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THE CALIBRATION OF EXPECTED SOYBEAN PRICE DISTRIBUTIONS: AN OPTION BASED APPROACH

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ABSTRACT

No-arbitrage option pricing models are used to recover complete probabilistic descriptions of expected soybean futures prices. The usefulness of the approach is examined via calibration tests. Results indicate that the estimated distributions are fairly reliable and that a three-parameter Burr distribution is useful in characterizing expected prices.

THE CALIBRATION OF EXPECTED SOYBEAN PRICE DISTRIBUTIONS: AN OPTION BASED APPROACH

Expectations of economic variables serve as primary inputs in business planning and decision making. One particularly important variable to agricultural producers, processors, and other market participants is the future price of a commodity. Because prices in the future are uncertain, decision makers would best be served by a characterization of the entire expected price distribution. Improvements in decision making methods and risk management techniques would be facilitated with a means of accurately describing *ex ante* price distributions rather than point forecasts. Whole probability distributions should be explored and estimated if risk averse users are to be well served (Anderson, Dillon and Hardaker).

In deriving estimates of an *ex ante* price distribution, *ex post* data are often used. However, the use of *ex post* data may lead to an inaccurate assessment of distribution parameters if they are non-stationary over time. Unfortunately, direct elicitation of *ex ante* parameter expectations from market participants is difficult, if not impossible.

This study investigates an approach to estimate a market-determined *ex ante* price distribution. The interaction of all participants in futures and options markets results in a collective expression of their beliefs about future prices. This seemingly innocuous observation provides an avenue toward the recovery of expected price distribution parameters without extensive surveys or direct elicitation. Rather than asking market participants about their expectations, these can be derived by observing market determined options prices. And, in deriving accurate assessments of probabilities of future prices, it is unlikely that one will find any better estimate than the market's (Gardner).

Options' payoffs are contingent on the possible outcomes of the underlying security's price. In the case of commodity futures options, the underlying security is a futures contract for that commodity. The option price (premium) therefore implicitly contains the assessments by market participants of the distribution of the underlying futures contract prices at expiration. The technique used in this study is to "invert" the process of valuing options by using observed premia to recover the implied *ex ante* futures contract price distribution. This methodology requires ancillary assumptions regarding market efficiency and the process of valuation. A no-arbitrage pricing technique is used that makes use of the simplifying assumptions of market efficiency and relies on relatively few restrictions regarding the process determining options prices.

OPTION PRICING: THE BLACK-SCHOLES MODEL

In the well known Black-Scholes (B-S) stock option pricing model and the Black futures option pricing model, an option's price depends on the underlying asset's price, the strike price, time to expiration, an assumed constant risk-free rate of interest, and the instantaneous volatility, σ, of the underlying asset's return stream. Of these variables, only the risk-free interest rate and the volatility are not easily observed. As a proxy for the risk-free rate, the yield on a treasury bill that expires near the option is often used.

One means of estimating volatility to search for the value of σ that equates the price computed by the B-S model to the option price observed in the market. The resulting estimate of volatility is termed the implied volatility (IV). If the current futures price of the underlying contract is taken as the mean of the expected future distribution, an entire two parameter distribution of expected prices may be described. Many studies have related the implied volatility to a broad range of economic variables and have studied the time series properties of these parameters (e.g. Schmalensee and Trippi; Beckers; Chiras and Manaster; Park and Sears; Jordan et al.; Shastri and Tandon). Unfortunately, the B-S model relies on a set of restrictive assumptions about the underlying price process and the ability to continuously form risk-free hedges that may make it inappropriate to recover complete probabilistic assessments of a future price distribution.

NO-ARBITRAGE OPTION PRICING MODEL

A widely accepted basis for asset pricing is the no-arbitrage pricing theory first proposed by Ross. Absence of arbitrage is a necessary condition for market equilibrium, so the assumption that assets trade at equilibrium assures that there is no arbitrage. Unlike the B-S model, the no-arbitrage pricing model makes no assumptions about the price dynamics of the underlying security.

