputs into the production process, and pollution, which is an output of production. A given environmental resource like air can absorb different types of unintended outputs like CO_2, SO_2, *etc.*, and its assimilative capacity can be different for different pollutants. See Murty [2010] for this distinction and its general equilibrium consequences. In this paper we abstract from the possible use of environmental resources as inputs into the production process.

[25] See the definition of T_2 in Section 3.2.

Further, we find that, in general, the assumptions made in the conventional literature about the disposability properties of the reduced-form technology are not borne out by $\tilde{T}$, which was derived in Section 3 from the intended production technology and the residual generation mechanism.[26] In particular, with respect to the reduced-form function $\tilde{f}$, note that

$$\tilde{f}_i(y, x^1, x^2) = f_a(y, y^a, x^1, x^2)h_i(z, x^2) + f_i(y, y^a, x^1, x^2),\ i = n_1 + 1, \ldots, n. \tag{4.3}$$

Given the sign conventions in (3.14), the sign of $\tilde{f}_i$ is ambiguous for a pollution-generating input i.

5. Implications of by-production for econometric and DEA modeling of technologies.

The foregoing analysis reveals that modeling the phenomenon of by-production requires more than one implicit production relation among inputs and outputs. One of these relations captures intended production activities of firms (that is, describes the set T_1), while the other captures the inevitability of residual generation when firms engage in intended production (that is, describes the set T_2). Technical efficiency of pollution-generating technologies requires both efficiency in intended production and technical efficiency in residual generation. The former identifies an upper bound for the intended outputs of firms for every given level of inputs, while the latter identifies a lower bound for pollution generation given every level of intended outputs and inputs that are responsible for causing pollution. Combined with appropriate disposability assumptions, the implications of by-production are clear for econometric and DEA models of pollution-generating technologies.

The econometric approach must involve simultaneous estimation of two (or more) production relations that have the above features. In particular the production relation associated with intended production will be the upper frontier of T_1 and the production relation

[26] This was apparent in Theorems 1 and 2 in Section 3, which showed that technologies that satisfy by-production do not satisfy free disposability in inputs that cause pollution and in the abatement output.

associated with residual generation will be the lower frontier of T_2. These production relations should satisfy the trade-offs implied by (3.14).

We now turn to understanding how we can capture multiple production relations involved in pollution-generating technologies using the activity analysis (DEA) approach. We consider a more general model than the one presented above, incorporating multiple pollution generating inputs and multiple pollutants.[27] We use the following notations.

(i) D decision making units, indexed by d.

(ii) M intended outputs, indexed by j, with quantity vector $y \in \mathbf{R}_{+}^{M}$. The $D \times M$ matrix of observations on intended output quantities is denoted by Y.

(iii) N inputs, indexed by i. The first N_1 are non-pollution-generating, while the remaining $N_2 = N - N_1$ are pollution generating. The quantity vector is $x = \langle x^1, x^2 \rangle \in \mathbf{R}_{+}^{N}$. The $D \times N$ matrix of observations on the input quantities is denoted by $X = \langle X^1, X^2 \rangle$.

(iv) M' pollutants, indexed by k, with quantity vector $z \in \mathbf{R}_{+}^{M'}$. The $D \times M'$ matrix of observations on pollutants is denoted by Z.

(v) The level of the abatement output is denoted by $y^a \in \mathbf{R}_{+}$. The $D \times 1$ matrix of observations on these is denoted by A.

As discussed above, by-production implies that the pollution-generating technology is $T = T_1 \cap T_2$. Thus, a data set coming from pollution-generating units must simultaneously belong to both T_1 and T_2.

[27] Extension to the case where some intended outputs also cause pollution is straightforward.

