Confidence Intervals for Elasticities and Flexibilities.

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This paper examines methods for constructing confidence intervals around elasticity and flexibility point estimates, including three bootstrap-based approaches, a Taylor's series approximation, and approaches proposed by Fieller and Scheffe. Results show that all methods except Scheffe's worked reasonably well, but the Fieller and Taylor's series methods outperformed the bootstrapped-generated intervals.

Key words: Bootstrapping, Confidence intervals, Elasticities, Flexibilities.

Confidence Intervals for Elasticities and Flexibilities.

This paper describes, analyzes, and compares alternative methods of constructing confidence intervals for elasticities and flexibilities. Confidence intervals for these important economic statistics are generally not reported because their statistical properties for many models are unknown. In recent years the use of first-order Taylor's series approximations of unknown elasticity variances has become popular (Toevs). Two potential problems exist with this approach; first, the approximations may not be accurate due either to bias from truncation of the Taylor's series or from small-sample bias in the asymptotic regression parameter variances used in the Taylor's series formulae (Krinsky and Robb; Green, Hahn, and Rocke). Second, the approximate variances may not be useful in constructing confidence intervals (Efron 1981, 1987) because of possible asymmetries in the small sample distributions of elasticity estimators based on nonlinear combinations of random regression coefficients.

Elasticities and flexibilities from the popular linear or translog models are distributed as ratios of normal variables (Anderson and Thursby) and Miller, Capps, and Wells have suggested that confidence intervals may be constructed in these cases using methods developed by Fieller. Also, a modification has been proposed by Scheffé to address problems caused by the possible unboundedness of the Fieller intervals. A generally applicable nonparametric alternative for constructing confidence intervals is Efron's bootstrap. Efron (1981, 1987) has suggested three bootstrap methods for constructing confidence intervals.

To compare and evaluate the alternative methods for constructing confidence intervals in the important ratio of normals case, this analysis focuses on elasticities and flexibilities from linear functions and uses data from Waugh's classic study of demand. Confidence intervals constructed by the Fieller, Scheffé, Taylor's series, and alternative bootstrap methods are evaluated.

Confidence Intervals for Elasticities Constructed as Ratios

Let the true model be linear and denoted as

(1) $Y_t = \beta_o + \beta_1 X_{1t} + \ldots + \beta_k X_{kt} + \epsilon_t, \quad t = 1, \ldots, T,$

and the estimated model as

(2) $\hat{Y}_t = b_o + b_1 X_{1t} + \ldots + b_k X_{kt}, \quad t = 1, \ldots, T,$

where $X_1, \ldots, X_k$ are nonstochastic regressors, $b_o, \ldots, b_k$ are estimates of the true but unknown parameters $\beta_o, \ldots, \beta_k$, and $\hat{Y}_t$ is the predicted value of the dependent variable, Y_t. The disturbance term ϵ_t is assumed i.i.d. with variance σ^2. To invoke the methods of Fieller and Scheffé, ϵ_t must also be assumed to be normally distributed in which case it follows that $b_o, \ldots, b_k$, Y_t, and $\hat{Y}_t$ are also normally distributed. The bootstrap is a distribution-free methodology that does not require this normality assumption.

Denote the true, unobserved mean of Y_t by μ. The true elasticity of Y with respect to X_i evaluated at the mean values is then $\eta_{\mu,i} = \beta_i X_i / \mu$, where $X_i = (1/T)\Sigma_t X_{it}$. Estimated elasticities are most often computed and reported for the sample mean quantities $\hat{Y} = (1/T)\Sigma_t \hat{Y}_t$ or $Y = (1/T)\Sigma_t Y_t$, where $\hat{Y} = Y$ for the linear model. Thus,

(3) $\eta_{\hat{Y},i} = b_i X_i / \hat{Y} = \eta_{Y,i} = b_i X_i / Y, \; i = 1, \ldots, k.$

For notational convenience the true elasticity is often denoted as η and the estimate from (3) as $\hat{\eta}$.