Ross and others (Breeden and Litzenberger; Banz and Miller; Cox and Ross) show that no-arbitrage implies the existence of a "supporting pricing function" denoted as f(s). The pricing function may be interpreted as a set of prices of pure contingent claims that pay \$1 if and only if their particular state, s, occurs. A \$1 risk free bond may be constructed by buying one pure contingent claim for each state. Hence, the price of a one dollar bond is equal to the sum of the state prices for all possible states. If the states are essentially continuous, f(s) corresponds to the state-price density rather than a discrete probability function, but the arguments are otherwise analogous in that the price of a bond that pays \$1 at time T regardless of what state occurs has a current price equal to $b(T) = \int_s r_s f(s)ds$ where r_s is the return in state s (in this case equal to a constant of \$1) and f(s) is the expected state price density at T. Given linearity of the pricing function across assets, any asset with return r_s at time T may be valued like the bond by simply taking the expectation of its returns with respect to f(s). The bond is the simplest because its returns are \$1 in all states. Conversely, if the price of a risk-free bond were known and all possible states were identified, a consistent f(s) could be located. Since the focus is on the use of the model, the discussion is somewhat limited. An excellent treatment of the complete set of no-arbitrage restrictions is given in Ingersoll.

Given the system of (1) asset prices, (2) states of the asset economy, and (3) distribution of state prices, any one of the three may be determined if the other two are known. The discussion above used payoffs and a supporting price distribution to determine no-arbitrage consistent prices, but observed asset prices and a payoff function could be used to solve for a set of state probabilities. Or if the state probabilities are parameterized as a continuous function, the parameters of that function can be estimated. Thus, a return stream r_s, and a distribution of time dependent state prices yield a current value consistent with no arbitrage. Any asset, V_i, with return stream $r_{i,s}$ in the economy may therefore be priced via the relation:

$$V_i = \int_s r_{i,s} f(s)ds. \quad (1)$$

Call and put options written on futures have clear return functions, $r_{i,s}$. For calls, the return at time T is simply $\max\{Y_T - x, 0\}$ where x is the exercise price and Y_T is the random futures price at time T. For puts, the function is $\max\{x - Y_T, 0\}$. The outcome of Y at T completely determines the relevant state for an option's payoff. By noting that for $Y_T > x$, the value of the put option (P) is zero, and that for $Y_T < x$, the value of the call option (C) is zero, current values of puts and calls can be expressed as their discounted expected payoff:

$$C_i = b(T)\int_{x_i}^{\infty}\{Y_T - x_i\}\ f(Y_T)dY_T, \tag{2}$$

$$\text{and } P_j = b(T)\int_{0}^{x_j}\{x_j - Y_T\}\ f(Y_T)dY_T\,. \tag{3}$$

A key distinction between this and the typical option pricing approach is that no assumptions have been made about the underlying price dynamics or changes in the economic environment prior to expiration. The only assumption made is that there are no arbitrage opportunities, thus guaranteeing the existence of $f(Y_T) = dF(Y_T)/dY_T$.

ESTIMATION OF EXPECTED DISTRIBUTION PARAMETERS

One objective of this research is to estimate parameters that will allow $F(Y_T)$, the implied pricing function or the *ex ante* price distribution, to be described. Distributions for F are chosen that are as unrestrictive as possible. One commonly used function is the lognormal distribution, but many studies find empirical distributions that are more leptokurtic and more or less skewed than that implied by a lognormal distribution of prices (Gordon; Hall, Brorsen, and Irwin).

This study uses two distributions as the primary candidates for $F(Y_T)$: (1) the two-parameter lognormal; and (2) the three-parameter Burr-12 or Singh-Maddala (SM) distribution.[1] Both distribution allow for only positive values of Y_T, and the SM may take on a wide range of skewness and kurtosis (Tadikamalla). The SM cumulative distribution function (CDF) is:

$F_{SM}(Y|\alpha,\lambda,\tau) = 1 - (\lambda/(Y^{\tau}+\lambda))^{\alpha}$ for α, λ, τ, $Y > 0$, and thus the density, or PDF is: (4)

$$f_{SM}(Y|\alpha,\lambda,\tau) = \alpha\lambda^{\alpha}\tau Y^{\tau-1}(Y^{\tau}+\lambda)^{-(\alpha+1)}. \tag{5}$$

The cumulative distribution function for the lognormal distribution is:

$$F_{LN}(Y|\mu,\sigma) = N(\ln(Y-\mu)/\sigma) \tag{6}$$

where $N(\bullet)$ is the cumulative normal density function. The lognormal density is:

$$f_{LN}(Y|\mu,\sigma) = (2\pi)^{-1/2}(\sigma Y)^{-1}\exp[-(\ln Y-\mu)^2/(2\sigma^2)] \tag{7}$$

A comparison of the two distributions is made to lift up possible improvements in moving to a three parameter distribution. There is no guarantee that the parameters of $F(Y_T)$ will conform to an *ex post* price distribution with either parameterization. The relative performance of the two candidate distributions is visited in a later section.