5.1. Constructing T_1.

We assume that T_1 satisfies free disposability of inputs, abatement output, and intended outputs (as defined in (2.2), (2.3), and (2.12)) and that it is closed, convex, and exhibits decreasing returns to scale.[28] In addition, T_1 satisfies the following assumption, which we call "independence of T_1 from z" and which states that pollution does not directly affect production of intended outputs:[29]

$$\langle x, y, y^a, z\rangle \in T_1 \Longrightarrow \langle x, y, y^a, \bar{z}\rangle \in T_1 \ \forall \ \bar{z} \in \mathbf{R}^{M'}. \tag{5.1}$$

The intended output technology T_1 that satisfies these assumptions is obtained in a standard way using DEA techniques as follows:

$$\begin{aligned} T_1 = \Big\{ \langle x, y, y^a, z\rangle \in \mathbf{R}^{N+M+M'+1} \mid \lambda X \le x,\ \lambda Y \ge y,\ \lambda A \ge y^a, \\ \sum_d \lambda_d \le 1 \text{ for some } \lambda \in \mathbf{R}^D_+ \Big\}. \end{aligned} \tag{5.2}$$

5.2. Constructing T_2.

We assume T_2 satisfies costly disposability of pollution, abatement output, and inputs that cause pollution (as defined in (3.15)). Also note that, since we have assumed that only x^2 and y^a affect residual generation, T_2 also satisfies "independence of T_2 from x^1 and y":

$$\langle x, y, y^a, z\rangle \in T_2 \Longrightarrow \langle \bar{x}, x^2, \bar{y}, y^a, z\rangle \in T_2 \ \forall \ \langle \bar{x}^1, \bar{y}\rangle \in \mathbf{R}^{N_1+M}. \tag{5.3}$$

The DEA version of T_2, which satisfies these assumptions, is obtained as

$$\begin{aligned} T_2 = \Big\{ \langle x^1, x^2, y, y^a, z\rangle \in \mathbf{R}^{N_1+N_2+M+M'+1} \mid \mu X^2 \ge x^2,\ \mu A \le y^a,\ \mu Z \le z, \\ \sum_d \mu_d \le 1 \text{ for some } \mu \in \mathbf{R}^D_+ \Big\}. \end{aligned} \tag{5.4}$$

[28] Extensions to constant or variable returns can be done in the usual way. (See, *e.g.*, Färe, Grosskopf, Lovell, and Pasurka [1989].)

[29] This assumption would have to be relaxed if, *e.g.*, the presence of pollution could adversely affect labor productivity in producing intended outputs.

The first inequality in (5.4) sreflects costly disposability of inputs that cause pollution, the second reflects costly disposability of abatement output, and the third reflects costly disposability of pollution. Since T_2 is independent of x^1 and y, no inequalities need to be specified for x^1 and y.

5.3. Constructing $T = T_1 \cap T_2$.

The overall technology T, the intersection of T_1 and T_2, is constructed as follows:

$$T = \Big\{ \langle x^1, x^2, y, y^a, z \rangle \in \mathbf{R}^{N_1+N_2+M+M'+1} \mid \lambda[X^1\ X^2] \leq \langle x^1,\ x^2 \rangle,\ \lambda Y \geq y,\ \lambda A \geq y^a, \\ \mu X^2 \geq x^2,\ \mu A \leq y^a \mu Z \leq z, \\ \sum_d \lambda_d \leq 1,\ \text{and} \sum_d \mu_d \leq 1 \\ \text{for some } \langle \lambda, \mu \rangle \in \mathbf{R}^{2D}_{+} \Big\}. \tag{5.5}$$

The above construction of T using activity analysis involves two sets of production relations. These are reflected in the two different intensity vectors λ and μ, each of which is applied to the same data set.

5.4. Examples and comparison with DEA technologies based on weak disposability and null jointness.

In this subsection, we compare our by-production approach with the standard DEA approach described in the literature cited in footnote 11. This comparison is facilitated by constructing a projection of our data-based technology into the space of intended and unintended outputs, holding input quantities fixed and allowing abatement output to vary in a way that is consistent with the full vector of intended and unintended output quantities, input quantities, and abatement activity belonging to the technology T. This is the space in which the reduced-form technology is constructed in the papers cited in footnote 11.