To begin elucidating the alternative methods for constructing confidence intervals, note that the nonlinear functions in (3) can be linearized via a first-order Taylor's series expansion about $E[b_i] = \beta_i$ and $E[Y] = E[\hat{Y}] = \mu$:

$$(4) \quad \eta_{Y,i} \approx \eta + \frac{X_i}{\mu}(b_i - \beta_i) - \frac{\beta_i X_i}{\mu^2}(Y - \mu).$$

Moving η to the LHS, squaring both sides of (4), and using the sample estimates to substitute for the unknown β_i and μ obtains:

$$(5)\quad Var(\eta_{Y,i}) \approx \frac{X_i^2}{Y^2} Var(b_i) + \frac{b_i^2 X_i^2}{Y^4} Var(Y) - \frac{2 b_i X_i^2}{Y^3} Cov(b_i, Y).$$

The expression for $Var(\eta_{\hat{Y},i})$ is identical to (5) except that the covariance term is zero (MCW).

To construct a confidence interval for η based on (5), the analyst might appeal to the asymptotic normality of $\hat{\eta}$ and use, e.g., $\pm\, 1.645[Var(\eta_{Y,i})]^{1/2}$ to construct a symmetric 90 percent interval. The validity of the normality assumption in finite samples for the ratio of normals case has been analyzed by Hayya, Armstrong, and Gressis who conclude from Monte Carlo simulations interpreted in our elasticity context that the normality assumption is appropriate if $|correlation(b_i,Y)| \leq 0.5$, $|CV(b_i)| > 0.19$, and $CV(Y) \leq 0.09$, where CV denotes the coefficient of variation.

Fieller's method assumes the ϵ_t are distributed normally and proceeds by creating the variable $Z = bX - \eta Y$, where subscripts are omitted for convenience. It follows that $E[Z] = 0$, and that Z is distributed normally with

$$Var(Z) = X^2 Var(b) + \eta^2 Var(Y) - 2X\eta Cov(b,X).$$

These results, in turn, imply that

$$(6)\quad Prob\left[\frac{Z^2}{\hat{Var}(Z)} \leq F_\alpha\right] = 1 - \alpha,$$

where $\hat{Var}(Z)$ is the sample estimate of Var(Z) and F_α is the α percentile value of the F statistic with 1, T-k-1 degrees of freedom. Substituting for Z and $\hat{Var}(Z)$ in (6), it follows that the bounds of the $(1 - \alpha)$ confidence interval for η are those values of η which satisfy the equality:

$$(7)\quad (bX - \eta Y)^2 = F_\alpha[X^2 \hat{Var}(b) + \eta^2 \hat{Var}(Y) - 2X\eta \hat{Cov}(b,Y)],$$

where hats again denote sample estimates of the variances and covariance.

The interval for η implicitly defined in (7) is exact <u>if</u> Y and ϵ are distributed as normals (Fieller), and $\hat{Var}(b)$, $\hat{Var}(Y)$, and $\hat{Cov}(b,Y)$ are maximum likelihood estimators of the unknown true variances and covariance. Scheffé has called Fieller's solution an improper interval because it includes with positive probability a statement that is trivially true, namely that η lies on the real line in the interval $(-\infty,\infty)$ whenever the $1 - \alpha$ percent confidence interval for Y includes 0.

The alternative procedure proposed by Scheffé is to first test the hypothesis that $Y = 0$. Conditional on rejecting this hypothesis, a proper interval may be obtained for η but the probability of containing η is now a conditional probability which in most cases will be less than $(1 - \alpha)$. To regain the stated level of confidence, Scheffé has suggested replacing the F statistic with 1, T-k-1 degrees of freedom with a statistic S^2 where for the elasticity application:

$$S = (F_{\alpha,(1,T-K-1)})^{1/2} + [F_{\alpha(2,T-K-1)}\hat{Var}(Y)/Y^2]^{3/2} \times [(2F_{\alpha(2,T-K-1)})^{1/2} - (F_{\alpha(1,T-K-1)})^{1/2}].$$

Scheffé's modification always widens the Fieller interval.