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[1] These are among several distributions first investigated in Fackler (1986).

EVALUATION OF ESTIMATED PARAMETERS

Calibration, or reliability, refers to the correspondence between a predicted and an actual event. In terms of distributions, calibration describes how close the predicted and resulting functions are. If there were a reason for the market's aggregation of individual expectations to yield estimated parameters that required an adjustment to correspond to the "true" parameters, then this adjustment is termed the calibration function. Specifically, if the true *ex ante* parameters of a distribution are $\phi(x)$ and the estimates are F(x), then K(F(x)) = $\phi(x)$ implicitly defines a transformation K(•) of F to generate estimates K(F(x)) that are well calibrated or reliable. The function K(•) is called the calibration function. Equivalently, given a subjective or implied p.d.f., the process generating the subjective or implied p.d.f. is said to be well calibrated if the proportion of times the realized value lies below the r^{th} fractile of the implied p.d.f. is equal to r (Curtis et al.). A calibration function accounts for more than a simple bias in that it corrects all moments of an estimated distribution. The result of calibration is to make the long run probabilities (density) of K(F(x)) = $\phi(x)$ for any level of x. If F(x) is already well calibrated, then K(•) will simply be an identity mapping. If, for example, F(x) places too much weight in the lower tail, K(•) will be lower than a uniform density at low values of x and higher at high values reflecting the re-weighting of F that is necessary to force a correspondence to $\phi(x)$. K(•) therefore re-weights F(•) and is itself a probability measure. The test for calibration then, is equivalent to testing the uniformity of K, (Fackler and King) for if F(•) is calibrated, K is simply a one-to-one mapping whose CDF is a straight line.

For the purposes of this study, the calibration function is based on the beta distribution with density

$$K(x) = x^{p-1}(1-x)^{q-1}/\beta(p,q), \qquad (8)$$

where ß(p,q) is the beta function with parameters p and q. Fackler and King outlines a means of using maximum likelihood estimates of the parameters of the Beta distribution to explicitly model the calibration function. Note that the uniform is a special case of the Beta with p=q=1 and would imply perfect calibration. Other shapes of the fitted calibration curve indicate the "reweighting" of the estimated distributions needed to correspond to those subsequently observed.

DATA

Soybean futures and options prices beginning on 10/31/84 (the inception of options trading) and ending on 9/30/88 were used. The data, provided by the Chicago Board of Trade, consist of all time stamped transactions at which a price changed and are thought to be highly accurate and free of errors. Some trades were excluded to alleviate induced biases. Trades that occurred more than one year prior to expiration were excluded. Also, deep in- or out-of-the-money options were scrutinized carefully although there is no theoretical reason for exclusion.

Based on Bookstaber's arguments, synchronous futures and option prices were used to avoid possible distortions found in closing prices. A point in time near the center of the trading day (11:00 a.m.) was chosen and one trade per strike price traded that day was chosen based on its proximity to 11:00. Then, the futures price nearest in time to each option was selected as the "matched" futures price. Option prices were required to have a matched futures observation within 90 seconds of the option price to be admissible. Days with less than three option trades in the time window were deleted. A description of the resulting sample is given in Table 1.

To solve for the parameters of the expected price distributions, a risk-free rate also was needed. The rate used is based on the daily discount-basis yield of three month treasury bills as provided by the Federal Reserve Bank of Cleveland.

Methodology and Results

The expression

$$\min_{\beta} \left[\sum_{i=1}^{n} \{(C_i - b(T)\int_{K_i}^{\infty} f(Y_T|\beta)(Y_T-K_i)dY_T)^2\} + \sum_{j=1}^{m} \{(P_j - b(T)\int_{0}^{K_j} f(Y_T|\beta)(K_j-Y_T)dY_T)^2\}\right] \qquad (9)$$

was used to solved for a set of implied distribution parameters, β, that most nearly results in the observed option prices for both the SM and LN distributions.[2] Daily samples of "n" puts and "m" calls were used subject to the requirement that (m + n) be greater than the dimension of β (greater than 3 for the SM and greater than 2 for the LN). A least squares penalty function was used. Implied *ex ante* price distributions were computed for a total of 1715 contract days (or 3430 distributions) over the 26 contracts with an average of 8.76 different options per day.