The DEA specification of a reduced-form, pollution-generating technology based on decreasing returns to scale, weak disposability, and null-jointness (see Färe, Grosskopf, Lovell, and Pasurka [1989]) is:

$$\tilde{T}_{WD} = \Big\{ \langle x, y, z \rangle \in \mathbf{R}^{N+M+M'} \mid \lambda X \leq x,\ \lambda Y \geq y,\ \lambda Z = z, \\ \sum_d \lambda_d \leq 1 \text{ for some } \lambda \in \mathbf{R}^D_+ \Big\}. \tag{5.6}$$

In the by-production approach, T_1, T_2, and T can be constructed from the data using (5.2), (5.4), and (5.5). We also construct restrictions (level sets) of T_1, T_2, and T to output space and denote them $P_1(x)$, $P_2(x)$, and $P(x)$, respectively. $P(x)$ is constructed as follows:

$$\begin{aligned} P(x) \equiv & \Big\{ \langle y, y^a, z \rangle \in \mathbf{R}^{M+M'+1} | \ \langle x^1, x^2, y, y^a, z \rangle \in T \Big\} \\ = & \Big\{ \langle y, y^a, z \rangle \in \mathbf{R}^{M+M'+1} | \ \lambda [X^1 \ X^2] \leq \langle x^1, x^2 \rangle,\ \lambda Y \geq y,\ \lambda A \geq y^a, \\ & \mu X^2 \geq x^2,\ \mu A \leq y^a \mu Z \leq z, \\ & \sum_d \lambda_d \leq 1, \text{ and } \sum_d \mu_d \leq 1 \\ & \text{for some } \langle \lambda, \mu \rangle \in \mathbf{R}^{2D}_+ \Big\}. \end{aligned} \tag{5.7}$$

The projection of $P_1(x)$ into $\langle y^a, y \rangle$ space is given by

$$\hat{P}_1(x) = \Big\{ \langle y, y^a \rangle \in \mathbf{R}^{M+1} | \ \langle y, y^a, z \rangle \in P_1(x) \text{ for some } z \in \mathbf{R}^{M'} \Big\}, \tag{5.8}$$

and the projection of $P_2(x)$ into $\langle y^a, z \rangle$ space is

$$\hat{P}_2(x) = \Big\{ \langle z, y^a \rangle \in \mathbf{R}^{M'+1} | \ \langle y, y^a, z \rangle \in P_1(x) \text{ for some } y \in \mathbf{R}^{M} \Big\}. \tag{5.9}$$

The projection of $P(x)$ into $\langle z, y \rangle$ space is then given by

$$\hat{P}(x) = \Big\{ \langle y, z \rangle \in \mathbf{R}^{M+M'} | \ \langle y, y^a \rangle \in \hat{P}_1(x) \ \wedge \ \langle z, y^a \rangle \in \hat{P}_2(x) \text{ for some } y^a \in \mathbf{R}_+ \Big\}. \tag{5.10}$$

In the DEA approach based on weak disposability and null jointness, the reduced form technology $\tilde{T}_{WD}$ can be constructed as in (5.6). We can then study its restriction:

$$\tilde{P}_{WD}(x) := \Big\{ \langle y, z \rangle \in \mathbf{R}^{M+M'} \mid \lambda X \leq x,\ \lambda Y \geq y,\ \lambda Z = z, \\ \sum_d \lambda_d \leq 1 \text{ for some } \lambda \in \mathbf{R}^D_+ \Big\}. \tag{5.11}$$

The difference between the two DEA approaches is elucidated by a comparison between $\hat{P}(x)$ and $\tilde{P}_{WD}(x)$.

In both of the following examples, $N_2 = 1,\ N_1 = 0,\ M = M' = 1$, and $x = 1$.

Example 1: $D = 5$. The data are as follows:

$$\begin{array}{cccc} x & y^a & y & z \\ 1 & 1 & 4 & 9 \\ 1 & 2 & 6 & 6 \\ 1 & 3 & 2 & 6 \\ 1 & 4 & 4 & 3 \\ 1 & 5 & 2 & 2 \end{array} \tag{5.12}$$