Bootstrapping provides an alternative, but computationally intensive, method to generate confidence intervals regardless of the distribution of the statistic of interest. The methodology proceeds in the following steps:

1. The residuals $e_t = Y_t - \hat{Y}_t$, $t = 1, \ldots, T$ from the regression in (2) are computed and used to create a distribution, ϕ. The residuals must be scaled by a factor of $(T/(T-k))^{0.5}$ before being inserted into ϕ to avoid a downward bias in the resulting bootstrap standard errors (Freedman and Peters). Each scaled e_t is assigned mass 1/T in ϕ.

2. A new vector of quantities $\{Y_1^*, \ldots, Y_T^*\}$ is generated from the formula:

$$(8)\ Y_t^* = b_0 + b_1X_{1t} + \ldots + b_kX_{kt} + e_t^*,$$

where e_t^* is chosen by random draw with replacement from ϕ.

3. New parameter estimates $\{b_0^*, b_1^*, \ldots, b_k^*\}$ are generated from regressing Y_t^* on $\{X_{1t}, \ldots, X_{kt}\}$ and used to create new elasticity estimates: $\hat{\eta}^* = b_i^* X_i / Y^* = b_i^* X_i / \hat{Y}^*$, where $Y^* = (1/T)\Sigma_t Y_t^* = \hat{Y}^* = (1/T)\Sigma_t \hat{Y}_t^*$.

4. Steps 2 and 3 are repeated a large number of times, N, by redrawing from ϕ to generate an empirical distribution for the $\hat{\eta}^*$. Let these density and cumulative distribution functions be denoted by θ and Θ, respectively.

A $(1 - \alpha)$ percent confidence interval for η can be obtained from Efron's percentile method by simply deleting the outer $\alpha/2$ tails from θ. Efron's bias-corrected percentile (BC) method (Efron 1981, 1987) uses the standard normal distribution to smooth the empirical distribution of the $\hat{\eta}^*$ and makes an explicit transformation for asymmetry based on the deviation of the median of the bootstrap distribution from its expected value, defined to be the point estimate $\hat{\eta}$. The bias-corrected percentile acceleration method (BC_a) uses this same bias correction factor and also an acceleration constant, a, which corrects the confidence interval for any skewness in the empirical distribution of the $\hat{\eta}^*$ (Efron, 1987).

Both the BC and BC_a methods derive from the assumption that, although $\hat{\eta} - \eta$ may not be distributed normally, a monotone transformation, g, exists such that $g(\hat{\eta}) - g(\eta)$ is distributed normally with mean and variance depending upon which interval-generating method is invoked (Efron 1987). Unlike intervals obtained from applying percentiles from the standard normal or Student's t distribution to the estimated standard error, the bootstrap methods are designed to reflect small sample asymmetries in the distribution of the elasticity about its point estimate.

Which of the various methods will work better in practice is an empirical question. We proceed now to develop and apply an empirical methodology to evaluate the alternative interval-generating methods for the case of elasticities and flexibilities believed to be distributed as ratios of normals.

Empirical Methodology

Waugh used OLS to estimate price and income flexibilities for seven foods: potatoes, sweet potatoes, tomatoes, grapefruit, apples, beef, and milk. The demand equations were of the form:

(9) $P_t = \beta_0 + \beta_1 Q_t + \beta_2 M_t + \epsilon_t$

where P, Q, and M denote price, quantity, and income, respectively, and t denotes annual time series observations from 1948 - 1962 or for tomatoes from 1950 - 1962.

Waugh's OLS results for each commodity were replicated. (Waugh's milk equation involved a time trend and, hence, was omitted for symmetry.) Whereas the true price flexibility is $\eta_{\mu p,Q} = \beta_1 Q/\mu_P$, where $Q=(1/T)\Sigma_t Q_t$ and μ_P is the true, unknown mean of P_t, the estimated price flexibility is $\eta_{P,Q} = b_1 Q/P$, where b_1 is the OLS estimate of β_1 and $P = (1/T)\Sigma_t P_t$. Henceforth denote $\eta_{\mu p,Q}$ as η and $\eta_{P,Q}$ as $\hat{\eta}$. Similar derivations apply to the income flexibilities.