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[2] The *Gauss* programming language was used on a DTK 386 machine. The algorithms were based on code originally written by Paul Fackler.

Figure 1 panel A illustrates the implied *ex ante* price densities for the July 1988 soybean futures contract that was implicit in the options prices on the 64th day prior to expiration of the option. Panel B shows the corresponding CDFs for both distributions.

Since the futures prices do not enter directly into the estimation of the distributions [equation (9)], comparisons of the first moment of the implied distributions with the current futures price gives an indication of the expected direction of futures price movements. For example, if the futures price is lower (higher) than the mean of the implicit *ex ante* price distribution, it indicates that the futures price is expected to increase (decrease). This type of information is not available if the futures price is used in the estimation of the *ex ante* distribution. Table 1 also summarizes the mean difference between average futures price and $E(Y_T)$ as reflected in the implied distributions. Note the increased differences during the near-harvest contracts. For the positive (negative) differences, it may indicate that the options market expectation is for the price to rise (fall) as contract expiration approaches.

The SM distribution yields negligible differences between futures prices and the first moments of the implied distributions, and these differences are smaller for the SM distribution than for the lognormal distribution. The true distribution of expected prices may be better reflected by the three parameter SM distribution than the more restrictive two parameter lognormal distribution.

CALIBRATION

For each of the non-overlapping 26 soybean contracts, *ex ante* distributions were examined at seven fixed intervals prior to expiration (7, 10, 20, 40, 60, 80, and 100 days), and comparisons were made with the contract prices at expiration. The fitted beta distribution [equation (8)] was examined as an indication of the shape of the calibration function. Figure 2 panel A shows a representative beta calibration function. The *ex ante* distribution [$F(Y_T)$], and the calibrated distribution [$K(F(Y_T)$] are given in panel B. The slope of the calibration function in A corresponds to the reweighting necessary to arrive at the calibrated function in B. The *ex ante* distribution shown is for the May 1988 soybean contract at 40 days prior to expiration parameterized as an SM distribution. The beta calibration function ($p = 1.41$ and $q = .93$) is derived by assessing the accuracy of 21 different contracts that traded 40 days out, and these experiences are used to compute a calibrated *ex ante* distribution.

Table 2 gives a summary of the estimated values of p and q for the seven fixed intervals used in calibration exercise. A "menu" for interpretation is included near the bottom of the table. For both the SM and lognormal distributions, there is some evidence that *ex ante* distributions are not well calibrated one week prior to contract expiration. The general shape of the beta function would be indicate a tendency for the futures prices to rise at expiration. More precisely, the option based estimates were drawn from a distribution that was over-dispersed and located to the left of the unknown distribution.

Over the range of 10-40 days prior to expiration, there is little evidence of miscalibration. At 80-100 days prior to expiration, the lognormal distribution appears to be miscalibrated, but the SM distribution appears reasonably well calibrated.

SUMMARY AND IMPLICATIONS

No-arbitrage pricing models can be used to derive *ex ante* price distributions from options price data. The SM distribution appears to be particularly useful in recovering information about expected price distributions.

Evidence from this study is that implied *ex ante* distributions are not accurate at times very near (seven days prior to) contract maturity. However, for the other time intervals investigated, a reasonably accurate *ex ante* distribution can be generated from options data with the SM distribution. The lognormal distribution, used in most options pricing studies, appears to be slightly less accurate than the SM distribution.

These techniques can be extended to provide probabilistic assessments of many uncertain future variables. The extensions would be simplest in markets for which contingent claims markets (like options markets) are already well functioning. For example, assessments of future interest rate and exchange rate probability distributions could be recovered from options on interest rate instruments and currency exchange futures. Options on stock indexes may foretell probability distributions of a performance measure of an aggregate economy. Metals, lumber, energy, fiber, food, and agricultural commodities each have reasonably well behaved options markets from which probabilistic information may be recoverable.