After plotting the data, we find that $\hat{P}_1(1)$ and $\hat{P}_2(1)$ can be represented functionally by piece-wise linear functions:

$$\begin{aligned} \hat{P}_1(1) &= \{\langle y^a, y\rangle \in \mathbf{R}^2_+ |\ y \leq \rho^1(y^a)\} \text{ and} \\ \hat{P}_2(1) &= \{\langle y^a, z\rangle \in \mathbf{R}^2_+ | z \geq \rho^2(y^a)\}, \end{aligned} \tag{5.13}$$

where

$$\begin{aligned} \rho^1(y^a) &= 6, & y^a &\in [0,2] \\ &= 8 - y^a, & y^a &\in [2,4] \\ &= 12 - 2y^a, & y^a &\in [4,5] \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \rho^2(y^a) &= 12 - 3y^a, & y^a &\in [1,2] \\ &= 9 - \frac{3}{2}y^a, & y^a &\in [2,4] \\ &= 7 - y^a, & y^a &\in [4,5] \\ &= 2, & y^a &\geq 5. \end{aligned} \tag{5.15}$$

The sets $\hat{P}_1(1)$ and $\hat{P}_2(1)$ are shown in Panels 1 and 2 of Figure 1. $\hat{P}(1)$, shown in Panel 3 of Figure 1, is constructed as follows:

$$\hat{P}(1) = \big\{\langle z, y\rangle \in \mathbf{R}^2_+ |\ z \geq \rho^2(y^a)\ \wedge\ y \leq \rho^1(y^a)\ \wedge\ y^a \in [0,5]\big\}. \tag{5.16}$$

Panel 4 of Figure 1 shows $\tilde{P}_{WD}(x)$. Note that the construction of $\hat{P}(x)$ involves explicit reference to the abatement output: in particular, we have been able to parametrize the frontier of the set $\hat{P}(x)$ in terms of the parameter y^a. No reference was made, however, to data on y^a in the construction of $\tilde{P}_{WD}(x)$.[30] As seen in Panel 4 of Figure 1, $\tilde{P}_{WD}(x)$ satisfies weak disposability and null jointness, but, the frontier has negatively sloped regions, indicating a negative correlation between intended and unintended outputs. The frontier of $\hat{P}(x)$, on the other hand, is everywhere non-negatively sloped.

Example 2: $D = 7$. The data are as follows:

x	y^a	y	z
1	0	8	9
1	1	7	6
1	2	6	8
1	3	6	3
1	4	1	2
1	5	4	0
1	6	2	0

(5.17)

Plotting the data reveals that $\hat{P}_1(1)$ and $\hat{P}_2(1)$ can be represented functionally by two piece-wise linear functions:

$$\begin{aligned} \hat{P}_1(1) &= \{\langle y^a, y\rangle \in \mathbf{R}^2_+ |\ y \leq \rho^1(y^a)\} \text{ and} \\ \hat{P}_2(1) &= \{\langle y^a, z\rangle \in \mathbf{R}^2_+ | z \geq \rho^2(y^a)\}, \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} \rho^1(y^a) &= 8 - \frac{2}{3}x, \qquad & y^a \in [0,3] \\ &= 9 - y^a, \qquad & y^a \in [3,5] \\ &= 14 - 2y^a, \qquad & y^a \in [5,6] \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} \rho^2(y^a) &= 9 - 3y^a, \qquad & y^a \in [0,1] \\ &= \frac{15}{2} - \frac{3}{2}y^a, \qquad & y^a \in [1,5] \\ &= 0, \qquad & y^a \geq 5. \end{aligned} \tag{5.20}$$

[30] In these examples, we have chosen to show the correlation between y and z via the abatement output so as to facilitate comparison with the conventional approaches. We have therefore held the level of the pollution-causing input x fixed. A similar example where the correlation between y and z is demonstrated via the pollution-causing input could also be constructed.