The OLS residuals for each commodity were used to create bootstrap data sets as described above. At each bootstrap trial j=1,...,N, for each commodity new OLS parameter estimates b^*_{0j}, b^*_{1j}, b^*_{2j} were computed along with estimated standard errors. These estimates were used to compute flexibilities and confidence intervals at each repetition using the Taylor's series, Fieller, and Scheffé methods. For example, the estimated price flexibilities are $\hat{\eta}^*_j = b^*_{1j} Q/P^*_j$, $P^*_j = 1/T\Sigma_t P^*_{jt}$. The $\hat{\eta}^*_j$, j=1,...,N comprise the empirical distribution, θ, of price flexibilities. This process was continued for N = 500 repetitions. This phase constituted the "outer loop" of the experiment (Freedman and Peters).

The key to the outer loop is that the bootstrap involves a fully defined simulation model where the parameters, b_0, b_1, b_2 and distribution of disturbances, ϕ, are known. Waugh's original price and income flexibilities evaluated at the data means, are the true mean flexibilities in this model. This observation suggests the following formal test of the intervals constructed at each trial via the Taylor's series, Fieller, and Scheffé methods: Does each contain the true flexibility, $\hat{\eta}$, the requisite number of times, e.g., 450 times for a 90 % interval when N = 500?

To put the various bootstrap methods of generating confidence intervals to the same formal test, the "inner loop" phase of the experiment was implemented (Freedman and Peters). This phase involved generating bootstrap trials for each of the N regressions produced in the outer loop. That is, for each outer loop regression, the residuals e_{jt} were computed, where

$$e_{jt} = P^*_{jt} - b^*_{0j} - b^*_{1j}Q_t - b^*_{2j}M_t, \quad j = 1,..., N, t = 1,...,T.$$

Let the distribution of these residuals, once again scaled by $(T/(T-k))^{0.5}$, be denoted by ϕ_j, j=1,...,N. Each ϕ_j was then used to create "double starred" data sets in the inner loop by redrawing with replacement from the appropriate ϕ_j. For example, on the j^{th} pass through the outer loop ϕ_j was created and used to compute the T-vector P^{**}_j, where

$$P^{**}_{jt} = b^*_{0j} + b^*_{1j}Q_t + b^*_{2j}M_t + e^*_{jt}, j=1,...,N, t=1,...,T,$$

where e^*_{jt} was drawn with replacement from ϕ_j. Double starred parameter estimates were then obtained from the regression of P^{**}_{jt} on Q_t and M_t and used to create double starred flexibilities, e.g., $\hat{\eta}^{**}_j = b^{**}_{1j}Q/ P^{**}_j$, where $P^{**}_j = (1/T)\Sigma_t P^{**}_t$. This completes one pass through the inner loop. Let the distribution ξ_j consist of the $\hat{\eta}^{**}_j$ obtained by repeating the inner loop procedure 500 times for each of the 500 repetitions of the outer loop. The inner loop, thus, generated 500x500 = 250,000 regressions for each commodity and 500 empirical distributions each consisting of 500 elements for each price and income flexibility.

Each inner loop empirical flexibility distribution ξ_j was then used to create bootstrap confidence intervals for the corresponding outer loop regression using Efron's three methods.

These intervals were then compared to the Taylor's series, Fieller, and Scheffé intervals computed in the outer loop and subjected to the same test of 90 % inclusion of the true flexibility.

Results

Results of the analysis are summarized in Tables 1 and 2. Table 1 contains Waugh's original price and income flexibilities evaluated at the sample means and provides average 90 % confidence intervals for them. All confidence interval bounds for the various methods are sample means based on the 500 replications.

Table 2 tests coverage for each interval. An interval's inclusion or exclusion of the true flexibility follows a binomial distribution which under the null hypothesis has mean $\rho = 0.9$ and variance $Var(\rho) = 0.9(1.0\text{-}0.9)/N$. Because a binomial distribution converges asymptotically to a normal distribution, T-statistics can be constructed to test for significant departures from 90 % inclusion of the true flexibilities using the known standard errors under the null hypothesis.

The Fieller and Taylor's series methods generated nearly identical intervals on average and both performed very well. From Table 2, in only two of the possible 12 cases was the null hypothesis of 90 % coverage rejected for each method.