The analysis of producer and processor marketing strategies often relies on a subjective description of price risk. Techniques used in this study offer a promising alternative for estimating these *ex ante* distributions.

Table 1.

Descriptive Statistics of Soybean Options Samples

Contract	Days	Total Obs.	Dif11	DifFO	% Calls	{--LOGNORMAL--} Cents/bu.	STD	{----------SM---------} Cents/bu.	STD
Jan-85	28	192	11.6	14.2	50.5	0.165	0.422	0.136	1.300
Mar-85	57	400	16.4	18.1	61.3	0.071	0.962	0.119	0.400
May-85	72	482	16.6	21.2	64.7	-0.069	1.667	-0.075	1.233
Jul-85	93	684	17.2	21.8	65.8	-0.300	0.996	-0.144	0.961
Aug-85	32	213	21.0	28.5	65.8	-1.258	1.768	-0.738	1.277
Sep-85	22	169	22.4	33.2	71.0	1.439	14.705	-5.031	6.347
Nov-85	138	1310	16.9	18.2	63.3	-0.694	2.316	-0.364	2.155
Jan-86	58	432	16.1	21.8	58.3	0.021	0.668	-0.017	1.371
Mar-86	79	619	14.8	19.3	64.8	0.355	1.806	0.250	0.684
May-86	86	569	18.8	24.5	65.9	-0.076	1.552	0.027	1.046
Jul-86	120	870	18.8	25.3	71.3	0.223	4.267	-0.531	2.588
Aug-86	11	157	23.9	35.1	70.7	6.797	22.345	1.438	3.248
Sep-86	20	184	23.8	39.5	66.1	-1.120	3.902	0.202	3.544
Nov-86	74	1338	17.2	19.7	60.4	6.116	1.065	2.041	0.745
Jan-87	44	326	19.0	26.5	61.3	-0.100	0.787	-0.072	4.889
Mar-87	32	310	19.9	28.9	62.0	-0.007	1.257	-2.585	0.909
May-87	20	242	21.9	33.4	67.4	-0.023	0.507	-0.093	0.589
Jul-87	87	687	18.1	25.2	68.1	1.169	5.934	0.008	2.526
Aug-87	49	420	18.6	33.0	67.2	2.833	11.145	1.662	9.328
Sep-87	42	335	19.4	35.5	58.5	0.095	3.803	-0.363	2.570
Nov-87	121	1365	14.1	13.7	64.0	0.338	2.314	0.267	1.959
Jan-88	81	653	14.2	20.5	64.0	-0.069	1.023	-0.088	5.519
Mar-88	106	953	14.5	16.8	67.2	0.545	6.517	0.362	0.798
May-88	92	787	15.2	17.3	62.9	0.525	4.963	1.317	6.925
Jul-88	128	1058	17.6	19.1	68.5	2.082	8.789	0.792	9.275
Aug-88	23	265	23.3	28.3	66.8	-0.911	2.359	0.088	1.548

Days.....Total number of trade days for which option premia were sufficient to recover parameters of implied distributions.

Total Obs....Total number of observations that remained in the contract after the deletion/estimation criteria.

Dif11.....Mean absolute difference in minutes of all strike prices used from 11:00.

DifFO.....Mean absolute difference in seconds between the option and futures prices.