The sets $\hat{P}_1(1)$ and $\hat{P}_2(1)$ are shown in Panels 1 and 2 of Figure 2. $\hat{P}(1)$, shown in Panel 3 of Figure 2, is constructed as follows:

$$\hat{P}(1) = \big\{\langle z, y\rangle \in \mathbf{R}^2_+ |\ z \geq \rho^2(y^a)\ \wedge\ y \leq \rho^1(y^a)\ \wedge\ y^a \in [0,5]\big\}. \tag{5.21}$$

The frontier of $\hat{P}(x)$, as seen in Panel 3 of Figure 2, is non-negatively sloped. As seen in Panel 4 of Figure 2, $\tilde{P}_{WD}(x)$ satisfies weak disposability but violates null jointness. In our example, this is rationalized by the fact that abatement output of a firm can completely mitigate pollution even when it is producing positive amounts of the intended outputs. Of course, in this example, we have only one output; if there were multiple outputs, some generating pollution and others not, the possibility of positive quantities of non-pollution-generating outputs combined with zero pollution would seem to be perfectly feasible even in the absence of abatement activities.

6. Conclusions.

Pollution is an unintended output that cannot be freely disposed of. Underlying its production are a set of chemical and physical reactions that take place in nature when firms engage in the production of intended outputs. These natural reactions define nature's residual generation mechanism, which is a relation between the residuals generated and some inputs that are used or some intended outputs that are produced by the firm: hence, the inevitability of a certain minimal amount of pollution being generated when firms engage in intended production. We call this phenomenon by-production of pollution. The larger is the scale of intended production, the more are the pollution causing inputs being used or the more are the pollution causing intended outputs being produced, and hence, the more is the pollution generated. This provides the fundamental explanation for the positive correlation that is observed between intended production and residual generation.[31]

[31] Some of the literature has adopted physical science terminology to describe these relationships in terms of the "material balance" condition (see Ayres and Kneese [1969] and, more recently, Coelli, Lauwers, and van Huylenbroeck [2007].

Most of the existing literature attributes the observed positive correlation to abatement options available to firms. We show that abatement options of firms add to the phenomenon of by-production in strengthening the observed positive correlation between pollution generation and the production of intended outputs. The existing literature usually does not explicitly model abatement options of firms, but considers a reduced form of the firm's technology, which satisfies standard disposability assumptions with respect to all inputs and intended outputs.

We show that more than one implicit production relation is needed to capture all the technological trade-offs that are implied by the phenomenon of by-production. In particular, we show that by-production can be modeled by decomposing the technology into an intended-production technology and a residual-generation technology. The latter must exhibit costly disposal of pollution, as discussed in Murty [2010]. Abatement activities of firms can be added to the model as an additional factor in both the intended-production technology and the residual-generation technology. From this general model, we are able to derive a reduced form of the technology in the space of inputs, intended outputs, and unintended outputs that is in the spirit of that usually studied in the literature. Contrary to the usual literature, however, we find that the reduced-form technology violates standard disposability assumptions with respect to inputs and intended outputs that cause pollution. We derive implications from the phenomenon of by-production for the econometric and DEA specifications of pollution-generating technologies. We derive a DEA specification of technologies that satisfies by-production. Such a specification can be used to study issues relating to the measurement of efficiency, marginal abatement costs, productivity *etc.* of firms with pollution-generating technologies.

APPENDIX

Implicit Function Theorem: Let $f : \mathbf{R}^n_+ \times \mathbf{R}^m_+ \to \mathbf{R}^m$ be a continuously differentiable vector valued function with image $f(x, y) = z$, where $x \in \mathbf{R}^n_+$ and $y \in \mathbf{R}^m_+$. Let $\langle \bar{x}, \bar{y} \rangle \in$

$\mathbf{R}_+^{n+m}$ be such that $f(\bar{x}, \bar{y}) = 0$ and the $m \times m$ matrix $\nabla_y f(\bar{x}, \bar{y})$ is full-row ranked (has a non-zero determinant). Then there exist neighborhoods U and V around $\bar{x}$ and $\bar{y}$ in $\mathbf{R}_+^n$ and $\mathbf{R}_+^m$, respectively, and a continuously differentiable function $\Phi : U \to V$ with image $\Phi(x) = y$ such that, for all $x \in U$, we have $f(x, \Phi(x)) = 0$ and

$$\nabla_x \Phi(x) = - \left[\nabla_y f(x, \Phi(x))\right]^{-1} \nabla_x f(x, \Phi(x)).$$