Scheffé's method necessarily widens the Fieller interval, but as Table 1 documents, the magnitude in this case was so dramatic as to preclude in most cases making any meaningful statements about the true flexibility. From Table 2, the coverage rate for the "90 %" Scheffé interval was nearly 100 % in all cases and significantly different from the purported 90 %.

The three bootstrap methods produced similar results which on average were somewhat inferior to those obtained by the Fieller and Taylor's series methods based on the coverage test. In general, the bootstrap methods produced tighter intervals than Fieller or Taylor's series and failed the test of 90 % coverage a number of times.

In further analyzing the results, it becomes apparent that similarity between the various intervals is the outcome of nearly normal flexibility distributions for each data set. This conclusion was documented by computing the Hayya, Armstrong, and Gressis conditions (see p. 3) for the ratio of normals to be distributed normally. These conditions were satisfied for 10 of the 12 flexibilities with rejections only for the beef price and income flexibility. If the elasticity or flexibility is distributed normally, Fieller and Taylor's series will produce the same interval, and the three bootstrap methods will also converge to the same interval.

An important question is the robustness of these results. In particular, are they artifacts of the Waugh data or might they hold generally? To address this question, a limited Monte Carlo experiment was conducted, wherein confidence intervals were constructed and evaluated for flexibilities generated from a series of simulated data sets.

Although space limitations preclude a detailed description of this analysis, it is noteworthy to report that despite the use of widely divergent error distributions to construct the simulated data, the Taylor's series and Fieller methods continued to peform well for flexibilities evaluated at the means and generally, though not always, outperformed the three bootstrap methods on the simulated data sets as well. The bootstrap methods produced intervals that were very similar to each other, again reflecting the underlying, symmetry of the intervals. The bootstrap intervals were, once again, somewhat tighter than their Fieller and Taylor's series counterparts.

Conclusions and Recommendations

This paper has illustrated and analyzed alternative methods for computing confidence intervals for elasticities and flexibilities in the ratio of normals case. While our results unequivocally reject Scheffé's method, the other five alternatives generally performed reasonably well and produced quite similar intervals, although exact coverage rates differed somewhat among the alternative methods.

The perfomance here of Fieller's method is an affirmation of MCW's recommendation for its use. However, the nearly identical performance by the simple Taylor's series interval calls into question criticism of this method by MCW, particularly since the approximate normality conditions which cause Taylor's series to perform well have been shown in this study to be quite pervasive and because the necessary variance calculations are provided by the major statistical packages.

Efron's alternative bootstrap methods all produced similar confidence intervals, again reflecting the pervasive symmetry of the flexibility distributions suggesting that it often may be safisfactory to use the simpler percentile or bias-corrected methods rather than the bias-corrected acceleration method which requires considerably more programming.

The bootstrap methods were modestly outperformed in the coverage test by the simpler Fieller and Taylor's series methods. However, this result is not inconsistent with the gist of preliminary empirical evidence on the bootstrap (see for example the comments following Efron (1987)). One explanation for bootstrapping's moderately poor performance in this study may be the relatively small sample sizes in the Waugh data.

Given the array of plausible interval-generating methods available to the analyst, we believe the bottom-line conclusion from this study is that the era of reporting only elasticity or flexibility point estimates should end.