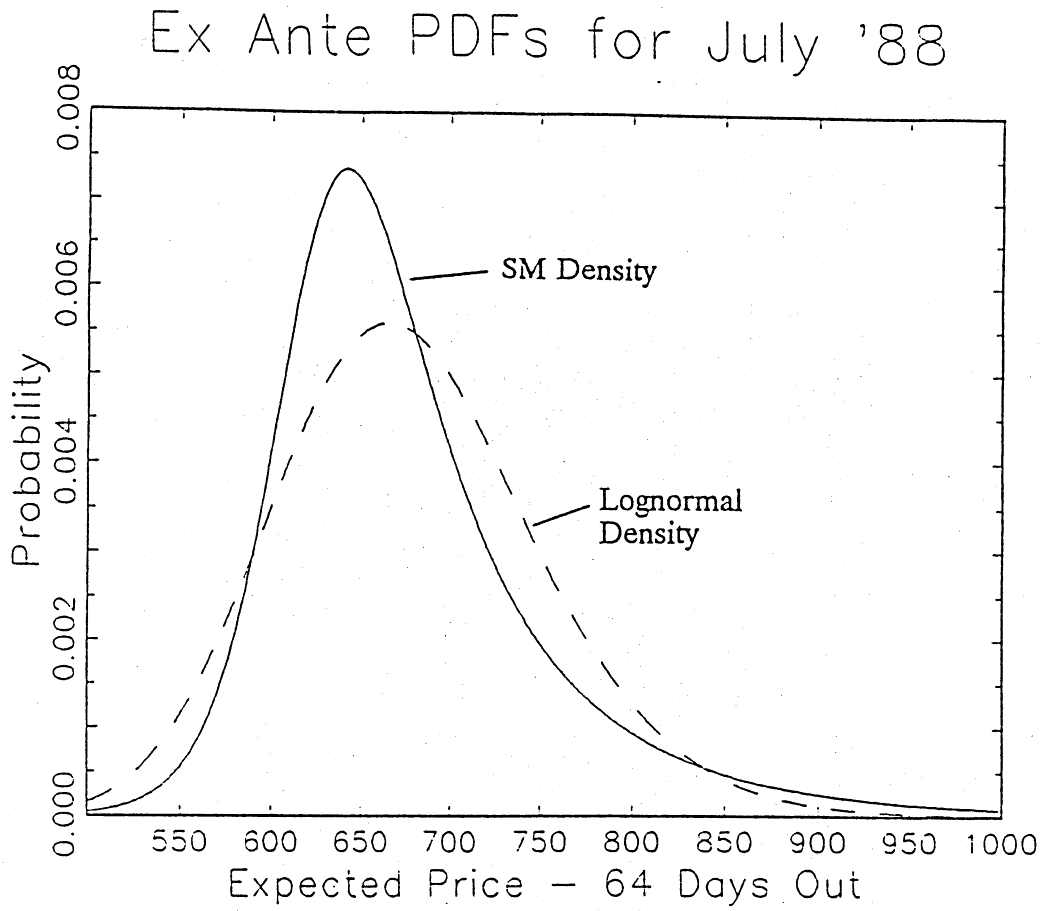
% Calls.....Percent of the sample represented by calls.

Cents/bu.....Mean of futures price minus first moment of implied distribution.

STD.....Standard deviation of daily Cents/bu. by contract.

Figure 1.

Panel A.



Panel B.