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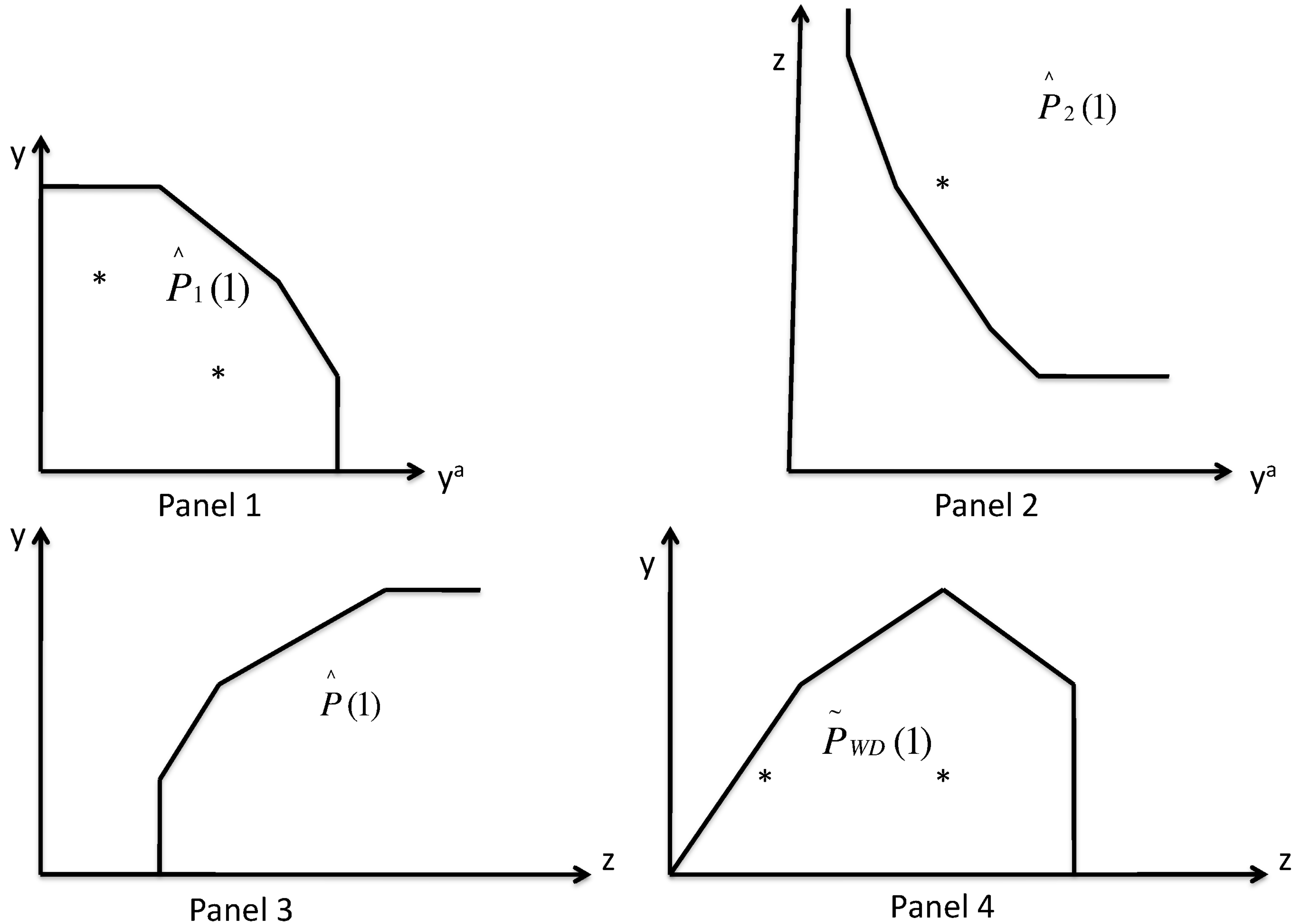
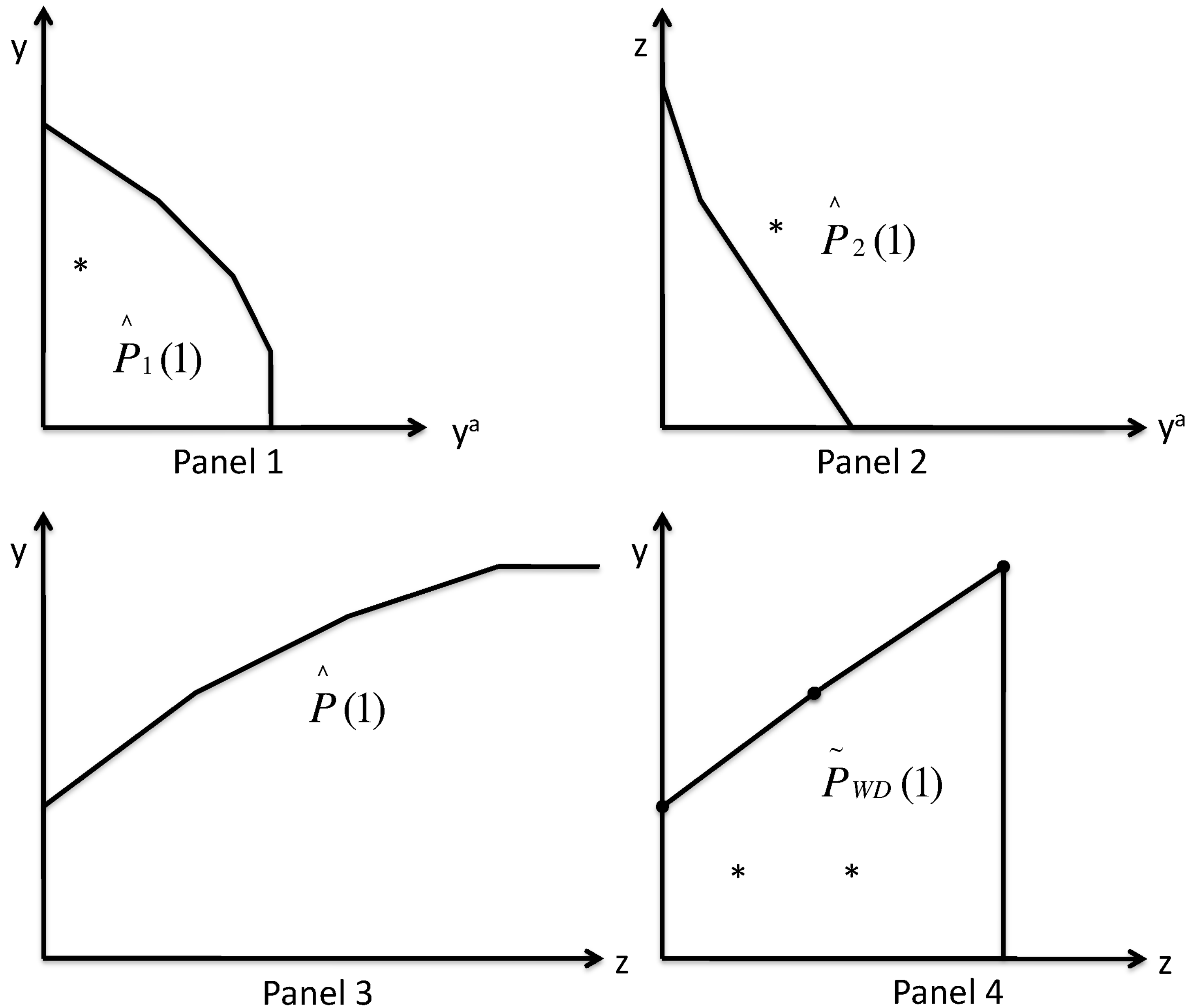


y
$\hat{P}_1(1)$
y^a
Panel 1
z
$\hat{P}_2(1)$
y^a
Panel 2
y
$\hat{P}(1)$
z
Panel 3
y
$\tilde{P}_{WD}(1)$
z
Panel 4

Figure 1



y
*
$\hat{P}_1(1)$
y^a
Panel 1
z
*
$\hat{P}_2(1)$
y^a
Panel 2
y
$\hat{P}(1)$
z
Panel 3
y
$\tilde{P}_{WD}(1)$
* *
z
Panel 4

Figure 2