Table 1

Mean Ninety Percent Confidence Intervals for Waugh's Flexibilities

								Bootstrap Methods					
		Fieller		Scheffé		Taylor's Series		Percentile		Bias Corrected		BC Acceleration	
Commodity	Point Estimate	lower	upper	lower	upper	lower	upper	lower	upper	lower	upper	lower	upper
						—Price Flexibilities—							
Potatoes	-2.567	-4.07	-1.17	-5.26	-0.03	-4.07	-1.17	-3.96	-1.26	-3.95	-1.25	-3.97	-1.25
Sweet Potatoes	-0.764	-1.20	-0.34	-1.55	0.00	-1.20	-0.34	-1.17	-0.37	-1.17	-0.37	-1.18	-0.38
Grapefruit	-0.822	-1.21	-0.44	-1.54	-1.36	-1.21	-0.44	-1.19	-0.48	-1.21	-0.49	-1.20	-0.48
Apples	-0.807	-1.10	-0.52	-1.35	-0.29	-1.10	-0.51	-1.08	-0.54	-1.09	-0.55	-1.09	-0.55
Tomatoes	-0.853	-1.67	0.05	-2.34	0.61	-1.67	-0.05	-1.60	-0.11	-1.60	-0.11	-1.60	-0.11
Beef	-1.444	-1.67	-1.24	-1.84	-1.08	-1.66	-1.24	-1.64	-1.27	-1.64	-1.26	-1.63	-1.26
						—Income Flexibilities—							
Potatoes	0.212	-0.14	0.55	-0.41	0.82	-0.14	0.55	-0.11	0.52	-0.11	0.52	-0.11	0.52
Sweet Potatoes	-0.627	-1.33	0.07	-1.90	0.63	-1.33	0.07	-1.28	0.02	-1.27	0.02	-1.28	-0.02
Grapefruit	0.545	0.27	0.83	0.05	1.06	0.27	0.83	0.28	0.79	0.27	0.78	0.27	0.79
Apples	0.325	0.10	0.55	-0.08	0.74	0.10	0.55	0.11	0.53	0.11	0.53	0.11	0.53
Tomatoes	0.360	0.14	0.57	-0.03	0.74	0.14	0.57	0.16	0.55	0.16	0.55	0.16	0.55
Beef	1.288	1.09	1.50	0.94	1.66	1.09	1.50	1.12	1.47	1.12	1.47	1.12	1.47

Table 2

Percentage of Time the True Flexibility is Captured in the "Ninety" Percent Interval

	Fieller		Scheffé		Taylor's Series		Bootstrap Methods: Percentile		Bootstrap Methods: Bias Corrected		Bootstrap Methods: BC Acceleration	
Commodity	Price Flexibility	Income Flexibility	Price Flexibility	Income Flexibility	Price Flexibility	Income Flexibility	Price Flexibility	Income Flexibility	Price Flexibility	Income Flexibility	Price Flexibility	Income Flexibility
Potatoes	92.0%	89.8%	99.8%	100.0%	92.0%	89.8%	90.0%	94.4%	90.2%	94.2%	89.2%	94.2%
	(1.49)	(0.15)	(7.30)*	(7.45)*	(1.49)	(0.15)	(0.00)	(3.27)*	(0.15)	(3.13)*	(0.60)	(3.13)*
Sweet Potatoes	91.2	93.4	99.2	99.2	91.2	93.4	89.2	92.8	89.6	93.0	88.6	92.8
	(0.89)	(2.53)*	(6.86)*	(6.86)*	(0.89)	(2.53)*	(0.60)	(2.09)*	(0.30)	(2.24)*	(1.04)	(2.09)*
Grapefruit	90.8	90.4	99.2	100.0	91.0	90.2	89.4	94.8	89.8	95.2	89.4	95.2
	(0.60)	(0.30)	(6.86)*	(7.45)*	(0.75)	(0.15)	(0.45)	(3.57)*	(0.15)	(3.88)*	(0.45)	(3.88)*
Apples	91.2	89.2	98.8	99.6	91.4	89.2	88.4	94.4	88.8	94.2	88.6	93.8
	(0.89)	(0.60)	(6.56)*	(7.15)*	(1.04)	(0.60)	(1.19)	(3.28)*	(0.89)	(3.13)*	(1.04)	(2.83)*
Tomatoes	89.6	90.2	100.0	99.2	89.6	90.4	87.4	93.6	87.2	92.8	88.4	93.0
	(0.30)	(0.15)	(7.45)*	(6.86)*	(0.30)	(0.30)	(1.93)*	(2.68)*	(2.08)*	(2.08)*	(1.19)	(2.23)*
Beef	91.2	93.0	99.8	100.0	91.2	92.8	84.8	93.4	84.8	93.8	84.4	94.2
	(0.89)	(2.23)*	(7.30)*	(7.45)*	(0.89)	(2.08)*	(3.88)*	(2.53)*	(3.88)*	(2.83)*	(4.17)*	(3.13)*

t statistics are in parentheses.
* denotes statistical significance at $\delta = 0.10$.

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