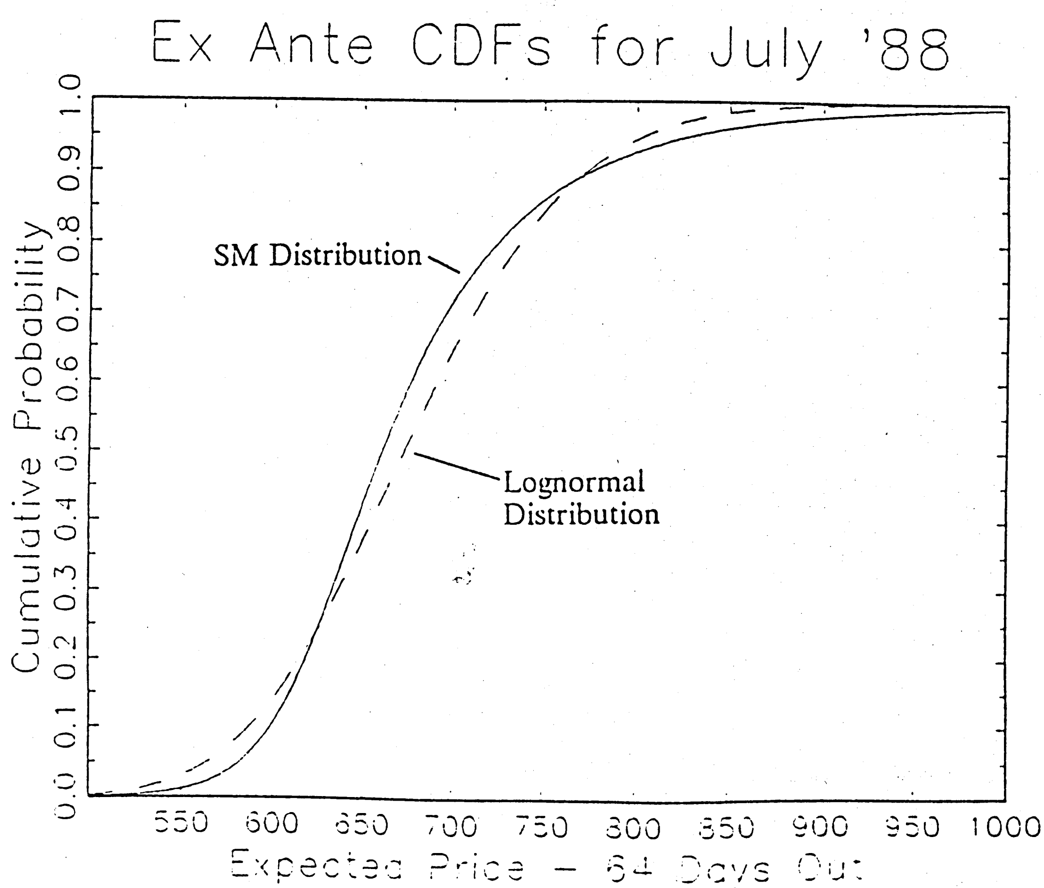
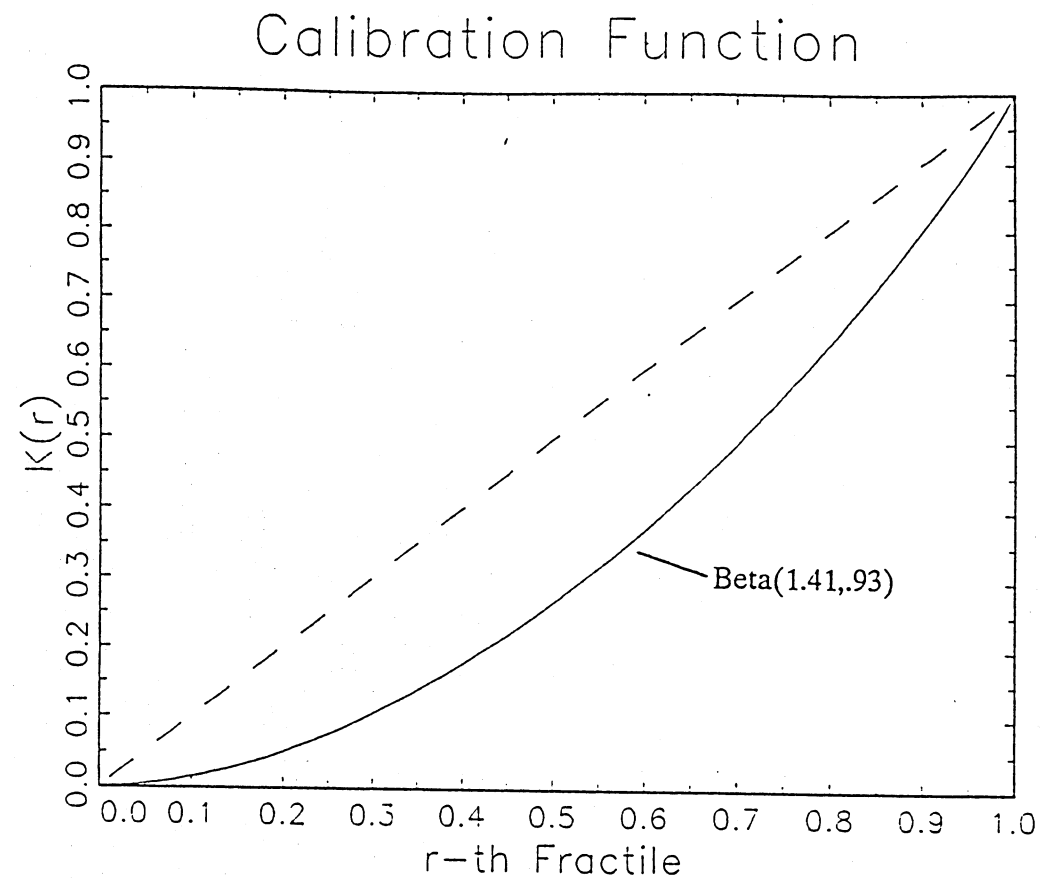


Figure 2.

Panel A.



Panel B.

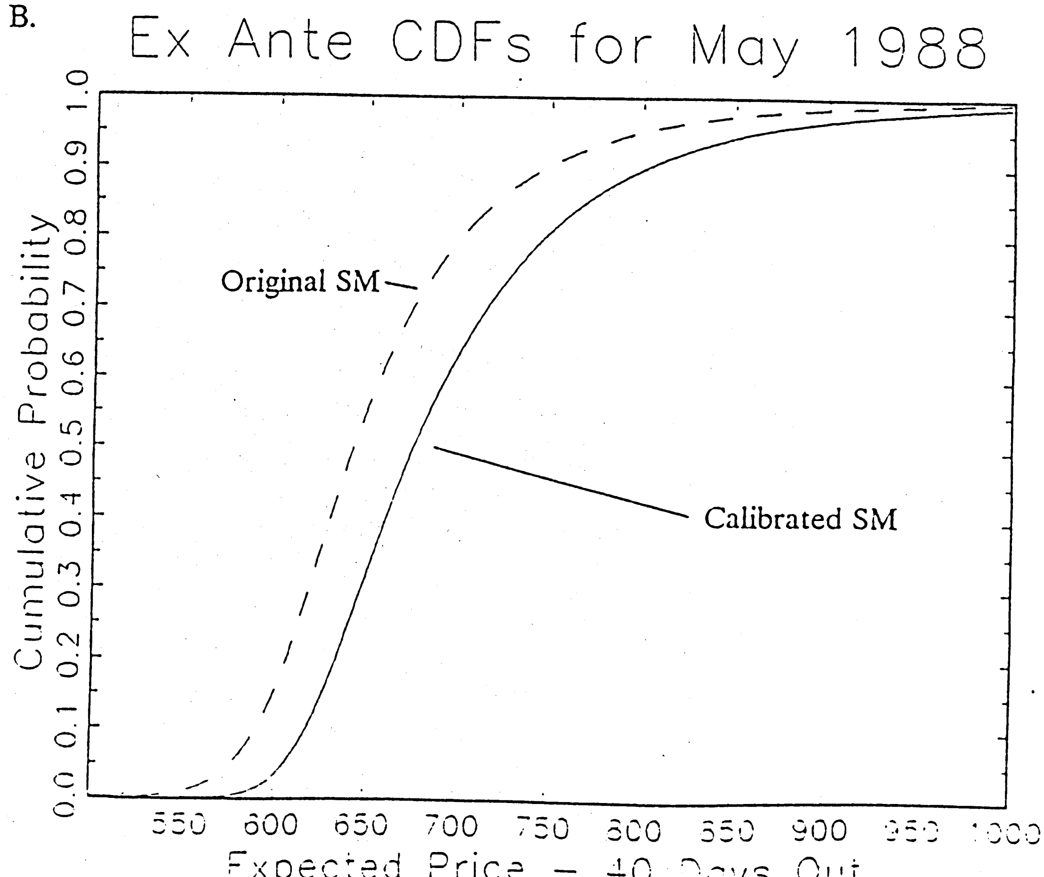


Table 2.

Calibration Statistics for Implied Soybean Distributions

Days to Maturity	Number of Observations	Parameters of Beta Calibration Function p	q	LR Statistic	prob > LR
Lognormal Distribution:					
5	22	5.274	3.660	19.978	0.000
10	24	1.455	1.117	2.067	0.356
20	25	1.331	1.228	1.070	0.586
40	21	1.184	0.724	4.480	0.106
60	20	1.288	0.791	3.854	0.146
80	19	0.898	0.538	7.544	0.023
100	19	0.894	0.506	9.444	0.009
SM Distribution:					
5	22	4.678	2.901	17.713	0.000
10	24	1.428	1.091	1.982	0.371
20	25	1.359	1.261	1.234	0.540
40	21	1.410	0.930	2.975	0.226
60	21	1.463	1.031	2.410	0.300
80	19	1.259	0.845	2.438	0.295
100	19	1.442	0.895	3.451	0.178

Note: A "Menu" to interpret shape of Beta calibration function is:

$p=q=1$	uniform distribution
$p=q>1$	"S"-shaped; (similar to normal CDF)
$p=q<1$	reverse "S"-shaped; (mirror image of "S" across 45° line)
$p<1, q>1$	"C"-shaped; (humped over uniform CDF)
$p>1, q<1$	"U"-shaped; (dropped under uniform CDF)

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